On negative bases

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Abstract. We study expansions in non-integer negative base $-\beta$ introduced by Ito and Sadahiro [7]. Using countable automata associated with $(-\beta)$-expansions, we characterize the case where the $(-\beta)$-shift is a system of finite type. We prove that, if β is a Pisot number, then the $(-\beta)$-shift is a sofic system. In that case, addition (and more generally normalization on any alphabet) is realizable by a finite transducer.

1 Introduction

Expansions in integer negative base $-b$, where $b \geqslant 2$, seem to have been introduced by Grünwald in [6], and rediscovered by several authors, see the historical comments given by Knuth [8]. The choice of a negative base $-b$ and of the alphabet $\{0, \ldots, b-1\}$ is interesting, because it provides a signless representation for every number (positive or negative). In this case it is easy to distinguish the sequences representing a positive integer from the ones representing a negative one: denoting $(w\boldsymbol{.})_{-b} := \sum_{i=0}^{k} w_k(-b)^k$ for any $w = w_k \cdots w_0$ in$\{0, \ldots, b-1\}^*$ with no leading 0's, we have $\mathbb{N} = \{(w\boldsymbol{.})_{-b} \mid |w| \text{ is odd}\}$. The classical monotonicity between the lexicographical ordering on words and the represented numerical values does not hold anymore in negative base, for instance $3 = (111\boldsymbol{.})_{-2}$, $4 = (100\boldsymbol{.})_{-2}$ and $111 >_{lex} 100$. Nevertheless it is possible to restore such a correspondence by introducing an appropriate ordering on words, in the sequel denoted by $\prec$, and called the *alternate order*.

Representations in negative base also appear in some complex base number systems, for instance base $\beta = 2i$ where $\beta^2 = -4$ (see [5] for a study of their properties from an automata theoretic point of view). Thus, beyond the interest in the problem in itself, the authors also wish the study of negative bases to be an useful preliminar step to better understanding the complex case.

Ito and Sadahiro recently introduced expansions in non-integer negative base $-\beta$ in [7]. They have given a characterization of admissible sequences, and shown that the $(-\beta)$-shift is sofic if and only if the $(-\beta)$-expansion of the number $-\frac{\beta}{\beta+1}$ is eventually periodic.

In this paper we pursue their work. The purpose of this contribution is to show that many properties of the positive base (integer or not) numeration systems extend to the negative base case, the main difference being the sets of numbers that are representable in the two different cases. The results could seem not surprising, but this study put into light the important role played by the order on words: the lexicographic order for the positive bases, the alternate order for the negative bases.

We start by a general result which is not related to numeration systems but to the alternate order, and which is of interest in itself. We define a symbolic dynamical system associated with a given infinite word s satisfying some properties with respect to the alternate order on infinite words. We design an infinite countable automaton recognizing it. We then are able to characterize the case when the symbolic dynamical system is sofic (resp. of finite type). Using this general construction we can prove that the $(-\beta)$-shift is a symbolic dynamical system of finite type if and only if the $(-\beta)$-expansion of $-\frac{\beta}{\beta+1}$ is purely periodic. We also show that the entropy of the $(-\beta)$-shift is equal to $\log \beta$.

We then focus on the case where β is a Pisot number, that is to say, an algebraic integer greater than 1 such that the modulus of its Galois conjugates is less than 1. The natural integers and the Golden Mean are Pisot numbers. We extend all the results known to hold true in the Pisot case for β-expansions to the $(-\beta)$-expansions. In particular we prove that, if β is a Pisot number, then every number from $\mathbb{Q}(\beta)$ has an eventually periodic $(-\beta)$-expansion, and thus that the $(-\beta)$-shift is a sofic system.

When β is a Pisot number, it is known that addition in base β — and more generally normalization in base β on an arbitrary alphabet — is realizable by a finite transducer [4]. We show that this is still the case in base $-\beta$.

2 Definitions and preliminaries

2.1 Words and automata

An *alphabet* is a totally ordered set. In this paper the alphabets are always finite. A finite sequence of elements of an alphabet A is called a *word*, and the set of words on A is the free monoid A^*. The empty word is denoted by ε. The set of infinite (resp. bi-infinite) words on A is denoted by $A^{\mathbb{N}}$ (resp. $A^{\mathbb{Z}}$). Let v be a word of A^*, denote by v^n the concatenation of v to itself n times, and by v^ω the infinite concatenation $vvv\cdots$. A word of the form uv^ω is said to be *eventually periodic*. A (purely) *periodic* word is an eventually periodic word of the form v^ω.

A finite word v is a *factor* of a (finite, infinite or bi-infinite) word x if there exists u and w such that $x = uvw$. When u is the empty word, v is a *prefix* of x. The prefix v is *strict* if $v \neq x$. When w is empty, v is said to be a *suffix* of x.

We recall some definitions on automata, see [2] and [13] for instance. An *automaton over* A, $\mathcal{A} = (Q, A, E, I, T)$, is a directed graph labelled by elements of A. The set of vertices, traditionally called *states*, is denoted by Q, $I \subset Q$ is the set of *initial* states, $T \subset Q$ is the set of *terminal* states and $E \subset Q \times A \times Q$ is the set of labelled *edges*. If $(p, a, q) \in E$, we write $p \xrightarrow{a} q$. The automaton is *finite* if Q is finite. The automaton $\mathcal{A}$ is *deterministic* if E is the graph of a (partial) function from $Q \times A$ into Q, and if there is a unique initial state. A subset H of A^* is said to be *recognizable by a finite automaton*, or *regular*, if there exists a finite automaton $\mathcal{A}$ such that H is equal to the set of labels of paths starting in an initial state and ending in a terminal state.

Recall that two words u and v are said to be *right congruent modulo* H if, for every w, uw is in H if and only if vw is in H. It is well known that H is recognizable by a finite automaton if and only if the congruence modulo H has finite index.

Let A and A' be two alphabets. A *transducer* is an automaton $\mathcal{T} = (Q, A^* \times A'^*, E, I, T)$ where the edges of E are labelled by couples in $A^* \times A'^*$. It is said to be *finite* if the set Q of states and the set E of edges are finite. If $(p, (u, v), q) \in E$, we write $p \xrightarrow{u|v} q$. The *input automaton* (resp. *output automaton*) of such a transducer is obtained by taking the projection of edges on the first (resp. second) component. A transducer is said to be *sequential* if its input automaton is deterministic.

The same notions can be defined for automata and transducer processing words from right to left : they are called *right* automata or transducers.

2.2 Symbolic dynamics

Let us recall some definitions on symbolic dynamical systems or subshifts (see [10, Chapter 1] or [9]). The set $A^{\mathbb{Z}}$ is endowed with the lexicographic order, denoted $<_{lex}$, the product topology, and the shift σ, defined by $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$. A set $S \subseteq A^{\mathbb{Z}}$ is a *symbolic dynamical system*, or *subshift*, if it is shift-invariant and closed for the product topology on $A^{\mathbb{Z}}$. A bi-infinite word z *avoids* a set of word $X \subset A^*$ if no factor of z is in X. The set of all words which avoid X is denoted S_X. A set $S \subseteq A^{\mathbb{Z}}$ is a subshift if and only if S is of the form S_X for some X.

The same notion can be defined for a one-sided subshift of $A^{\mathbb{N}}$.

Let $F(S)$ be the set of factors of elements of S, let $I(S) = A^+ \setminus F(S)$ be the set of words avoided by S, and let $X(S)$ be the set of elements of $I(S)$ which have no proper factor in $I(S)$. The subshift S is *sofic* if and only if $F(S)$ is recognizable by a finite automaton, or equivalently if $X(S)$ is recognizable by a finite automaton. The subshift S is of *finite type* if $S = S_X$ for some finite set X, or equivalently if $X(S)$ is finite.

The topological entropy of a subshift S is

$$h(S) = \lim_{n\to\infty} \frac{1}{n} \log(B_n(S))$$

where $B_n(S)$ is the number of elements of $F(S)$ of length n. When S is sofic, the entropy of S is equal to the logarithm of the spectral radius of the adjacency matrix of the finite automaton recognizing $F(S)$.

2.3 Numeration systems

The reader is referred to [10, Chapter 7] for a detailed presentation on these topics. Representations of real numbers in a non-integer base β were introduced by Rényi [12] under the name of *β-expansions*. Let x be a real number in the interval $[0,1]$. A *representation in base β* (or a β-representation) of x is an infinite word $(x_i)_{i\geqslant 1}$ such that

$$x = \sum_{i\geqslant 1} x_i \beta^{-i}.$$

A particular β-representation — called the *β-expansion* — can be computed by the "greedy algorithm" : denote by $\lfloor y \rfloor$, $\lceil y \rceil$ and $\{y\}$ the lower integer part, the upper integer part and the fractional part of a number y. Set $r_0 = x$ and let for $i \geqslant 1$, $x_i = \lfloor \beta r_{i-1} \rfloor$, $r_i = \{\beta r_{i-1}\}$. Then $x = \sum_{i\geqslant 1} x_i \beta^{-i}$. The digits x_i are elements of the canonical alphabet $A_\beta = \{0, \ldots, \lceil \beta \rceil - 1\}$.

The β-expansion of $x \in [0,1]$ will be denoted by $\mathsf{d}_\beta(x) = (x_i)_{i\geqslant 1}$. If $x > 1$, there exists some $k \geqslant 1$ such that x/β^k belongs to $[0,1)$. If $\mathsf{d}_\beta(x/\beta^k) = (y_i)_{i\geqslant 1}$ then by shifting $x = (y_1 \cdots y_k \centerdot y_{k+1} y_{k+2} \cdots)_\beta$.

An equivalent definition is obtained by using the *β-transformation* of the unit interval which is the mapping

$$T_\beta : x \mapsto \beta x - \lfloor \beta x \rfloor.$$

Then $\mathsf{d}_\beta(x) = (x_i)_{i\geqslant 1}$ if and only if $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$.

If a representation ends in infinitely many zeros, like $v0^\omega$, the ending zeros are omitted and the representation is said to be *finite*.

In the case where the β-expansion of 1 is finite, there is a special representation playing an important role. Let $\mathsf{d}_\beta(1) = (t_i)_{i\geqslant 1}$ and set $\mathsf{d}^*_\beta(1) = \mathsf{d}_\beta(1)$ if $\mathsf{d}_\beta(1)$ is infinite and $\mathsf{d}^*_\beta(1) = (t_1 \cdots t_{m-1}(t_m - 1))^\omega$ if $\mathsf{d}_\beta(1) = t_1 \cdots t_{m-1} t_m$ is finite.

Denote by D_β the set of β-expansions of numbers of $[0,1)$. It is a shift-invariant subset of $A_\beta^{\mathbb{N}}$. The *β-shift* S_β is the closure of D_β and it is a subshift of $A_\beta^{\mathbb{Z}}$. When β is an integer, S_β is the full β-shift $A_\beta^{\mathbb{Z}}$.

Theorem 1 (Parry[11]). *Let $\beta > 1$ be a real number. A word $(w_i)_{i\geqslant 1}$ belongs to D_β if and only if for all $n \geqslant 1$*

$$w_n w_{n+1} \cdots <_{lex} \mathsf{d}^*_\beta(1).$$

A word $(w_i)_{i\in\mathbb{Z}}$ belongs to S_β if and only if for all n

$$w_n w_{n+1} \cdots \leqslant_{lex} \mathsf{d}^*_\beta(1).$$

The following results are well-known (see [10, Chapt. 7]).

Theorem 2. *1. The β-shift is sofic if and only if $\mathsf{d}_\beta(1)$ is eventually periodic.*
2. The β-shift is of finite type if and only if $\mathsf{d}_\beta(1)$ is finite.

It is known that the entropy of the β-shift is equal to $\log \beta$.

If β is a Pisot number, then every element of $\mathbb{Q}(\beta) \cap [0,1]$ has an eventually periodic β-expansion, and the β-shift S_β is a sofic system [1, 14].

Let C be an arbitrary finite alphabet of integer digits. The *normalization function* in base β on C

$$\nu_{\beta,C} : C^{\mathbb{N}} \to \mathcal{A}_\beta^{\mathbb{N}}$$

is the partial function which maps an infinite word $y = (y_i)_{i\geqslant 1}$ over C, such that $0 \leqslant y = \sum_{i\geqslant 1} y_i \beta^{-i} \leqslant 1$, onto the β-expansion of y. It is known [4] that, when β is a Pisot number, normalization is computable by a finite transducer on any alphabet C. Note that addition is a particular case of normalization, with $C = \{0, \ldots, 2(\lceil\beta\rceil - 1)\}$.

3 Symbolic dynamical systems and the alternate order

Define the *alternate order* $\prec$ on infinite words or finite words with same length on an alphabet A:

$$x_1x_2x_3\cdots \prec y_1y_2y_3\cdots$$

if and only if there exists $k \geqslant 1$ such that

$$x_i = y_i \text{ for } 1 \leqslant i < k \quad \text{and} \quad (-1)^k(x_k - y_k) < 0.$$

This order was implicitely defined in [6].

Let A be a finite alphabet, and let $s = s_1s_2\cdots$ be a word in $A^{\mathbb{N}}$ such that $s_1 = \max A$ and for each $n \geqslant 1$, $s \preceq s_ns_{n+1}\cdots$. Let

$$S = \{w = (w_i)_{i\in\mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n,\ s \preceq w_nw_{n+1}\cdots\}.$$

We construct a countable infinite automaton $\mathcal{A}_S$ as follows (see Fig.1, where $[a, b]$ denotes $\{a, a+1, \ldots, b\}$ if $a \leqslant b$, ε else. It is assumed in Fig. 1 that $s_1 > s_j$ for $j \geqslant 2$.) The set of states is $\mathbb{N}$. For each state $i \geqslant 0$, there is an edge $i \xrightarrow{s_{i+1}} i+1$. Thus the state i is the name corresponding to the path labelled $s_1 \cdots s_i$. If i is even, then for each a such that $0 \leqslant a \leqslant s_{i+1} - 1$, there is an edge $i \xrightarrow{a} j$, where j is such that $s_1 \cdots s_j$ is the suffix of maximal length of $s_1 \cdots s_i a$. If i is odd, then for each b such that $s_{i+1} + 1 \leqslant b \leqslant s_1 - 1$, there is an edge $i \xrightarrow{b} j$ where j is maximal such that $s_1 \cdots s_j$ is a suffix of $s_1 \cdots s_i b$; and if $s_{i+1} < s_1$ there is one edge $i \xrightarrow{s_1} 1$. By contruction, the deterministic automaton $\mathcal{A}_S$ recognizes exactly the words w such that every suffix of w is $\succeq s$ and the result below follows.

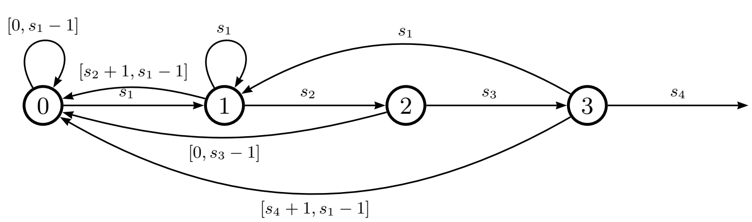

Fig. 1: The automaton $\mathcal{A}_S$

Proposition 1. *The subshift* $S = \{w = (w_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n,\ s \preceq w_n w_{n+1} \cdots\}$ *is recognizable by the countable infinite automaton* $\mathcal{A}_S$*.*

Proposition 2. *The subshift* $S = \{w = (w_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n,\ s \preceq w_n w_{n+1} \cdots\}$ *is sofic if and only if* s *is eventually periodic.*

Proof. The subshift S is sofic if and only if the set of its finite factors $F(S)$ is recognizable by a finite automaton. Given a word u of A^*, denote by $[u]$ the right class of u modulo $F(S)$. Then in the automaton $\mathcal{A}_S$, for each state $i \geqslant 1$, $i = [s_1 \cdots s_i]$, and $0 = [\varepsilon]$. Suppose that s is eventually periodic, $s = s_1 \cdots s_m (s_{m+1} \cdots s_{m+p})^{\omega}$, with m and p minimal. Thus, for each $k \geqslant 0$ and each $0 \leqslant i \leqslant p-1$, $s_{m+pk+i} = s_{m+i}$.
Case 1: p is even. Then $m + i = [s_1 \cdots s_{m+i}] = [s_1 \cdots s_{m+pk+i}]$ for every $k \geqslant 0$ and $0 \leqslant i \leqslant p-1$. Then the set of states of $\mathcal{A}_S$ is $\{0, 1, \ldots, m+p-1\}$.
Case 2: p is odd. Then $m + i = [s_1 \cdots s_{m+i}] = [s_1 \cdots s_{m+2pk+i}]$ for every $k \geqslant 0$ and $0 \leqslant i \leqslant 2p-1$. The set of states of $\mathcal{A}_S$ is $\{0, 1, \ldots, m+2p-1\}$.
Conversely, suppose that s is not eventually periodic. Then there exists an infinite sequence of indices $i_1 < i_2 < \cdots$ such that the sequences $s_{i_k} s_{i_k+1} \cdots$ are all different for all $k \geqslant 1$. Take any pair (i_j, i_ℓ), $j, \ell \geqslant 1$. If i_j and i_ℓ do not have the same parity, then $s_1 \cdots s_{i_j}$ and $s_1 \cdots s_{i_\ell}$ are not right congruent modulo $F(S)$. If i_j and i_ℓ have the same parity, there exists $q \geqslant 0$ such that $s_{i_j} \cdots s_{i_j+q-1} = s_{i_\ell} \cdots s_{i_\ell+q-1} = v$ and, for instance, $(-1)^{i_j+q}(s_{i_j+q} - s_{i_\ell+q}) > 0$ (with the convention that, if $q = 0$ then $v = \varepsilon$). Then $s_1 \cdots s_{i_j-1} v s_{i_j+q} \in F(S)$, $s_1 \cdots s_{i_\ell-1} v s_{i_\ell+q} \in F(S)$, but $s_1 \cdots s_{i_j-1} v s_{i_\ell+q}$ does not belong to $F(S)$. Hence $s_1 \cdots s_{i_j}$ and $s_1 \cdots s_{i_\ell}$ are not right congruent modulo $F(S)$, so the number of right congruence classes is infinite and $F(S)$ is thus not recognizable by a finite automaton. □

Proposition 3. *The subshift* $S = \{w = (w_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n,\ s \preceq w_n w_{n+1} \cdots\}$ *is a subshift of finite type if and only if* s *is purely periodic.*

Proof. Suppose that $s = (s_1 \cdots s_p)^{\omega}$. Consider the finite set $X = \{s_1 \cdots s_{n-1} b \mid b \in A,\ (-1)^n (b - s_n) < 0,\ 1 \leqslant n \leqslant p\}$. We show that $S = S_X$. If w is in S, then w avoids X, and conversely. Now, suppose that S is of finite type. It is thus sofic, and by Proposition 2 s is eventually periodic. If it is not purely periodic, then $s = s_1 \cdots s_m (s_{m+1} \cdots s_{m+p})^{\omega}$, with m and p minimal, and $s_1 \cdots s_m \neq \varepsilon$. Let $I = \{s_1 \cdots s_{n-1} b \mid b \in A,\ (-1)^n (b - s_n) < 0,\ 1 \leqslant n \leqslant m\} \cup \{s_1 \cdots s_m (s_{m+1} \cdots s_{m+p})^{2k}\ s_{m+1} \cdots s_{m+n-1} b \mid b \in A,\ k \geqslant 0, (-1)^{m+2kp+n}(b - s_{m+n}) < 0,\ 1 \leqslant n \leqslant 2p\}$. Then $I \subset A^+ \setminus F(S)$. First, suppose there exists $1 \leqslant j \leqslant p$ such that $(-1)^j (s_j - s_{m+j}) < 0$ and $s_1 \cdots s_{j-1} = s_{m+1} \cdots s_{m+j-1}$. For $k \geqslant 0$ fixed, let $w^{(2k)} = s_1 \cdots s_m (s_{m+1} \cdots s_{m+p})^{2k} s_1 \cdots s_j \in I$. We have $s_1 \cdots s_m (s_{m+1} \cdots s_{m+p})^{2k} s_{m+1} \cdots s_{m+j-1} \in F(S)$. On the other hand, for $n \geqslant 2$, $s_n \cdots s_m (s_{m+1} \cdots s_{m+p})^{2k}$ is $\succ$ than the prefix of s of same length, thus $s_n \cdots s_m (s_{m+1} \cdots s_{m+p})^{2k} s_1 \cdots s_j \in F(S)$. Hence any strict factor of $w^{(2k)}$ is in $F(S)$. Therefore for any $k \geqslant 0$, $w^{(2k)} \in X(S)$, and $X(S)$ is thus infinite: S is not of finite type. Now, if such a j does not exist, then for every $1 \leqslant j \leqslant p$, $s_j = s_{m+j}$, and $s = (s_1 \cdots s_m)^{\omega}$ is purely periodic. □

Remark 1. Let $s' = s'_1 s'_2 \cdots$ be a word in $A^{\mathbb{N}}$ such that $s'_1 = \min A$ and, for each $n \geqslant 1$, $s'_n s'_{n+1} \cdots \preceq s'$. Let $S' = \{w = (w_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n,\ w_n w_{n+1} \cdots \preceq s'\}$. The statements in Propositions 1, 2 and 3 are also valid for the subshift S' (with the automaton $\mathcal{A}_{S'}$ constructed accordingly).

4 Negative real base

4.1 The $(-\beta)$-shift

Ito and Sadahiro [7] introduced a greedy algorithm to represent any real number in real base $-\beta$, $\beta > 1$, and with digits in $A_{-\beta} = \{0, 1, \ldots, \lfloor \beta \rfloor\}$. Remark that, when β is not an integer, $A_{-\beta} = A_\beta$.

A transformation on $I_{-\beta} = \left[-\frac{\beta}{\beta+1}, \frac{1}{\beta+1}\right)$ is defined as follows:

$$T_{-\beta}(x) = -\beta x - \lfloor -\beta x + \frac{\beta}{\beta+1} \rfloor.$$

For every real number $x \in I_{-\beta}$ denote $\mathsf{d}_{-\beta}(x)$ the $(-\beta)$-expansion of x. Then $\mathsf{d}_{-\beta}(x) = (x_i)_{i \geqslant 1}$ if and only if $x_i = \lfloor -\beta T_{-\beta}^{i-1}(x) + \frac{\beta}{\beta+1} \rfloor$, and $x = \sum_{i \geqslant 1} x_i (-\beta)^{-i}$. When this last equality holds, we may also write:

$$x = (\boldsymbol{.} x_1 x_2 \cdots)_{-\beta}.$$

Since for every $x \in \mathbb{R} \backslash I_{-\beta}$ there exists an integer $k \geqslant 1$ such that $x/(-\beta)^k \in I_{-\beta}$, the sequence $\mathsf{d}_{-\beta}(x/(-\beta)^k) = (y_i)_{i \geqslant 1}$ satisfies $x = (y_1 \cdots y_k \boldsymbol{.} y_{k+1} y_{k+2} \cdots)_{-\beta}$. Thus, allowing an opportune shift on the digits, every real number has a $(-\beta)$-expansion.

We show that the alternate order $\prec$ on $(-\beta)$-expansions gives the numerical order.

Proposition 4. *Let x and y be in $I_{-\beta}$. Then*

$$x < y \iff \mathsf{d}_{-\beta}(x) \prec \mathsf{d}_{-\beta}(y).$$

Proof. Suppose that $\mathsf{d}_{-\beta}(x) \prec \mathsf{d}_{-\beta}(y)$. Then there exists $k \geqslant 1$ such that $x_i = y_i$ for $1 \leqslant i < k$ and $(-1)^k (x_k - y_k) < 0$. Suppose that k is even, $k = 2q$. Then $x_{2q} \leqslant y_{2q} - 1$. Thus $x - y \leqslant -\beta^{-2q} + \sum_{i \geqslant 2q+1} x_i (-\beta)^{-i} - \sum_{i \geqslant 2q+1} y_i (-\beta)^{-i} < 0$, since $\sum_{i \geqslant 1} x_{2q+i} (-\beta)^{-i}$ and $\sum_{i \geqslant 1} y_{2q+i} (-\beta)^{-i}$ are in $I_{-\beta}$. The case $k = 2q+1$ is similar. The converse is immediate. □

Example 1. In base -2, $3 = (111\boldsymbol{.})_{-2}$, $4 = (100\boldsymbol{.})_{-2}$ and $111 \prec 100$.

A word $(x_i)_{i \geqslant 1}$ is said to be $(-\beta)$-*admissible* if there exists a real number $x \in I_{-\beta}$ such that $\mathsf{d}_{-\beta}(x) = (x_i)_{i \geqslant 1}$. The $(-\beta)$-*shift* $S_{-\beta}$ is the closure of the set of $(-\beta)$-admissible words, and it is a subshift of $A_\beta^{\mathbb{Z}}$.

Define the sequence $\mathsf{d}^*_{-\beta}(\frac{1}{\beta+1})$ as follows:

– if $\mathsf{d}_{-\beta}(-\frac{\beta}{\beta+1}) = d_1 d_2 \cdots$ is not a periodic sequence with odd period,

$$\mathsf{d}^*_{-\beta}(\frac{1}{\beta+1}) = \mathsf{d}_{-\beta}(\frac{1}{\beta+1}) = 0 d_1 d_2 \cdots$$

– otherwise if $\mathsf{d}_{-\beta}(-\frac{\beta}{\beta+1}) = (d_1 \cdots d_{2p+1})^\omega$,

$$\mathsf{d}^*_{-\beta}(\frac{1}{\beta+1}) = (0 d_1 \cdots d_{2p}(d_{2p+1}-1))^\omega.$$

Theorem 3 (Ito-Sadahiro [7]). *A word $(w_i)_{i \geqslant 1}$ is $(-\beta)$-admissible if and only if for each $n \geqslant 1$*

$$\mathsf{d}_{-\beta}(-\frac{\beta}{\beta+1}) \preceq w_n w_{n+1} \cdots \prec \mathsf{d}^*_{-\beta}(\frac{1}{\beta+1}).$$

A word $(w_i)_{i \in \mathbb{Z}}$ is an element of the $(-\beta)$-shift if and only if for each n

$$\mathsf{d}_{-\beta}(-\frac{\beta}{\beta+1}) \preceq w_n w_{n+1} \cdots \preceq \mathsf{d}^*_{-\beta}(\frac{1}{\beta+1}).$$

Theorem 3 can be restated as follows.

Lemma 1. *Let $\mathsf{d}_{-\beta}(-\frac{\beta}{\beta+1}) = d_1 d_2 \cdots$ and let*

$$S = \{(w_i)_{i \in \mathbb{Z}} \in A_\beta^{\mathbb{Z}} \mid \forall n,\ d_1 d_2 \cdots \preceq w_n w_{n+1} \cdots\}.$$

If $\mathsf{d}_{-\beta}(-\frac{\beta}{\beta+1})$ is not a periodic sequence with odd period, then $S_{-\beta} = S$.
If $\mathsf{d}_{-\beta}(-\frac{\beta}{\beta+1})$ is a periodic sequence of odd period, $\mathsf{d}_{-\beta}(-\frac{\beta}{\beta+1}) = (d_1 \cdots d_{2p+1})^\omega$, then $S_{-\beta} = S \cap S'$ where

$$S' = \{(w_i)_{i \in \mathbb{Z}} \in A_\beta^{\mathbb{Z}} \mid \forall n,\ w_n w_{n+1} \cdots \preceq (0 d_1 \cdots d_{2p}(d_{2p+1}-1))^\omega\}.$$

Theorem 4. *The $(-\beta)$-shift is a system of finite type if and only if $\mathsf{d}_{-\beta}(-\frac{\beta}{\beta+1})$ is purely periodic.*

Proof. If $\mathsf{d}_{-\beta}(-\frac{\beta}{\beta+1})$ is purely periodic with an even period, the result follows from Theorem 3, Lemma 1 and Proposition 3. If $\mathsf{d}_{-\beta}(-\frac{\beta}{\beta+1})$ is purely periodic with an odd period, the result follows from Theorem 3, Lemma 1, Proposition 3, Remark 1, and the fact that the intersection of two finite sets is finite. □

By Theorem 3, Lemma 1, Proposition 2, Remark 1, and the fact that the intersection of two regular sets is again regular the following result follows.

Theorem 5 (Ito-Sadahiro [7]). *The $(-\beta)$-shift is a sofic system if and only if $\mathsf{d}_{-\beta}(-\frac{\beta}{\beta+1})$ is eventually periodic.*

Fig. 2: Finite automata for the G-shift (left) and for the $(-G)$-shift (right)

Example 2. Let $G = \frac{1+\sqrt{5}}{2}$; then $\mathsf{d}_{-G}(-\frac{G}{G+1}) = 10^\omega$, and the $(-G)$-shift is a sofic system which is not of finite type.
Let $\beta = G^2 = \frac{3+\sqrt{5}}{2}$. Then $\mathsf{d}_{-\beta}(-\frac{\beta}{\beta+1}) = (21)^\omega$ and the $(-\beta)$-shift is of finite type: the set of minimal forbidden factors is $X(S_{-\beta}) = \{20\}$.

Example 3. The automaton in Fig. 2 (right) recognizing the $(-G)$-shift is obtained by minimizing the result of the construction of Proposition 1. Remark that it is the automaton which recognizes the celebrated even shift (see [9]).

This example suggests that the entropy of the $-\beta$-shift is the same as the entropy of the β-shift. Using results from Fotiades and Boudourides [3], we can prove the following result.

Proposition 5. *The entropy of the $(-\beta)$-shift is equal to $\log\beta$.*

4.2 The Pisot case

We first prove that the classical result saying that if β is a Pisot number, then every element of $\mathbb{Q}(\beta)\cap[0,1]$ has an eventually periodic β-expansion is still valid for the base $-\beta$.

Theorem 6. *If β is a Pisot number, then every element of $\mathbb{Q}(\beta)\cap I_{-\beta}$ has an eventually periodic $(-\beta)$-expansion.*

Proof. Let $M_\beta(X) = X^d - a_1X^{d-1} - \cdots - a_d$ be the minimal polynomial of β and denote by $\beta = \beta_1, \ldots, \beta_d$ the conjugates of β. Let x be arbitrarily fixed in $\mathbb{Q}(\beta)\cap I_{-\beta}$. Since $\mathbb{Q}(\beta) = \mathbb{Q}(-\beta)$, x can be expressed as $x = q^{-1}\sum_{i=0}^{d-1} p_i(-\beta)^i$ with q and p_i in $\mathbb{Z}$, $q > 0$ as small as possible in order to have uniqueness.

Let $(x_i)_{i\geqslant 1}$ be the $(-\beta)$-expansion of x, and write

$$r_n = r_n^{(1)} = r_n^{(1)}(x) = \frac{x_{n+1}}{-\beta} + \frac{x_{n+2}}{(-\beta)^2} + \cdots = (-\beta)^n\left(x - \sum_{k=1}^{n} x_k(-\beta)^{-k}\right).$$

Since $r_n = T^n_{-\beta}(x)$ belongs to $I_{-\beta}$ then $|r_n| \leqslant \frac{\beta}{\beta+1} < 1$. For $2 \leqslant j \leqslant d$, let

$$r_n^{(j)} = r_n^{(j)}(x) = (-\beta_j)^n\left(q^{-1}\sum_{i=0}^{d-1} p_i(-\beta_j)^i - \sum_{k=1}^{n} x_k(-\beta_j)^{-k}\right).$$

Let $\eta = \max\{|\beta_j| \mid 2 \leqslant j \leqslant d\}$: since β is a Pisot number, $\eta < 1$. Since $x_k \leqslant \lfloor\beta\rfloor$ we get

$$|r_n^{(j)}| \leqslant q^{-1}\sum_{i=0}^{d-1} |p_i|\eta^{n+i} + \lfloor\beta\rfloor\sum_{k=0}^{n-1}\eta^k$$

and since $\eta < 1$, $\max_{1 \leqslant j \leqslant d}\{\sup_n\{|r_n^{(j)}|\}\} < \infty$.

We need a technical result. Set $R_n = (r_n^{(1)}, \ldots, r_n^{(d)})$ and let B the matrix $B = ((-\beta_j)^{-i})_{1 \leqslant i,j \leqslant d}$.

Lemma 2. *Let $x = q^{-1}\sum_{i=0}^{d-1} p_i(-\beta)^i$. For every $n \geqslant 0$ there exists a unique d-uple $Z_n = (z_n^{(1)}, \ldots, z_n^{(d)})$ in $\mathbb{Z}^d$ such that $R_n = q^{-1}Z_nB$.*

Proof. By induction on n. First, $r_1 = -\beta x - x_1$, thus

$$r_1 = q^{-1}\left(\sum_{i=0}^{d-1} p_i(-\beta)^{i+1} - qx_1\right) = q^{-1}\left(\frac{z_1^{(1)}}{-\beta} + \cdots + \frac{z_1^{(d)}}{(-\beta)^d}\right)$$

using the fact that $(-\beta)^d = -a_1(-\beta)^{d-1} + a_2(-\beta)^{d-2} + \cdots + (-1)^d a_d$. Now, $r_{n+1} = -\beta r_n - x_{n+1}$, hence

$$r_{n+1} = q^{-1}\left(z_n^{(1)} + \frac{z_1^{(2)}}{-\beta} + \cdots + \frac{z_n^{(d)}}{(-\beta)^{d-1}} - qx_{n+1}\right) = q^{-1}\left(\frac{z_{n+1}^{(1)}}{-\beta} + \cdots + \frac{z_{n+1}^{(d)}}{(-\beta)^d}\right)$$

since $z_n^{(1)} - qx_{n+1} \in \mathbb{Z}$. Thus for every n there exists $(z_n^{(1)}, \ldots, z_n^{(d)})$ in $\mathbb{Z}^d$ such that

$$r_n = q^{-1}\sum_{k=1}^{d} z_n^{(k)}(-\beta)^{-k}.$$

Since the latter equation has integral coefficients and is satisfied by $-\beta$, it is also satisfied by $-\beta_j$, $2 \leqslant j \leqslant d$, and

$$r_n^{(j)} = (-\beta_j)^n\left(q^{-1}\sum_{i=0}^{d-1}\bar{p}_i(-\beta_j)^i - \sum_{k=1}^{n} x_k(-\beta_j)^{-k}\right) = q^{-1}\sum_{k=1}^{d} z_n^{(k)}(-\beta_j)^{-k}.$$

□

We go back to the proof of Theorem 6. Let $V_n = qR_n$. The $(V_n)_{n \geqslant 1}$ have bounded norm, since $\max_{1 \leqslant j \leqslant d}\{\sup_n\{|r_n^{(j)}|\}\} < \infty$. As the matrix B is invertible, for every $n \geqslant 1$,

$$\|Z_n\| = \|(z_n^{(1)}, \ldots, z_n^{(d)})\| = \max\{|z_n^{(j)}| \ : 1 \leqslant j \leqslant d\} < \infty$$

so there exist p and $m \geqslant 1$ such that $Z_{m+p} = Z_p$, hence $r_{m+p} = r_p$ and the $(-\beta)$-expansion of x is eventually periodic. □

As a corollary we get the following result.

Theorem 7. *If β is a Pisot number then the $(-\beta)$-shift is a sofic system.*

The *normalization* in base $-\beta$ is the function which maps any $(-\beta)$-representation on an alphabet C of digits of a given number of $I_{-\beta}$ onto the admissible $(-\beta)$-expansion of that number.

Let $C = \{-c, \dots, c\}$, where $c \geqslant \lfloor \beta \rfloor$ is an integer. Denote

$$Z_{-\beta}(2c) = \Big\{ (z_i)_{i \geqslant 0} \in \{-2c, \dots, 2c\}^{\mathbb{N}} \;\Big|\; \sum_{i \geqslant 0} z_i (-\beta)^{-i} = 0 \Big\}.$$

The set $Z_{-\beta}(2c)$ is recognized by a countable infinite automaton $\mathcal{A}_{-\beta}(2c)$: the set of states $Q(2c)$ consists of all $s \in \mathbb{Z}[\beta] \cap [-\frac{2c}{\beta-1}, \frac{2c}{\beta-1}]$. Transitions are of the form $s \xrightarrow{e} s'$ with $e \in \{-c, \dots, c\}$ such that $s' = -\beta s + e$. The state 0 is initial; every state is terminal.

Let $M_\beta(X)$ be the minimal polynomial of β, and denote by $\beta = \beta_1, \dots, \beta_d$ the conjugates of β. We define a norm on the discrete lattice of rank d, $\mathbb{Z}[X]/(M_\beta)$, as

$$||P(X)|| = \max_{1 \leqslant i \leqslant d} |P(\beta_i)|.$$

Proposition 6. *If β is a Pisot number then the automaton $\mathcal{A}_{-\beta}(2c)$ is finite for every $c \geqslant \lfloor \beta \rfloor$.*

Proof. Every state s in $Q(2c)$ is associated with the label of the shortest path $f_0 f_1 \cdots f_n$ from 0 to s in the automaton. Thus $s = f_0(-\beta)^n + f_1(-\beta)^{n-1} + \cdots + f_n = P(\beta)$, with $P(X)$ in $\mathbb{Z}[X]/(M_\beta)$. Since $f_0 f_1 \cdots f_n$ is a prefix of a word of $Z_{-\beta}(2c)$, there exists $f_{n+1} f_{n+2} \cdots$ such that $(f_i)_{i \geqslant 0}$ is in $Z_{-\beta}(2c)$. Thus $s = |P(\beta)| < \frac{2c}{\beta-1}$. For every conjugate β_i, $2 \leqslant i \leqslant d$, $|\beta_i| < 1$, and $|P(\beta_i)| < \frac{2c}{1-|\beta_i|}$. Thus every state of $Q(2c)$ is bounded in norm, and so there is only a finite number of them. □

The *redundancy transducer* $\mathcal{R}_{-\beta}(c)$ is similar to $\mathcal{A}_{-\beta}(2c)$. Each transition $s \xrightarrow{e} s'$ of $\mathcal{A}_{-\beta}(2c)$ is replaced in $\mathcal{R}_{-\beta}(c)$ by a set of transitions $s \xrightarrow{a|b} s'$, with $a, b \in \{-c, \dots, c\}$ and $a - b = e$. Thus one obtains the following proposition.

Proposition 7. *The redundancy transducer $\mathcal{R}_{-\beta}(c)$ recognizes the set*

$$\big\{ (x_1 x_2 \cdots, y_1 y_2 \cdots) \in C^{\mathbb{N}} \times C^{\mathbb{N}} \;\big|\; \sum_{i \geqslant 1} x_i (-\beta)^{-i} = \sum_{i \geqslant 1} y_i (-\beta)^{-i} \big\}.$$

If β is a Pisot number, then $\mathcal{R}_{-\beta}(c)$ is finite.

Theorem 8. *If β is a Pisot number, then normalization in base $-\beta$ on any alphabet C is realizable by a finite transducer.*

Proof. The normalization is obtained by keeping in $\mathcal{R}_{-\beta}(c)$ only the outputs y that are $(-\beta)$-admissible. By Theorem 7 the set of admissible words is recognizable by a finite automaton $\mathcal{D}_{-\beta}$. The finite transducer $\mathcal{N}_{-\beta}(c)$ doing the normalization is obtained by making the intersection of the output automaton of $\mathcal{R}_{-\beta}(c)$ with $\mathcal{D}_{-\beta}$. □

Proposition 8. *If β is a Pisot number, then the conversion from base $-\beta$ to base β is realizable by a finite transducer. The result is β-admissible.*

Proof. Let $x \in I_{-\beta}$, $x \geqslant 0$, such that $\mathsf{d}_{-\beta}(x) = x_1 x_2 x_3 \cdots$. Denote $\bar{a}$ the signit digit $(-a)$. Then $\overline{x_1} x_2 \overline{x_3} \cdots$ is a β-representation of x on the alphabet $\widetilde{A_{-\beta}} = \{-\lfloor\beta\rfloor, \ldots, \lfloor\beta\rfloor\}$. Thus the conversion is equivalent to the normalization in base β on the alphabet $\widetilde{A_{-\beta}}$, and when β is a Pisot number, it is realizable by a finite transducer by [4]. □

Remark 2. In the case where the base is a negative integer, conversion from base b to base $-b$ is realizable by a finite right sequential transducer. In a forthcoming paper we show that conversion from base β to base $-\beta$ — with the result in non-admissible form — is realizable by a finite left sequential transducer when β is a Pisot number.

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