Non-standard number representation: computer arithmetic, beta-numeration and quasicrystals

Christiane Frougny[a,1],
[a] *LIAFA UMR 7089 CNRS, and Université Paris 8*

Abstract. The purpose of this survey is to present the main concepts and results in non-standard number representation, and to give some examples of practical applications. This domain lies at the interface between discrete mathematics (dynamical systems, number theory, combinatorics) and computer science (computer arithmetic, cryptography, coding theory, algorithms). It also plays an important role in the modelization of physical structures like quasicrystals.

Keywords. Number representation, computer arithmetic, quasicrystal

1. Introduction

Non-standard number representation is emerging as a new research field, with many difficult open questions, and several important applications. The notions presented in this contribution are strongly related to the chapters of this volume written by Akiyama, Pelantová and Masáková, and Sakarovitch.

Our purpose is to explain how the simplest way of representing numbers — an integer base β and a canonical set of digits $\{0, 1, \dots, \beta - 1\}$ — is not sufficient for solving some problems.

In computer arithmetic, the challenge is to perform fast arithmetic. We will see how this task can be achieved by using a different set of digits. This will also allow on-line arithmetic, where it is possible to pipe-line additions, subtractions, multiplications and divisions.

Beta-numeration consists in the use of a base β which is an irrational number. This field is closely related to symbolic dynamics, as the set of β-expansions of real numbers of the unit interval forms a dynamical system. In this survey, we will present results connected with finite automata theory. Pisot numbers, which are algebraic integers with Galois conjugates lying inside the open unit disk, play a key role, as they generalize nicely the integers.

[1]Correspondence to: Christiane Frougny, LIAFA UMR 7089 CNRS, 2 place Jussieu, 75251 Paris cedex 05, France. E-mail: Christiane.Frougny@liafa.jussieu.fr.

Quasicrystals are a kind of solid in which the atoms are arranged in a seemingly regular, but non-repeating structure. The first one, observed by Shechtman in 1982, presents a five-fold symmetry, which is forbidden in classical crystallography. In a quasicrystal, the pattern of atoms is only quasiperiodic. The first observed quasicrystal is strongly related to the golden mean, and in this theory also, Pisot numbers are deeply rooted. I will explain how beta-numeration is an adequate tool for the modelization of quasi-crystals.

2. Preliminaries

We refer the reader to [17] and to [42]. An *alphabet* A is a finite set. A finite sequence of elements of A is called a *word*, and the set of words on A is the free monoid A^*. The *empty word* is denoted by ε. The set of infinite sequences or infinite words on A is denoted by $A^{\mathbb{N}}$. Let v be a word of A^*, denote by v^n the concatenation of v to itself n times, and by v^ω the infinite concatenation $vvv\cdots$. A word is said to be *eventually periodic* if it is of the form uv^ω.

An *automaton over* A, $\mathcal{A} = (Q, A, E, I, T)$, is a directed graph labelled by elements of A. The set of vertices, traditionally called *states*, is denoted by Q, $I \subset Q$ is the set of *initial* states, $T \subset Q$ is the set of *terminal* states and $E \subset Q \times A \times Q$ is the set of labelled *edges*. If $(p, a, q) \in E$, we note $p \xrightarrow{a} q$. The automaton is *finite* if Q is finite. The automaton $\mathcal{A}$ is *deterministic* if E is the graph of a (partial) function from $Q \times A$ into Q, and if there is a unique initial state. A subset H of A^* is said to be *recognizable by a finite automaton* if there exists a finite automaton $\mathcal{A}$ such that H is equal to the set of labels of paths starting in an initial state and ending in a terminal state. A subset K of $A^{\mathbb{N}}$ is said to be *recognizable by a finite automaton* if there exists a finite automaton $\mathcal{A}$ such that K is equal to the set of labels of infinite paths starting in an initial state and going infinitely often through a terminal state (Bűchi acceptance condition, see [17]).

We are also interested in 2-tape automata or transducers. Let A and B be two alphabets. A *transducer* is an automaton over the non-free monoid $A^* \times B^*$: $\mathcal{A} = (Q, A^* \times B^*, E, I, T)$ is a directed graph the edges of which are labelled by elements of $A^* \times B^*$. Words of A^* are referred to as *input words*, as words of B^* are referred to as *output words*. If $(p, (f, g), q) \in E$, we note $p \xrightarrow{f|g} q$. The transducer is finite if Q and E are finite.

A relation R of $A^* \times B^*$ is said to be *computable by a finite transducer* if there exists a finite transducer $\mathcal{A}$ such that R is equal to the set of labels of paths starting in an initial state and ending in a terminal state. A function is computable by a finite transducer if its graph is computable by a finite transducer. These definitions extend to relations and functions of infinite words as above.

A left *sequential* transducer is a finite transducer where edges are labelled by elements of $A \times B^*$, and such that the *underlying input automaton* obtained by taking the projection over A of the label of every edge is deterministic. For finite words, there is a *terminal* partial function $\omega : Q \longrightarrow B^*$, whose value is concatenated to the output word corresponding to a computation in $\mathcal{A}$. The same definition works for functions of infinite words, considering infinite paths in $\mathcal{A}$, but there is no terminal function ω in that case. The notion of a *right* sequential transducer is defined similarly.

3. Computer arithmetic

Computer arithmetic is the field which gathers techniques to build fast, efficient and robust arithmetic processors and algorithms. In what follows, we focus on the problems concerning number representation.

3.1. Standard number representation

We consider here only positional number systems, defined by a *base* β and a set of *digits* A_β. In the standard number representation, β is an integer greater than one, $\beta > 1$, and $A_\beta = \{0, 1, \ldots, \beta - 1\}$ is also called the *canonical alphabet* of digits.

A *β-representation* of a positive integer N is a sequence of digits from A_β, that is to say a *word* $a_k \cdots a_0$ on A_β, such that $N = \sum_{i=0}^{k} a_i \beta^i$. It is denoted $\langle N \rangle_\beta = a_k \cdots a_0$, most significant digit first. This representation is unique (called *normal*) if $a_k \neq 0$.

A *β-representation* of a number x in $[0, 1]$ is an infinite sequence (word) $(x_i)_{i \geq 1}$ of elements of A_β such that $x = \sum_{i \geq 1} x_i \beta^{-i}$. It is denoted $\langle x \rangle_\beta = .x_1 x_2 \cdots$ This representation is unique if it does not end in $(\beta - 1)^\omega$, in which case it is said to be the *β-expansion* of x. By shifting, any real $x > 1$ can be given a representation.

We now recall some properties satisfied by the standard number system, see [31] for the proofs. Let C be an alphabet of positive or negative digits containing $A_\beta = \{0, \ldots, \beta - 1\}$. The *numerical value* in base β is the fonction $\pi_\beta : C^* \to \mathbb{Z}$ such that $\pi_\beta(c_k \cdots c_0) = \sum_{i=0}^{k} c_i \beta^i$. Define the *digit-set conversion* on C as the function $\chi_\beta : C^* \to A_\beta^*$ such that $\chi_\beta(c_k \cdots c_0) = a_n \cdots a_0$, where $a_n \cdots a_0$ is a β-representation on A_β of the number $\pi_\beta(c_k \cdots c_0)$.

PROPOSITION **1** [17] *For any alphabet C the digit-set conversion on C is a right sequential function.*

Addition can be seen as a digit-set conversion on the alphabet $\{0, \ldots, 2(\beta - 1)\}$, subtraction is a digit-set conversion on $\{-(\beta - 1), \ldots, (\beta - 1)\}$, and multiplication by a fixed positive integer m is a digit-set conversion on $\{0, m, \ldots, m(\beta - 1)\}$. Notice that arbitrary multiplication of two integers is not computable by a finite automaton.

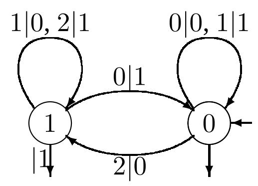


Figure 1. Addition of integers in base 2

On the contrary, division by a fixed positive integer m is a left sequential function.

As mentioned above, the representation of the real numbers is not unique, since, for $0 \leq d \leq \beta - 2$, the words $d(\beta - 1)^\omega$ and $(d + 1)0^\omega$ have the same value.

PROPOSITION **2** *The normalization function* $\nu : A_\beta^{\mathbb{N}} \longrightarrow A_\beta^{\mathbb{N}}$ *which transforms improper representations ending in* $(\beta-1)^\omega$ *into normal expansions ending in* 0^ω *is computable by a finite transducer.*

The picture on Fig. 2 shows a transducer for the normalization base 2. Infinitely repeated states are indicated by double circles.

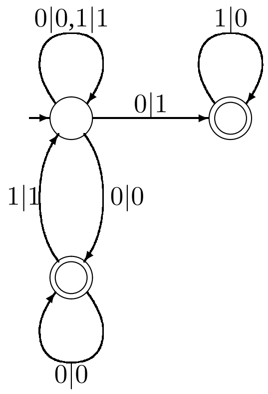


Figure 2. Normalization of real numbers in base 2

Notice that this transducer is not left sequential.

3.2. Redundant representations

Redundant representations are popular in computer arithmetic. Let us take for alphabet of digits a set $C = \{c, c+1, \ldots, c+h-1\}$. Fix n to be the number of positions, that is to say the length of the representations of the integers. Then the following result is folklore.

THEOREM **1** *Let* $I = [c\frac{\beta^n-1}{\beta-1}, (c+h-1)\frac{\beta^n-1}{\beta-1}]$.
If $|C| < \beta$ *some integers in* I *have no representation in base* β *with* n *positions.*
If $|C| = \beta$ *every integer in* I *has a unique representation.*
If $|C| > \beta$*, every integer in* I *has a representation, non necessarily unique.*

The same result has a real number version, with $I = [c\frac{1}{\beta-1}, (c+h-1)\frac{1}{\beta-1}]$.

When $|C| > \beta$, the system is said to be *redundant*. Cauchy [12] already considered the case $\beta = 10$ and $C = \{-5, -4, \ldots, 4, 5\}$. In computer arithmetic, the most interesting cases are $\beta = 10$ and $C = \{-6, \ldots, 6\}$, introduced by Avizienis [3], and $\beta = 2$ with $C = \{-1, 0, 1\}$, see Chow et Robertson [13].

In a redundant number system, it is possible to design fast algorithms for addition. More precisely, take an integer $a \geq 1$ and let $C = \{-a, -a+1, \ldots, a\}$ be a signed digit alphabet. Since the alphabet is symmetric, the opposite of a number is simply obtained by taking opposite digits. From the result above, there is redundancy when $2a \geq \beta$. To be able to determine the sign of a number represented as a word $c_{n-1} \cdots c_0$ only by looking at the sign of the most significant digit c_{n-1}, we must take $a \leq \beta - 1$. Under these hypotheses, it is possible to perform addition in constant time in parallel, since there is no propagation of the carry. The idea is the following. First suppose that $\beta/2 < a \leq \beta - 1$.

Take two representations $c_{n-1} \cdots c_0$ and $d_{n-1} \cdots d_0$ on C, of the numbers M and N respectively. For $0 \leq i \leq n-1$ set $z_i = c_i + d_i$. Then,

1. if $a \leq z_i \leq 2a$, set $r_{i+1} = 1$ and $s_i = z_i - \beta$
2. if $-2a \leq z_i \leq -a$, set $r_{i+1} = -1$ and $s_i = z_i + \beta$
3. if $-a+1 \leq z_i \leq a-1$, set $r_{i+1} = 0$ and $s_i = z_i$.

Then set $r_0 = 0$, and, for $0 \leq i \leq n-1$, $e_i = c_i + r_i$, and $e_n = c_n$. Thus $e_n \ldots e_0$ is a β-representation of $M+N$, with all the digits e_i belonging to C.

A slightly more complicated algorithm works in the case $\beta = 2a$, where a window is used to look at the sign of the right neighbour of the current position:

1. if $a+1 \leq z_i \leq 2a$, set $r_{i+1} = 1$ and $s_i = z_i - \beta$
2. if $-2a \leq z_i \leq -a-1$, set $r_{i+1} = -1$ and $s_i = z_i + \beta$
3. if $-a+1 \leq z_i \leq a-1$, set $r_{i+1} = 0$ and $s_i = z_i$
4. if $z_i = a$ then if $z_{i-1} \leq 0$ set $r_{i+1} = 0$ and $s_i = z_i$; if $z_{i-1} > 0$ set $r_{i+1} = 1$ and $s_i = z_i - \beta$
5. if $z_i = -a$ then if $z_{i-1} \geq 0$, set $r_{i+1} = 0$ and $s_i = z_i$; if $z_{i-1} < 0$ set $r_{i+1} = -1$ and $s_i = z_i + \beta$.

Special representations in base 2 with digit-set $\{-1, 0, 1\}$ such that the number of non-zero digits is minimal where considered by Booth [10]. It is a right-to-left recoding of a standard representation: every factor of form 01^n, with $n \geq 2$, is transformed into $10^{n-1}\bar{1}$, where $\bar{1}$ denotes the signed digit -1. The Booth recoding is a right sequential function from $\{0,1\}^*$ to $\{-1,0,1\}^*$ realized by the transducer depicted on Fig. 3.

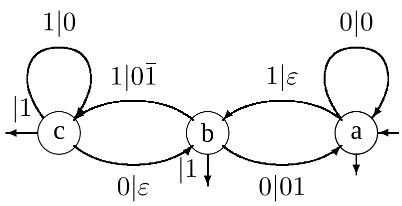


Figure 3. Booth right sequential recoding

The applications of the Booth normal form are multiplication, internal representation for dividers in base 4 with digits in $\{-3, \ldots, 3\}$, and computations on elliptic curves, see [33].

Another widely used representation is the so-called *carry-save* representation. Here the base is $\beta = 2$, and the alphabet of digits is $D = \{0, 1, 2\}$. Addition of a representation with digits in D and a representation with digits in $\{0, 1\}$ with result on D can be done in constant time in parallel. This has important applications for the design of internal adders in multipliers, see [30,20].

3.3. On-line computability

In computer arithmetic, on-line computation consists of performing arithmetic operations in Most Significant Digit First (MSDF) mode, digit serially after a certain latency delay. This allows the pipelining of different operations such as addition, multiplication and division. It is also appropriate for the processing of real numbers having infinite expansions. It is well known that when multiplying two real numbers, only the left part of the result is significant. To be able to perform on-line addition, it is necessary to use a redundant number system (see [44], [19]).

We now give a formal definition of on-line computability. Let A and B be two finite digit sets. Let

$$\varphi : A^{\mathbb{N}} \to B^{\mathbb{N}}$$
$$(a_j)_{j\geq 1} \mapsto (b_j)_{j\geq 1}$$

The function φ is said to be *on-line computable with delay* δ if there exists a natural number δ such that, for each $j \geq 1$ there exists a function $\Phi_j : A^{j+\delta} \to B$ such that $b_j = \Phi_j(a_1 \cdots a_{j+\delta})$, where $A^{j+\delta}$ denotes the set of sequences of length $j+\delta$ of elements of A. This definition extends readily to functions of several variables.

Recall that a distance ρ can be defined on $A^{\mathbb{N}}$ as follows: let $v = (v_j)_{j\geq 1}$ and $w = (w_j)_{j\geq 1}$ be in $A^{\mathbb{N}}$, set $\rho(v,w) = 2^{-r}$ where $r = \min\{j \mid v_j \neq w_j\}$ if $v \neq w$, $\rho(v,w) = 0$ otherwise. The set $A^{\mathbb{N}}$ is then a compact metric space. This topology is equivalent to the product topology. Then any function from $A^{\mathbb{N}}$ to $B^{\mathbb{N}}$ which is on-line computable with delay δ is 2^{δ}-Lipschitz, and is thus uniformly continuous [23].

It is well known that some functions are not on-line computable, like addition in the standard binary system with canonical digit set $\{0,1\}$. When the representation is redundant, addition and multiplication can be computed on-line. More precisely, in integer base β, addition on the alphabet $\{-a,\ldots,a\}$ is on-line computable with delay 1 if $\beta/2 < a \leq \beta-1$, and with delay 2 if $\beta = 2a$.

Multiplication of two numbers represented in integer base $\beta > 1$ with digits in $C = \{-a,\ldots,a\}$, $\beta/2 \leq a \leq \beta-1$, is computable by an on-line algorithm with delay δ, where δ is the smallest positive integer such that

$$\frac{\beta}{2} + \frac{2a^2}{\beta^{\delta}(\beta-1)} \leq a + \frac{1}{2}.$$

Thus for current cases, the delay is as follows. If $\beta = 2$ and $a = 1$, $\delta = 2$. If $\beta = 3$ and $a = 2$, $\delta = 2$. If $\beta = 2a \geq 4$ then $\delta = 2$. If $\beta \geq 4$ and if $a \geq \lfloor \beta/2 \rfloor + 1$, $\delta = 1$.

A left *on-line finite automaton* is a particular left sequential transducer, which is defined as follows:
- there is a transient part: during a time δ (the delay) the automaton reads without writing
- and there is a synchronous part where the transitions are letter-to-letter.

The following result follows easily from the properties recalled above.

PROPOSITION **3** *Let $\beta > 1$ be an integer. Every affine function with rational coefficients is computable in base β by a left on-line finite automaton on $C = \{-a, -a+1, \ldots, a\}$, with $\beta/2 \leq a \leq \beta-1$.*

The following result which is a kind of a converse has been proved by Muller [35]. Again let a such that $\beta/2 \leq a \leq \beta - 1$, and take $D = \{-d, \ldots, d\}$ with $d \geq a$. Set $I = [-a/(\beta-1), a/(\beta-1)]$, $J = [-d/(\beta-1), d/(\beta-1)]$. Let χ be a function such that there exists a function $\chi_{\mathbb{R}}$ making the following diagram to commute

$$\begin{array}{ccc} D^{\mathbb{N}} & \xrightarrow{\ \chi\ } & B^{\mathbb{N}} \\ {\scriptstyle \pi_\beta}\downarrow & & \downarrow{\scriptstyle \pi_\beta} \\ J & \xrightarrow[\ \chi_{\mathbb{R}}\ ]{} & I \end{array}$$

The function $\chi_{\mathbb{R}}$ is called the *real interpretation* of the function χ.

PROPOSITION **4** *Let χ be a function as above. Suppose that χ is computed by a left on-line finite automaton. If the second derivative $\chi_{\mathbb{R}}^{"}$ is piecewise continuous, then, in each interval where $\chi_{\mathbb{R}}^{"}$ is continuous, $\chi_{\mathbb{R}}$ is affine with rational coefficients.*

3.4. *Complex base*

To represent complex numbers, complex bases have been introduced in order to handle a complex number as a sequence of integer digits.

3.4.1. *Knuth number system*

Knuth [29] used base $\beta = i\sqrt{b}$, with b integer ≥ 2 and digit set $A_\beta = \{0, \ldots, b-1\}$. In this system every complex number has a representation.

If $b = c^2$, every Gaussian integer has a unique finite representation of the form $a_k \cdots a_0 \raisebox{0.3ex}{.} a_{-1}$.

EXAMPLE **1** Let $\beta = 2i$, then $A_\beta = \{0, \ldots, 3\}$ and $z = 4 + i$ is represented as $10310{\cdot}2$. □

The following results are derived from the ones valid in integer base. On A_β addition in base $\beta = i\sqrt{b}$ is right sequential. On $C = \{-a, -a+1, \ldots, a\}$ with $b/2 \leq a \leq b-1\}$, addition is computable in constant time in parallel, and realizable by an on-line finite automaton, see [36,23,43].

3.4.2. *Penney number system*

In this complex number system, the base is of the form $\beta = -b + i$, with b integer ≥ 1, and digit set $A_\beta = \{0, \ldots, b^2\}$. The case $b = 1$ was introduced by Penney [39].

We summarize the main results. Every complex number has a representation. Every Gaussian integer has a unique integer representation of the form $a_k \cdots a_0 \in A_\beta^*$. On A_β addition in base $\beta = -b + i$ is right subsequential [41].

The case $\beta = -1 + i$ and $A_\beta = \{0, 1\}$ has received a lot of attention in computer arithmetics for implementation in arithmetic processors. On $C = \{-a, -a+1, \cdots, a\}$, with $a = 1$, 2 or 3, addition in base $-1 + i$ is computable in constant time in parallel, and realizable by an on-line finite automaton, see [15,36,23,43].

3.5. Real basis

Muller in [34] introduced an original way of representing real numbers, for application to the CORDIC algorithms for computation of elementary functions. Let $U = (u_n)_{n\geq 0}$ be a decreasing sequence of positive real numbers, summable, and let D be a finite alphabet of integer digits. Under certain conditions a real number x can be represented as

$$x = \sum_{n\geq 0} d_n u_n$$

with $d_n \in D$ by a greedy algorithm.

For instance, take $u_n = \log(1 + 2^{-n})$, and $D = \{0, 1\}$. Then every positive real number has a representation. If $x = \sum_{n\geq 0} d_n \log(1 + 2^{-n})$ then

$$e^x = \prod_{n\geq 0} \log(1 + 2^{-n})^{d_n}$$

is obtained with no computation.

4. Beta-numeration

When the base β is not an integer, numbers may have more than one representation. This natural redundancy raises questions on the problem of normalization. Here we focus on computations by finite automata. For more details on the relations with symbolic dynamics, see [31] and [1]. There is a nice survey by Berthé and Siegel [8] on the connections with tilings.

4.1. Beta-expansions

Let $\beta > 1$ be a real number and let D be an alphabet of digits. A *β-representation* on D of a number x of $[0, 1]$ is an infinite sequence $(d_j)_{j\geq 1}$ of $D^{\mathbb{N}}$ such that $\sum_{j\geq 1} d_j \beta^{-j} = x$.

Any real number $x \in [0, 1]$ can be represented in base β by the following greedy algorithm [40]:
Denote by $\lfloor . \rfloor$ and by $\{.\}$ the integral part and the fractional part of a number. Let $x_1 = \lfloor \beta x \rfloor$ and let $r_1 = \{\beta x\}$. Then iterate for $j \geq 2$, $x_j = \lfloor \beta r_{j-1} \rfloor$ and $r_j = \{\beta r_{j-1}\}$. Thus $x = \sum_{j\geq 1} x_j \beta^{-j}$, where the digits x_j are elements of the *canonical* alphabet $A_\beta = \{0, \ldots, \lfloor \beta \rfloor\}$ if $\beta \notin \mathbb{N}$, $A_\beta = \{0, \ldots, \beta - 1\}$ otherwise. The sequence $(x_j)_{j\geq 1}$ of $A_\beta^{\mathbb{N}}$ is called the *β-expansion* of x. When β is an integer, it is the standard β-ary number system. When β is not an integer, a number x may have several different β-representations on A_β: this system is naturally redundant. The β-expansion obtained by the greedy algorithm is the greatest one in the lexicographic order. When a β-representation ends with infinitely many zeroes, it is said to be *finite*, and the 0's are omitted.

Let $d_\beta(1) = (t_j)_{j\geq 1}$ be the β-expansion of 1. If $d_\beta(1)$ is finite, $d_\beta(1) = t_1 \cdots t_N$, set $d^*_\beta(1) = (t_1 \cdots t_{N-1}(t_N - 1))^\omega$, otherwise set $d^*_\beta(1) = d_\beta(1)$. We recall the following result of Parry [37]. An infinite word $s = (s_j)_{j\geq 1}$ is the β-expansion of a number x of $[0, 1[$ if and only if for every $p \geq 1$, $s_p s_{p+1} \cdots$ is smaller in the lexicographic order than $d^*_\beta(1)$.

EXAMPLE **2** Consider the golden ratio $\tau = (1+\sqrt{5})/2$. Then $A_\tau = \{0,1\}$, $d_\tau(1) = 11$ and $d^*_\tau(1) = (10)^\omega$. The number $x = 3 - \sqrt{5}$ has for greedy τ-expansion $\langle x \rangle_\tau = 1001$. Other τ-representations of x are 0111, $100(01)^\omega$, $011(01)^\omega$, $\ldots$

It is easily seen that the factor 11 is forbidden in the greedy expansion $\langle x \rangle_\tau$ for any x. □

A number β such that $d_\beta(1)$ is eventually periodic is called a *Parry number*. If $d_\beta(1)$ is finite it is a *simple Parry number*. If β is a Parry number the set of β-expansions of numbers of $[0,1]$ is recognizable by a finite automaton.

A *Pisot number* is an algebraic integer > 1 such that all its algebraic conjugates are smaller than 1 in modulus. The natural integers and the golden ratio are Pisot numbers. Recall that if β is a Pisot number then it is a Parry number [9].

Let D be a digit set. The *numerical value* in base β on D is the function $\pi_\beta : D^{\mathbb{N}} \longrightarrow \mathbb{R}$ such that $\pi_\beta((d_j)_{j\geq 1}) = \sum_{j\geq 1} d_j \beta^{-j}$.

The *normalization* on D is the function $\nu_D : D^{\mathbb{N}} \longrightarrow A_\beta^{\mathbb{N}}$ which maps any sequence $(d_j)_{j\geq 1} \in D^{\mathbb{N}}$ where $x = \pi_\beta((d_j)_{j\geq 1})$ belongs to $[0,1]$ onto the β-expansion of x.

A *digit set conversion* in base β from D to A_β is a function $\chi : D^{\mathbb{N}} \longrightarrow A_\beta^{\mathbb{N}}$ such that for each sequence $(d_j)_{j\geq 1} \in D^{\mathbb{N}}$ where $x = \pi_\beta((d_j)_{j\geq 1})$ belongs to $[0,1]$, there exists a sequence $(a_j)_{j\geq 1} \in A_\beta^{\mathbb{N}}$ such that $x = \pi_\beta((a_j)_{j\geq 1})$. Remark that the image $\chi((d_j)_{j\geq 1})$ belongs to $A_\beta^{\mathbb{N}}$, but need not be the greedy β-expansion of x.

Some of the results which hold true in the case where β is an integer can be extended to the case where β is not an integer.

Let $D = \{0, \ldots, d\}$ be a digit set containing A_β, that is, $d \geq \lfloor \beta \rfloor$.

THEOREM **2** [24] *There exists a digit set conversion $\chi : D^{\mathbb{N}} \longrightarrow A_\beta^{\mathbb{N}}$ in base β which is on-line computable with delay δ, where δ is the smallest positive integer such that*

$$\beta^{\delta+1} + d \leq \beta^\delta(\lfloor \beta \rfloor + 1).$$

If β is a Pisot number then the digit set conversion χ is computable by a left on-line finite automaton.

Note that multiplication in real base β is also on-line computable [26].

We now consider the problem of normalization, see [22,7,25].

THEOREM **3** *If β is a Pisot number then for every alphabet D of non-negative digits normalization ν_D on D is computable by a finite transducer.*
Conversely, if β is not a Pisot number, then for any alphabet D of non-negative digits, $D \supseteq \{0, \ldots, \lfloor \beta \rfloor, \lfloor \beta \rfloor + 1\}$, the normalization ν_D on D is not computable by a finite transducer.

The transducer realizing normalization cannot be sequential.

4.2. U-representations

Let $U = (u_n)_{n\geq 0}$ be a strictly increasing sequence of integers with $u_0 = 1$. A *U-representation* of an integer $N \geq 0$ is a finite sequence of integers $(d_i)_{k\geq i\geq 0}$ such that $N = \sum_{i=0}^{k} d_i u_i$. It is denoted $(N)_U = d_k \cdots d_0$.

A *normal* or *greedy U-representation* of N is obtained by the following greedy algorithm [21]: denote $q(m,p)$ and $r(m,p)$ the quotient and the remainder of the Euclidean division of m by p. Let k such that $u_k \leq N < u_{k+1}$. Put $d_k = q(N, u_k)$ and $r_k = r(N, u_k)$, and, for $k-1 \geq i \geq 0$, $d_i = q(r_{i+1}, u_i)$ and $r_i = r(r_{i+1}, u_i)$. Then $N = d_k u_k + \cdots + d_0 u_0$. The word $d_k \cdots d_0$ is called the *normal U-representation* of N, and is denoted $\langle N \rangle_U = d_k \cdots d_0$. Each digit d_i is element of the canonical alphabet A_U.

EXAMPLE **3** Let $U = \{1, 2, 3, 5, 8, \ldots\}$ be the set of Fibonacci numbers. Then $A_U = \{0, 1\}$ and $\langle 6 \rangle_U = 1001$. □

The results in this domain are linked to those on β-expansions. Let $G(U)$ be the set of greedy or normal U-representations of all the non-negative integers. If U is linearly recurrent such that its characteristic polynomial is exactly the minimal polynomial of a Pisot number then $G(U)$ is recognizable by a finite automaton. Under the same hypothesis, normalization on every alphabet is computable by a finite transducer, see [31].

A set $S \subset \mathbb{N}$ is said to be U-recognizable if the set $\{< n >_U \mid n \in S\}$ is recognizable by a finite automaton.

Recall the beautiful theorem of Cobham [14] in standard number systems. Two numbers $p > 1$ and $q > 1$ are said to be *multiplicatively dependent* if there exist positive integers k and ℓ such that $p^k = q^\ell$. If a set S is both p- and q-recognizable, where p and q are multiplicatively independent, then S is a finite union of arithmetic progressions.

A generalization of Cobham theorem is the following: let β and γ two multiplicatively independent Pisot numbers. Let U and Y two linear sequences with characteristic polynomial equal to the minimal polynomial of β and γ respectively. The only sets of integers that are both U-recognizable and Y-recognizable are unions of arithmetic progressions [6]. A generalization of Cobham theorem for substitutions was given in [16].

5. Quasicrystals

For definitions and more results see the survey by Pelantová and Masáková in this volume. We are interested here with the connexion with beta-numeration.

A set $X \subset \mathbb{R}^d$ is *uniformly discrete* if there exists a positive real r such that for any $x \in \mathbb{R}^d$, the open ball of center x and radius r contains at most one point of X. A set $X \subset \mathbb{R}^d$ is *relatively dense* if there exists a positive real R such that for any $x \in \mathbb{R}^d$, the open ball of center x and radius R contains at least one point of X. A *Delaunay set* is a set which is both uniformly discrete and relatively dense.

A set X of $\mathbb{R}^d$ is a *Meyer set* if it is a *Delaunay set* and if there exists a finite set F such that the set of differences $X - X$ is a subset of $X + F$. Meyer [32] shown that if X is a Meyer set and if $\beta > 1$ is a real number such that $\beta X \subset X$ then β must be a Pisot or a Salem number [1]. Conversely for each d and for each Pisot or Salem number β, there exists a Meyer set $X \subset \mathbb{R}^d$ such that $\beta X \subset X$.

[1] A *Salem number* is an algebraic integer such that every conjugate has modulus smaller than or equal to 1, and at least one of them has modulus 1.

5.1. Beta-integers

Let $\beta > 1$ be a real number. The set $\mathbb{Z}_\beta$ of β-*integers* is the set of real numbers such that the β-expansion of their absolute value has no fractional part, that is,

$$\mathbb{Z}_\beta = \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_k \cdots x_0\}.$$

Then

$$\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta \ , \ \mathbb{Z}_\beta = -\mathbb{Z}_\beta$$

Denote $\mathbb{Z}_\beta^+$ the set of non-negative β-integers, and $\mathbb{Z}_\beta^- = -(\mathbb{Z}_\beta^+)$.

PROPOSITION **5** [11] *If β is a Pisot number then $\mathbb{Z}_\beta$ is a Meyer set.*

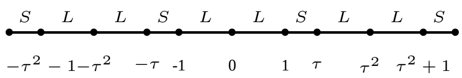
EXAMPLE **4** Let τ be the golden ratio.

$$\begin{aligned}\mathbb{Z}_\tau &= \mathbb{Z}_\tau^+ \cup (-\mathbb{Z}_\tau^+)\\ &= \{0,\ 1,\ \tau,\ \tau^2,\ \tau^2+1,\ \ldots\} \cup \{-1,\ -\tau,\ -\tau^2,\ -\tau^2-1,\ \ldots\}\end{aligned}$$

The set $\mathbb{Z}_\tau^+$ is generated by the Fibonacci substitution

$$L \mapsto LS$$
$$S \mapsto L$$

and $\mathbb{Z}_\tau$ is obtained by symmetry for the negative part.



$\mathbb{Z}_\tau$ is a Meyer set which is not a model set, see [38] for the definition.

The τ-expansions of elements of $\mathbb{Z}_\tau^+$ are exactly the expansions in the Fibonacci numeration system of the non-negative integers, that is to say, $\{0, 1, 10, 100, 101, 1000, \ldots\}$. $\square$

It is an open problem to characterize the minimal finite sets F such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$, see in particular [11,28,2] for partial answers.

5.2. Cyclotomic Pisot numbers

Bravais lattices are used as mathematical models for crystals. A Bravais lattice is an infinite discrete point-set such that the neighborhoods of a point are the same whichever point of the set is considered. Geometrically, a Bravais lattice is characterized by all Euclidean transformations (translations and possibly rotations) that transform the lattice into itself. The condition $2\cos(2\pi/N) \in \mathbb{Z}$, which implies that $N = 1, 2, 3, 4, 6$, characterizes Bravais lattices which are invariant under rotation of $2\pi/N$, the N-fold Bravais lattices, in $\mathbb{R}^2$ (and in $\mathbb{R}^3$). For these values, N is said to be *crystallographic*.

Let us set $\zeta = e^{i\frac{2\pi}{N}}$. The *cyclotomic* ring of order N in the plane is the $\mathbb{Z}$-module:

$$\mathbb{Z}[\zeta] \ = \mathbb{Z}[2\cos\frac{2\pi}{N}] + \mathbb{Z}[2\cos\frac{2\pi}{N}]\zeta,$$

This N-fold structure is generically dense in $\mathbb{C}$, except precisely for the crystallographic cases. Indeed $\mathbb{Z}[\zeta] = \mathbb{Z}$ for $N = 1$ or 2, $\mathbb{Z}[\zeta] = \mathbb{Z} + \mathbb{Z}i$ for $N = 4$ (square lattice), and $\mathbb{Z}[\zeta] = \mathbb{Z} + \mathbb{Z}e^{i\frac{\pi}{3}}$ for the triangular and hexagonal cases $N = 3$ and $N = 6$. Note that a Bravais lattice is a Meyer set such that $F = \{0\}$.

For a general non-crystallographic N, the number $2\cos\frac{2\pi}{N}$ is an algebraic integer of degree $m = \varphi(N)/2 \leq \lfloor (N-1)/2 \rfloor$ where φ is the Euler function.

A *cyclotomic Pisot number* with symmetry of order N is a Pisot number β such that

$$\mathbb{Z}[2\cos\frac{2\pi}{N}] = \mathbb{Z}[\beta].$$

What is striking is the fact that, up to now, all the quasicrystals really obtained by the physicists are linked to cyclotomic quadratic Pisot units. More precisely, denote M_β the minimal polynomial of β. Then

- $N = 5$ or $N = 10$: $\beta = \frac{1+\sqrt{5}}{2} = 2\cos\frac{\pi}{5}$, $M_\beta(X) = X^2 - X - 1$
- $N = 8$: $\beta = 1 + \sqrt{2} = 1 + 2\cos\frac{\pi}{4}$, $M_\beta(X) = X^2 - 2X - 1$
- $N = 12$: $\beta = 2 + \sqrt{3} = 2 + 2\cos\frac{\pi}{6}$, $M_\beta(X) = X^2 - 4X + 1$.

Other cyclotomic Pisot units are

- $N = 7$ or $N = 14$: $\beta = 1 + 2\cos\frac{\pi}{7}$, $M_\beta(X) = X^3 - 2X^2 - X + 1$
- $N = 9$ or $N = 18$: $\beta = 1 + 2\cos\frac{\pi}{9}$, $M_\beta(X) = X^3 - 3X^2 + 1$.

A complete classification of cyclotomic Pisot numbers of degree ≤ 4 was given by Bell and Hare in [5].

5.3. *Beta-lattices in the plane*

Let β be a cyclotomic Pisot number with order N symmetry. Then $\mathbb{Z}[\zeta] = \mathbb{Z}[\beta] + \mathbb{Z}[\beta]\zeta$, with $\zeta = e^{i\frac{2\pi}{N}}$, is a ring invariant under rotation of order N (see [4]). This ring is the natural framework for two-dimensional structures having β as scaling factor, and $2\pi/N$ as rotational symmetry.

Generically, let β be a Pisot number; a *beta-lattice* is a point set

$$\Gamma = \sum_{i=1}^{d} \mathbb{Z}_\beta \mathbf{e}_i$$

where $(\mathbf{e}_i)$ is a basis of $\mathbb{R}^d$. Such a set is a Meyer set with self-similarity factor β. Observe that β-lattices are based on β-integers as lattices are based on integers. So β-lattices are good frames for the study of quasiperiodic point-sets and tilings, see [18].

Examples of beta-lattices in the plane are point-sets of the form

$$\Gamma_q(\beta) = \mathbb{Z}_\beta + \mathbb{Z}_\beta \zeta^q\,,$$

with β a cyclotomic Pisot unit of order N, for $1 \leq q \leq N-1$. Note that the latter are not rotationally invariant. Examples of rotationally invariant point-sets based on beta-integers are

$$\Lambda_q = \bigcup_{j=0}^{N-1} \Gamma_q(\beta)\zeta^j\,, \quad 1 \leq q \leq N-1\,,$$

and

$$\mathbb{Z}_\beta[\zeta] = \sum_{j=0}^{N-1} \mathbb{Z}_\beta \zeta^j\,.$$

All these sets are Meyer sets.

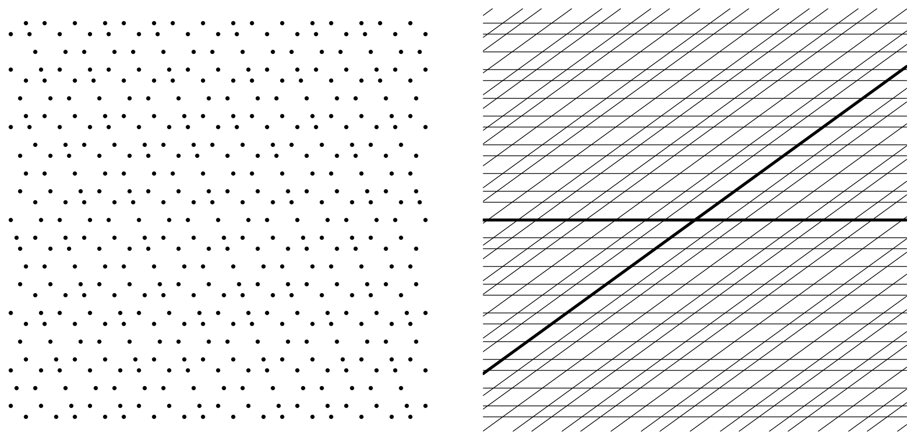


Figure 4. The τ-lattice $\Gamma_1(\tau)$ with points (left), and its trivial tiling made by joining points along the horizontal axis, and along the direction defined by ζ.

In the particular case where β is a quadratic Pisot unit, the set of β-integers $\mathbb{Z}_\beta$ can be equipped with an internal additive law, which gives it an abelian group structure [11].

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