On-line odometers for two-sided symbolic dynamical systems

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Abstract. We consider biinfinite sequences on a finite alphabet of digits which satisfy a constraint of finite type. Such sequences are perturbed by adding a 1 in position 0. The odometer is the function which transforms the initial sequence into an admissible sequence equivalent to the perturbed one. It is shown that the odometer can be realized by an on-line finite automaton when the constraint is linked to numeration in base β, where β is a Pisot number satisfying the equation $\beta^m = t_1\beta^{m-1} + \cdots + t_m$, where $t_1 \geq t_2 \geq \cdots t_m \geq 1$ are integers.

Keywords : Finite automata, symbolic dynamical systems, numeration systems, odometer.

1 Introduction

We present the problem by an example. Consider the set V of biinfinite sequences of 0's and 1's satisfying the constraint that no factor 11 occurs. Denote such a sequence by $a = (a_n)_{n\in\mathbb{Z}} = \cdots a_2 a_1 a_0 . a_{-1} a_{-2} a_{-3} \cdots$. The set V is a symbolic dynamical system of finite type, called the golden ratio shift, see [14]. Now we introduce a small perturbation consisting of adding 1 at coordinate of index 0, and we want to eliminate the non-admissible factors, namely those containing 11 or 2, in order to obtain an "equivalent" sequence $b = (b_n)_{n\in\mathbb{Z}}$ belonging to V. The *odometer*, denoted by τ, is the function that maps the sequence a onto the sequence b. Here we use the fact that $011 \sim 100$ and that $0200 \sim 1001$ in base $\beta = (1+\sqrt{5})/2$. We give two examples.

Let $a = \cdots 0000101.01010000 \cdots$. Then

$$\begin{aligned} a+1 &= \cdots 0000102.01010000 \cdots \\ &\sim \cdots 0000110.02010000 \cdots \\ &\sim \cdots 0001000.10020000 \cdots \\ &\sim \cdots 0001000.10100100 \cdots \end{aligned}$$

Thus $\tau(a) = b = \cdots 0001000.10100100 \cdots$.

Let $a' = \cdots 010101010101.010101010101 \cdots$. Then

$$\begin{aligned} a'+1 &= \cdots 010101010102.010101010101\cdots \\ &\sim \cdots 010101010110.020101010101\cdots \\ &\sim \cdots 010101011000.100201010101\cdots \\ &\sim \cdots 010101100000.101002010101\cdots \\ &\cdots \\ &\sim \cdots 000000000000.101010101010\cdots \end{aligned}$$

Thus $\tau(a') = b' = \cdots 000000000000.101010101010 \cdots$.

In this paper we show that the odometer can be realized by an on-line algorithm starting from the middle of the input, going simultaneously in both directions, and needing only a finite auxiliary storage memory, that is to say by a bidirectional *on-line finite automaton*, see Section 2.5 for definitions. On-line finite automata are a special kind of sequential finite automata, processing data from left to right, see [16]. In integral base b on the canonical digit set $\{0, \ldots, b-1\}$, addition is not on-line computable, but with a balanced alphabet of signed digits of the form $\{-a, \ldots, a\}$ with $b/2 \leq a \leq b-1$, using the algorithms of Avizienis [1] and Chow and Robertson [4], addition is computable by an on-line finite automaton.

Now we introduce the problem in full generality. Let β be a non-integer real number > 1. For any $\beta > 1$, by the greedy algorithm of Rényi [18], one can compute a representation in base β of any real number x from the interval $[0,1]$, called its β-expansion and denoted by $d_\beta(x)$, and where the digits are elements of the canonical digit set $A_\beta = \{0, \ldots, \lfloor\beta\rfloor\}$. In such a representation, not all the digits patterns are allowed (see [17] for instance), and the numeration system defined by β is thus redundant. The β-expansion of 1 plays a special role. If $d_\beta(1) = t_1 \cdots t_m$ is finite (meaning that it ends with infinitely many zeroes), the associated dynamical system is said to be of *finite type*.

The *normalization* is the fonction ν which maps an infinite sequence of digits $(d_k)_{k\geq 1}$ onto $d_\beta(\sum_{k\geq 1} d_k \beta^{-k})$. Recall that a Pisot number (also called a Pisot-Vijayaraghavan number) is an algebraic integer such that all its algebraic conjugates have modulus less than 1. The natural numbers and the golden ratio are Pisot numbers. It is known that if β is a Pisot number, then normalization on any alphabet is computable by a finite automaton [6], and that non-normalized digit set conversion is computable by an on-line finite automaton [8].

Two-sided beta-expansions have been considered in [20] for the golden ratio case, and in full generality in [19]. In [20] two-sided normalization is introduced and studied in relation with the Erdős measure.

The one-sided odometer has been introduced in [12]. Roughly speaking, to a number $\beta > 1$ one can canonically associate a numeration system defined by a sequence of integers U_β, allowing the representation of the natural numbers [2]. The odometer is the left-infinite extension of the successor function in the numeration system U_β. It is shown in [12] that the one-sided odometer associated with β is continuous if and only if the β-expansion of 1 is finite. We have shown in [7] that the successor function in U_β is realizable by a right subsequential finite automaton if and only if $d_\beta(1)$ is finite. From this follows, by a result of [10], that the associated one-sided odometer can be realized by a right on-line finite automaton. If $d_\beta(1) = t_1 \cdots t_m$, one can give a direct construction of a right on-line finite automaton with delay $m-1$, [9].

Since we are interested in on-line functions, which are continuous by the result just recalled, $d_\beta(1)$ must be finite, so we define the two-sided odometer only in this case. Let β such that $d_\beta(1) = t_1 \cdots t_m$. We consider the set V_β of biinfinite sequences on the alphabet A_β such that no factor greater than or equal in the lexicographic order to $t_1 \cdots t_m$ occurs. This defines a two-sided symbolic dynamical system of finite type. Admissibility of such sequences can be checked on finite factors of length m. These sequences are perturbed by adding a 1 in position 0. The two-sided odometer can be seen as the biinfinite extension of the successor function. It is the function which transforms a sequence a of V_β into a sequence b of V_β equivalent to the sequence $a+1$ (see below for a formal definition of equivalence).

From the previous results, it is natural to consider numbers β that are Pisot numbers. In this paper we consider numbers β such that $d_\beta(1) = t_1 \cdots t_m$, with $t_1 \geq t_2 \geq \cdots \geq t_m \geq 1$. It is proved in [3] that they are Pisot numbers. These Pisot numbers appear in several contexts, *e.g.* [11] and [13]. We show that, for these numbers β, the two-sided odometer associated with β is computable by a bidirectional on-line finite automaton with delay m, hence it is a continuous function. This delay is optimum. There exist other Pisot numbers with $d_\beta(1) = t_1 \cdots t_m$ such that the two-sided odometer is computable by an on-line finite automaton, but the delay is larger than m. In fact the major difficulty is the determination of the exact delay in general. For instance, for the Pisot number β such that $d_\beta(1) = 201$, with similar arguments one can show that the two-sided odometer is computable by an on-line finite automaton with delay 4.

The paper is organized as follows. After the definitions, the result is proved, and the two-sided odometer associated with the golden ratio is developed in Example 1.

2 Preliminaries

2.1 Words

An *alphabet* A is a finite set. A finite sequence of elements of A is called a *word*, and the set of words on A is the free monoid A^*. The *length* of a word u is equal to the number of its letters, and is denoted by $|u|$. The *empty word* is denoted by ε. Let v be a word of A^*, denote by v^n the concatenation of n times the word v, and by A^n the set of words on A of length n.

The set of *(right) infinite* words on A is denoted by $A^{\mathbb{N}}$, and by v^ω the infinite concatenation $vvv\cdots$. The set of *left infinite* words on A is denoted by ${}^{\mathbb{N}}A$, and the sequence $\cdots vvv$ is denoted by ${}^{\omega}v$.

One defines a distance ρ on $A^{\mathbb{N}}$ as follows: let $v = (v_i)_{i\in\mathbb{N}}$ and $w = (w_i)_{i\in\mathbb{N}}$ be in $A^{\mathbb{N}}$, $\rho(v,w) = 2^{-r}$ where $r = \min\{i \geq 0 \mid v_i \neq w_i\}$ if $v \neq w$, and $\rho(v,w) = 0$ otherwise. The set $A^{\mathbb{N}}$ equipped with this metric becomes a compact metric space. The topology induced by the metric is the same as the product topology of $A^{\mathbb{N}}$ where A is equipped with the discrete topology.

The set of biinfinite words is denoted by $A^{\mathbb{Z}}$, and $\cdots vvv \cdots$ by ${}^{\omega}v^{\omega}$. In this paper, a biinfinite word $v \in A^{\mathbb{Z}}$ is denoted by $\cdots v_2 v_1 v_0 \cdot v_{-1} v_{-2} v_{-3} \cdots$ with v_i in A for every i in $\mathbb{Z}$.

One defines a distance ρ on $A^{\mathbb{Z}}$ as follows: let $v = (v_i)_{i\in\mathbb{Z}}$ and $w = (w_i)_{i\in\mathbb{Z}}$ be in $A^{\mathbb{Z}}$, $\rho(v,w) = 2^{-r}$ if $v \neq w$ and $r = \max\{i \mid v_i \cdots v_{-i} = w_i \cdots w_{-i}\}$, and $\rho(v,w) = 0$ if $v = w$. The set $A^{\mathbb{Z}}$ is then a compact metric space. This topology is equal to the product topology.

2.2 On-line functions

Let A and B be two alphabets, and let $\varphi : (a_n)_{n\in\mathbb{N}} \mapsto (b_n)_{n\in\mathbb{N}}$ be a function from $A^{\mathbb{N}}$ to $B^{\mathbb{N}}$. Then φ is said to be *on-line computable with delay* δ if there exists an integer δ such that for every $n \geq 0$ there exists a function $\Phi_n : A^{n+\delta} \to B$ such that $b_n = \Phi_n(a_0 \cdots a_{n+\delta-1})$. An on-line computable function with delay δ is 2^{δ}-Lipschitz and is thus uniformously continuous [8].

A function $\varphi : (a_n)_{n\in\mathbb{Z}} \mapsto (b_n)_{n\in\mathbb{Z}}$ from $A^{\mathbb{Z}}$ to $B^{\mathbb{Z}}$ is said to be *on-line computable with delay* δ if there exists an integer δ such that for every $n \geq 0$ there exists a function $\Phi_n : A^{n+\delta} \to B$ and for every $p \geq 1$ a function $\Psi_p : A^{p+\delta} \to B$ such that $b_n = \Phi_n(a_{n+\delta-1} \cdots a_0)$ and $b_{-p} = \Psi_p(a_{-1} \cdots a_{-p-\delta})$.

As for infinite words, we have that

Proposition 1. *Any function $\varphi : A^{\mathbb{Z}} \longrightarrow B^{\mathbb{Z}}$ which is on-line computable with delay δ is 2^{δ}-Lipschitz and is thus uniformously continuous.*

2.3 Beta-Representations

We refer the reader to [15, Chap. 7] for results and proofs on these topics. Let $\beta > 1$ be a real number. A *β-representation* on a digit set D of a number x of $[0,1]$ is an infinite sequence $(d_k)_{k\geq 1}$ of digits d_k in D such that $\sum_{k\geq 1} d_k \beta^{-k} = x$.

Any real number $x \in [0,1]$ can be represented in base β by the following greedy algorithm [18]. Denote by $\lfloor . \rfloor$ and by $\{.\}$ the integral part and the fractional part of a number, respectively. Let $r_0 = x$ and for $k \geq 1$, let $x_k = \lfloor \beta r_{k-1} \rfloor$ and $r_k = \{\beta r_{k-1}\}$.
Thus $x = \sum_{k\geq 1} x_k \beta^{-k}$, where the digits x_k are elements of the *canonical* alphabet $A_\beta = \{0, \ldots, \lfloor \beta \rfloor\}$ if $\beta \notin \mathbb{N}$, and $A_\beta = \{0, \ldots, \beta - 1\}$ otherwise. The sequence $(x_k)_{k\geq 1}$ of $A_\beta^{\mathbb{N}}$ is called the *β-expansion* of x, and is denoted by $d_\beta(x)$. When β is not an integer, a number x may have several different β-representations on A_β: this system is naturally redundant. The β-expansion obtained by the greedy algorithm is the greatest one in the lexicographic order. A β-representation is said to be *finite* if it ends with infinitely many zeroes, and the zeroes are omitted.

The β-expansion of an $x > 1$ is obtained as follows: let $n \geq 0$ such that $\beta^n \leq x < \beta^{n+1}$. The β-expansion of x is $x_n \cdots x_0 . x_{-1} x_{-2} \cdots$ if and only if $x_n \cdots x_0 x_{-1} x_{-2} \cdots$ is the β-expansion of x/β^{n+1}.

The set $A_\beta^{\mathbb{N}}$ is endowed with the lexicographic order, denoted by $\prec$, and the shift σ, defined by $\sigma((x_i)_{i\geq 1}) = (x_{i+1})_{i\geq 1}$. Denote by D_β the set of β-expansions of numbers of $[0,1[$. It is a shift-invariant subset of $A_\beta^{\mathbb{N}}$. The *β-shift* S_β is the topological closure of D_β in $A_\beta^{\mathbb{N}}$.

Let us introduce the following notation. Let $d_\beta(1) = (t_i)_{i\geq 1}$ and set $d_\beta^*(1) = d_\beta(1)$ if $d_\beta(1)$ is infinite and $d_\beta^*(1) = (t_1 \cdots t_{m-1}(t_m - 1))^\omega$ if $d_\beta(1) = t_1 \cdots t_{m-1} t_m$ is finite. Accordingly $d_\beta^*(x) = d_\beta(x)$ if the latter is infinite, $d_\beta^*(x) = x_1 \cdots x_{k-1}(x_k - 1) d_\beta^*(1)$ if $d_\beta(x) = x_1 \cdots x_k$ is finite.

By a result from [17], one has that

$$D_\beta = \{s \in A_\beta^{\mathbb{N}} \mid \forall p \geq 0 \ \sigma^p(s) \prec d_\beta^*(1)\}$$

and

$$S_\beta = \{s \in A_\beta^{\mathbb{N}} \mid \forall p \geq 0 \ \sigma^p(s) \preceq d_\beta^*(1)\}.$$

We now give the definition of the two-sided β-shift, denoted by V_β (see [19])

$$V_\beta = \{s \in A_\beta^{\mathbb{Z}} \mid \forall k \in \mathbb{Z} \ (s_n)_{n\leq k} \in S_\beta\}.$$

Denote by L_β the set $F(D_\beta)$ of finite factors of D_β. Clearly $L_\beta = F(S_\beta) = F(V_\beta)$. Set $V_\beta^+ = \{...s_3 s_2 s_1 s_0 \mid s \in V_\beta\}$ and $V_\beta^- = \{s_{-1} s_{-2} s_{-3} \cdots \mid s \in V_\beta\}$.

When $d_\beta(1) = t_1 \cdots t_m$, the β-shift is a system of *finite type*, and to determine if a word belongs to L_β can be checked on factors of length m. The set of maximal words in the lexicographic order is $M_\beta = (t_1 \cdots t_{m-1}(t_m - 1))^* \{\varepsilon, t_1, t_1 t_2, \ldots, t_1 t_2 \cdots t_{m-1}\}$.

The set L_β is recognized by a finite automaton $\mathcal{A}_\beta$ with m states $q_1, \ldots, q_m$ defined as follows. For each i, $1 \leq i < m$, there is an edge labelled t_i from q_i to q_{i+1}. For $1 \leq i \leq m$, there are edges labelled by $0, 1, \ldots, t_i - 1$ from q_i to q_1. Let q_1 be the only initial state, and all states be terminal. The β-shift S_β is thus the set of labels of infinite paths in $\mathcal{A}_\beta$, and the two-sided β-shift V_β is the set of labels of biinfinite paths in $\mathcal{A}_\beta$.

2.4 Two-sided odometer associated with β

Let β such that $d_\beta(1) = t_1 \cdots t_m$. We introduce some notations : let $s = \cdots s_2 s_1 s_0 . s_{-1} s_{-2} \cdots$, set $s + 1 = \cdots s_2 s_1 (s_0 + 1) . s_{-1} s_{-2} \cdots$, and similarly for finite or infinite words. Let $v = v_j v_{j-1} \cdots$, $x = \sum_{i\leq j} v_i \beta^i$, and let $\lambda(v) = d_\beta(x)$ if v does not end with $(t_1 \cdots t_{m-1}(t_m - 1))^\omega$, $\lambda(v) = d_\beta^*(x)$ otherwise.

The *two-sided odometer* associated with β is the function $\tau : V_\beta \to V_\beta$ defined as follows. Let $s = \cdots s_2 s_1 s_0 . s_{-1} s_{-2} \cdots \in V_\beta$. There are two cases.
1. There exists $N_s \geq 0$ such that for every $j \geq N_s$, $s_j \cdots s_0 \notin M_\beta$. This means that the carry does not propagate infinitely far to the left. For $j \geq N_s$ let $\tau(s) = \cdots s_{j+2} s_{j+1} \lambda(s_j \cdots s_1 (s_0 + 1) . s_{-1} s_{-2} \cdots)$.

2. The carry does propagate infinitely far to the left.
If $\cdots s_2 s_1 s_0 = {}^\omega(t_1 \cdots t_{m-1}(t_m - 1))$ then $\tau(s) = {}^\omega 0{\bf .}s_{-1}s_{-2}\cdots$. If $\cdots s_2 s_1 s_0 = {}^\omega(t_1 \cdots t_{m-1}(t_m - 1))t_1 \cdots t_k$, with $1 \leq k \leq m-1$, then $\tau(s) = {}^\omega 0 y_k y_{k-1}\cdots$, where $y_{k+1}y_k \cdots = \lambda(0t_1 \cdots t_{k-1}(t_k+1){\bf .}s_{-1}s_{-2}\cdots)$.

The one-sided odometer is thus the restriction of the two-sided odometer to the set V_β^+.

If u is a word equal to $u_1 \cdots u_n$, we denote by $\pi_\beta(u) = u_1\beta^{n-1} + \cdots + u_n$ and by $\mu_\beta(u) = u_1\beta^{-1} + \cdots + u_n\beta^{-n}$.

2.5 Automata

We refer the reader to [5]. An *automaton over* A, $\mathcal{A} = (Q, A, E, I, T)$, is a directed graph labelled by elements of A. The set of vertices, traditionally called *states*, is denoted by Q, $I \subset Q$ is the set of *initial* states, $T \subset Q$ is the set of *terminal* states and $E \subset Q \times A \times Q$ is the set of labelled *edges*. If $(p, a, q) \in E$, we note $p \xrightarrow{a} q$. The automaton is *finite* if Q is finite. The automaton $\mathcal{A}$ is *deterministic* if E is the graph of a (partial) function from $Q \times A$ into Q, and if there is a unique initial state. A subset H of A^* is said to be *recognizable by a finite automaton* if there exists a finite automaton $\mathcal{A}$ such that H is equal to the set of labels of paths starting in an initial state and ending in a terminal state. A subset K of $A^{\mathbb{N}}$ is said to be *recognizable by a finite automaton* if there exists a finite automaton $\mathcal{A}$ such that K is equal to the set of labels of infinite paths starting in an initial state and going infinitely often through a terminal state (Bűchi acceptance condition, see [5]).

On-line automata are a special kind of 2-tape automata or transducers. Let A and B be two alphabets. An *on-line finite automaton* with delay δ, $\mathcal{A} = (Q, A \times (B \cup \varepsilon), E, \{i_0\})$, is a finite automaton labelled by elements of $A \times (B \cup \varepsilon)$ which is deterministic when projected on the input tape A. Moreover it is composed of a transient part and of a synchronous part (see [16]). The set of states is equal to $Q = Q_t \cup Q_s$, where Q_t is the set of transient states and Q_s is the set of synchronous states. In the transient part, every path of length δ starting in the initial state i_0 is of the form

$$i_0 \xrightarrow{x_1/\varepsilon} i_1 \xrightarrow{x_2/\varepsilon} \cdots \xrightarrow{x_\delta/\varepsilon} i_\delta$$

where $i_0, \ldots, i_{\delta-1}$ are in Q_t, x_j in A, for $1 \leq j \leq \delta$, and every edge arriving in a state of Q_t is as above. In the synchronous part, edges are labelled by elements of $A \times B$. This means that the automaton starts reading words of length $\leq \delta$ outputting nothing, and after δ readings, outputs serially one digit for each input digit. In practice we shall not consider transient states and edges, but we will replace a transient path $i_0 \xrightarrow{x_1/\varepsilon} i_1 \xrightarrow{x_2/\varepsilon} \cdots \xrightarrow{x_\delta/\varepsilon} i_\delta$ by defining $x_1 \cdots x_\delta$ as an *initial state*. Edges will then be all synchronous.

A function $\varphi : A^{\mathbb{N}} \longrightarrow B^{\mathbb{N}}$ is computable by an on-line finite automaton if its graph is recognizable by an on-line finite automaton.

This definition is implicitly the one of a *left* automaton, where words are processed from left to right. An automaton processing words from right to left will be called a *right* automaton. A *bidirectional on-line finite automaton* with delay δ is composed of a left on-line finite automaton with delay δ and of a right on-line finite automaton with delay δ. Such a device starts from the middle, and then goes synchronously in both directions. A function $\varphi : A^{\mathbb{Z}} \longrightarrow B^{\mathbb{Z}}$ is computable by an on-line finite automaton if its graph is recognizable by a bidirectional on-line finite automaton.

3 On-line two-sided odometer

Our purpose is to show the following result.

Theorem 1. *Let β be a Pisot number such that $d_\beta(1) = t_1 \cdots t_m$ with $t_1 \geq t_2 \geq \cdots t_m \geq 1$. The two-sided odometer associated with β is computable by a bidirectional on-line finite automaton with delay m.*

The proof is composed of three parts. First we define the central initial states, then the right on-line finite automaton $\mathcal{P}$ working on V_β^+ and finally the left on-line finite automaton $\mathcal{N}$ working on V_β^-. States of $\mathcal{P}$ and of $\mathcal{N}$ will be composed of pairs (u, v) where v is the last factor of length $m-1$ of the input having been read before arriving to that state. This allows the checking of the admissibility of the input (see Example 1 below). The input is a word $a = (a_k)_{k\in\mathbb{Z}}$ belonging to V_β and the output is a word $b = (b_k)_{k\in\mathbb{Z}}$ belonging to V_β such that $b = \tau(a)$. Two words u and v are *equivalent*, denoted by $u \sim v$, if v is obtained from u by replacing every factor $0t_1 \cdots t_m$ by the equivalent factor 10^m and $0t_1 \cdots t_{j-1}(t_j + 1)0^m$, for $1 \leq j \leq m-1$, by $10^j(t_1 - t_{j+1}) \cdots (t_{m-j} - t_m)t_{m-j+1} \cdots t_m$. The delay m is taken into account in the length m of the states.

First we give a fundamental lemma, which uses the hypothesis of decrease of the coefficients t_i's.

Lemma 1. *Let $w = 0t_1 \cdots t_{j-1}t_j{\centerdot}t_{j+1} \cdots t_{j+k-1}(t_{j+k} - h)a_{-k-1} \cdots a_{-m-1}$ be in L_β, with $1 \leq j \leq m-1$, $1 \leq k \leq m$, $1 \leq h \leq t_{j+k}$, and let $v = w + 1$. Then $v \sim y = 10^j{\centerdot}b_{-1}u$ with b_{-1} in A_β, $|u| = m$, and $0 \leq \mu_\beta(u) \leq 1 - \beta^{-m}$.*

Proof of Theorem 1. Let $(a_k)_{k\in\mathbb{Z}}$ be fixed in V_β.
1. Central initial states
A *central initial state* is a word of the form $c = a_{m-1} \cdots a_1(a_0 + 1){\centerdot}a_{-1} \cdots a_{-m}$. Let $w = a_m c a_{-m-1}$. The general scheme is the following : there are two edges outgoing from c, one going into $\mathcal{P}$ and the other one going into $\mathcal{N}$

$$p_0 \overset{a_m/b_0}{\longleftarrow} c \overset{a_{-m-1}/b_{-1}}{\longrightarrow} n_1$$

where $p_0 = (s, t)$ and $n_1 = (u, v)$, where s and u are words of length m such that $sb_0{\centerdot}b_{-1}u \sim w$. The words t and v represent the past of s and u respectively. We say that p_0 is an initial state of $\mathcal{P}$ and n_1 is an initial state of $\mathcal{N}$.
Denote by R_p the set of words of L_β of length p. There are three cases.
1. If w is in L_β, then we set $b_0 = a_0+1$ and $b_{-1} = a_{-1}$. Thus $p_0 = (a_m \cdots a_1, a_m \cdots$

a_2) and $n_1 = (a_{-2} \cdots a_{-m-1}, a_{-3} \cdots a_{-m-1})$ are elements of $R_m \times R_{m-1}$.
2. If w contains a factor $t_1 \cdots t_m$, we take the leftmost factor. There exists j, $1 \le j \le m$ such that

$$w = a_m \cdots a_j t_1 \cdots t_j . t_{j+1} \cdots t_m a_{-m+j-1} \cdots a_{-m-1}.$$

Thus

$$w \sim a_m \cdots a_{j+1}(a_j + 1)0^j . 0^{m-j} a_{-m+j-1} \cdots a_{-m-1}.$$

Set $b_0 = 0$ and $p_0 = (a_m \cdots a_{j-1}(a_j+1)0^{j-1}, a_m \cdots a_j t_1 \cdots t_{j-2})$ which belongs to $R_m \times R_{m-1}$ since $a_m \cdots a_j \preceq t_1 \cdots t_{m-j}(t_{m-j} - 1)$, except if $j = 1$ and $w = t_1 \cdots t_{m-1}(t_m - 1)t_1 . t_2 \cdots t_m a_{-m-1}$, in which case $p_0 = (t_1 \cdots t_m, t_1 \cdots t_{m-1})$. If $j < m$ then $b_{-1} = 0$ and $n_1 = (0^{m-j} a_{-m+j-1} \cdots a_{-m-1}, t_{j+2} \cdots t_m a_{-m+j-1} \cdots a_{-m-1})$ is in $R_m \times R_{m-1}$. If $j = m$ then $b_{-1} = a_{-1}$ and $n_1 = (a_{-2} \cdots a_{-m-1}, a_{-3} \cdots a_{-m-1})$ is in $R_m \times R_{m-1}$.
3. If w contains a factor $\succ t_1 \cdots t_m$, we take the leftmost one, and there exist $1 \le j \le m-1$, $1 \le k \le m$ and $1 \le h \le t_{j+k}$ such that

$$w = a_m \cdots a_j t_1 \cdots t_{j-1}(t_j + 1) . t_{j+1} \cdots t_{j+k-1}(t_{j+k} - h) a_{-k-1} \cdots a_{-m-1}.$$

By Lemma 1, $w \sim a_m \cdots a_{j-1}(a_j + 1)0^j . b_{-1} u$ with $\mu_\beta(u) \le 1 - \beta^{-m}$. Thus we set $b_0 = 0$, and $p_0 = (a_m \cdots a_{j-1}(a_j+1)0^{j-1}, a_m \cdots a_j t_1 \cdots t_{j-1})$; then $p_0 \in R_m \times R_{m-1}$ except if $j = 1$ and $w = t_1 \cdots t_{m-1}(t_m - 1)t_1 . t_2 \cdots t_m a_{-m-1}$, in which case $p_0 = (t_1 \cdots t_m, t_1 \cdots t_{m-1})$. We set $n_1 = (u, t_{j+2} \cdots t_{j+k-1}(t_{j+k} - h) a_{-k-1} \cdots a_{-m-1})$.

2. Right on-line finite automaton $\mathcal{P}$
There are four classes of states of $\mathcal{P}$:
$C_1 = \{p_k = (a_{m+k} \cdots a_{k+1}, a_{m+k} \cdots a_{k+2}) \in R_m \times R_{m-1} \mid k \ge 0\}$,
$C_2 = \{p_k = (a_{m+k} \cdots a_{j+k-1}(a_{j+k} + 1)0^{j-1}, a_{m+k} \cdots a_{j+k} t_1 \cdots t_{j-2}) \in R_m \times R_{m-1},\ 2 \le j \le m \mid k \ge 0\}$,
$C_3 = \{p_k = (a_{m+k} \cdots a_{k+2}(a_{k+1} + 1), a_{m+k} \cdots a_{k+2}) \in R_m \times R_{m-1} \mid k \ge 0\}$,
$C_4 = \{(t_1 \cdots t_m, t_1 \cdots t_{m-1})\}$.
The set of initial states is $\{p_0 \in \cup_{1 \le i \le 4} C_i\}$.
There are two types of transitions, the reduction edges and the shift edges. First we describe the shift edges.
Let $p_k \in C_1$. There is an edge $p_k \xrightarrow{a_{m+k+1}/a_{k+1}} p_{k+1} \in C_1$.
Let $p_k \in C_2$. There is an edge $p_k \xrightarrow{a_{m+k+1}/0} p_{k+1} = (a_{m+k+1} \cdots a_{j+k}(a_{j+k+1} + 1)0^{j-2}, a_{m+k+1} \cdots a_{j+k+1} t_1 \cdots t_{j-3})$, and p_{k+1} belongs to C_2 if $j \ge 3$. If $j = 2$, p_{k+1} belongs to C_3, or to C_4 if p_k is of the form $p_k = (t_2 \cdots t_m 0, t_2 \cdots t_{m-1})$ and the edge is $p_k \xrightarrow{t_1/0} p_{k+1} = (t_1 \cdots t_m, t_1 \cdots t_{m-1}) \in C_4$.
Let $p_k \in C_3$. There is an edge $p_k \xrightarrow{a_{m+k+1}/a_{k+1}+1} p_{k+1} \in C_1$.
The only reduction edge is the following one $(t_1 \cdots t_m, t_1 \cdots t_{m-1}) \xrightarrow{a_{m+k+1}/0} p_{k+1} = ((a_{m+k+1} + 1)0^{m-1}, a_{m+k+1} t_1 \cdots t_{m-2})$ which belongs to C_2.

Clearly, if $p_k = (u_k, v_k)$ and if $a_{m+k+1} v_k \in R_m$ then there is an edge $p_k \xrightarrow{a_{m+k+1}/b_{k+1}} p_{k+1}$ with $p_{k+1} = (u_{k+1}, v_{k+1})$ and $\pi_\beta(a_{m+k+1} u_k) = \pi_\beta(u_{k+1} b_{k+1})$.

Let $p_0 \stackrel{a_{m+1}/b_1}{\longrightarrow} p_1 \stackrel{a_{m+2}/b_2}{\longrightarrow} \cdots p_{n-1} \stackrel{a_{m+n}/b_n}{\longrightarrow} p_n \cdots$ be an infinite path in $\mathcal{P}$. We have to show that the sequence $(b_i)_{i\geq 0}$ belongs to V_β^+.

First suppose there exists infinitely many k's such that p_k belongs to C_4. Thus the input is of the form ${}^\omega(t_1 \cdots t_{m-1}(t_m-1))z.a_{-1}\cdots$ with $z = t_1 \cdots t_m$ or $z = t_1 \cdots t_{j-1}(t_j+1)$. The carry propagates infinitely to the left, and the positive part of the output is ${}^\omega 0$.

Second, there exists a finite number of indices k such that p_k belongs to C_4, and let K be the greatest of them. This means that the carry does not propagate after state p_{K+1}. After that, there are only shift edges, which preserve admissibility.

3. Left on-line finite automaton $\mathcal{N}$

The set of states of $\mathcal{N}$ is taken as $S = \{n_k = (u_k, v_k) \mid k \geq 1,\ |u_k| = m,\ 0 \leq \mu_\beta(u_k) < 1,\ v_k \in R_{m-1}\}$. Note that, when u_k belongs to R_m then $\mu_\beta(u_k) \leq 1-\beta^{-m}$, so the set of initial states is $I = \{n_1 = (u_1, v_1) \mid |u_1| = m,\ 0 \leq \mu_\beta(u_1) \leq 1-\beta^{-m},\ v_1 = a_{-3}\cdots a_{-m-1}\}$. The proof that, if β is a Pisot number then the set S is finite, is the same as the one given in Theorem 2 of my paper [8], so I do not give it here. This implies that the component u_k of state n_k is written on a finite alphabet of digits.

Let $k \geq 1$. There is an edge

$$n_k = (u_k, v_k) \stackrel{a_{-m-k-1}/b_{-k-1}}{\longrightarrow} n_{k+1} = (u_{k+1}, v_{k+1})$$

if $v_k a_{-m-k-1} \in R_m$, and b_{-k-1} is chosen as follows.
If $\mu_\beta(u_k a_{-m-k-1}) < 1$ take $b_{-k-1} = \lfloor \beta\mu_\beta(u_k a_{-m-k-1}) \rfloor$ and let $y_{k+1} = \{\beta\mu_\beta(u_k a_{-m-k-1})\}$. Hence $b_{-k-1} \in A_\beta$ and $y_{k+1} < 1$. Take u_{k+1} of length m such that $\mu_\beta(u_{k+1}) = y_{k+1}$.
If $\mu_\beta(u_k a_{-m-k-1}) \geq 1$ take $b_{-k-1} = t_1$ and u_{k+1} of length m such that $u_k a_{-m-k-1} \sim t_1 u_{k+1}$. Then $\mu_\beta(u_{k+1}) = \beta\mu_\beta(u_k) + a_{-m-k-1}\beta^{-m} - t_1 < \beta + t_1\beta^{-m} - t_1 \leq 1$.

So the relation $\mu_\beta(u_k a_{-m-k-1}) = \mu_\beta(b_{-k-1}u_{k+1})$ holds for each k, with n_{k+1} in S and b_{-k-1} in A_β.

Let $n_1 \stackrel{a_{-m-2}/b_{-2}}{\longrightarrow} n_2 \stackrel{a_{-m-3}/b_{-3}}{\longrightarrow} \cdots n_{k-1} \stackrel{a_{-m-k}/b_{-k}}{\longrightarrow} n_k \cdots$ be an infinite path in $\mathcal{N}$. We have to show that the sequence $(b_{-k})_{k\geq 1}$ belongs to V_β^-.
1. First note that if $n_k = (u_k, v_k)$ is such that u_k belongs to R_m, then the sequence $b_{-k-1}b_{-k-2}\cdots$ belongs to V_β^-, because edges are shift edges (there is no carry propagating to the right).
2. Suppose that there exists a minimal k such that u_k does not belong to R_m and $\mu_\beta(u_k) \leq 1-\beta^{-m}$, and the output sequence $b_{-k-1}\cdots b_{-k-m}$ is not admissible. Then it follows that $\mu_\beta(u_k a_{-m-k-1}\cdots a_{-2m-k}) > 1$, which is impossible.
3. Suppose that there exists a minimal k such that u_k does not belong to R_m and $1-\beta^{-m} < \mu_\beta(u_k) < 1$, and the output sequence $b_{-k-1}\cdots b_{-k-m}$ is not admissible. Since $\mu_\beta(u_k) > 1-\beta^{-m}$, the β-expansion of u_k is of the form $t_1\cdots t_{m-1}(t_m-1)w$ where w is not 0. Thus there exists $z_1\cdots z_m$ in R_m such that $\mu_\beta(u_k z_1\cdots z_m) > 1$. The existence of a minimal j, $1 \leq j \leq m$, such that

$\mu_\beta(u_k z_1 \cdots z_j) > 1$ is a contradiction since, in the automaton $\mathcal{N}$, no path starting from n_k with input $z_1 \cdots z_j$ can exist. $\square$

Example 1. Take $\beta = \frac{1+\sqrt{5}}{2}$ the golden ratio. Then $A_\beta = \{0, 1\}$, $d_\beta(1) = 11$ and the factor 11 is forbidden. The on-line finite odometer with delay 2 associated with the golden ratio is shown on Figure 1, Figure 2 and Figure 3 below. A state of the form (u, v) is represented as $\genfrac{}{}{0pt}{}{u}{v}$. On Figure 2 and Figure 3 initial states of $\mathcal{P}$ and $\mathcal{N}$ are indicated by an ingoing arrow. The signed digit -1 is denoted by $\bar{1}$.

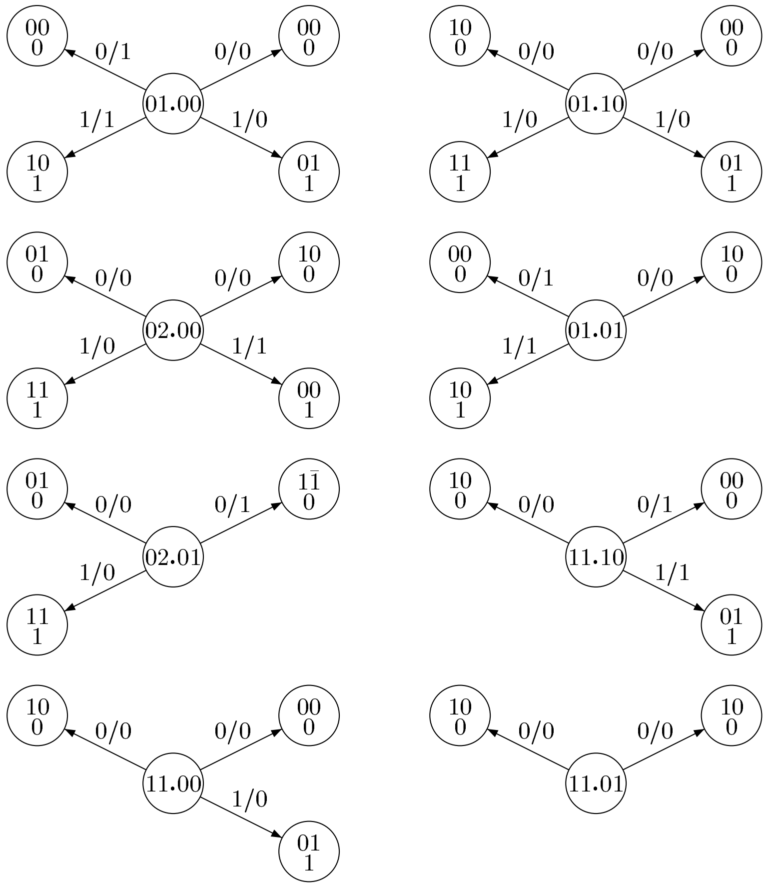

Fig. 1. Central part of the two-sided odometer with delay 2 associated with the golden ratio

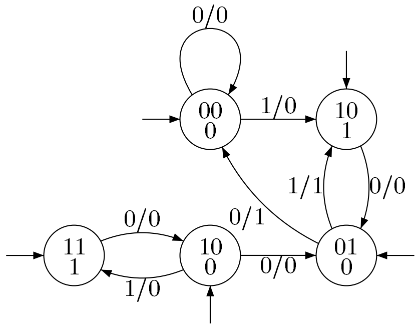

Fig. 2. Right on-line finite automaton $\mathcal{P}$ with delay 2 for the two-sided odometer associated with the golden ratio

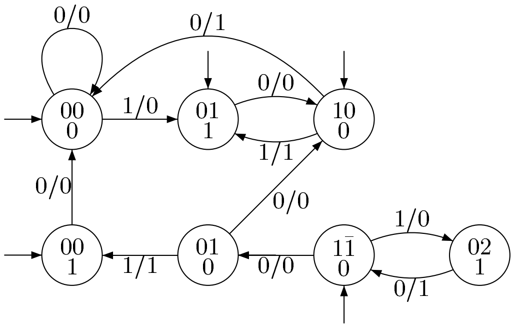

Fig. 3. Left on-line finite automaton $\mathcal{N}$ with delay 2 for the two-sided odometer associated with the golden ratio

References

1. A. Avizienis, Signed-digit number representations for fast parallel arithmetic. *IRE Transactions on electronic computers* **10** (1961), 389–400.
2. A. Bertrand-Mathis, Comment écrire les nombres entiers dans une base qui n'est pas entière. *Acta Math. Acad. Sci. Hungar.* **54** (1989), 237–241.
3. A. Brauer, On algebraic equations with all but one root in the interior of the unit circle. *Math. Nachr.* **4** (1951), 250-257.
4. C.Y. Chow and J.E. Robertson, Logical design of a redundant binary adder. *Proc. 4th Symposium on Computer Arithmetic*, I.E.E.E. Computer Society Press (1978), 109–115.
5. S. Eilenberg, *Automata, Languages and Machines*, vol. A, Academic Press, 1974.
6. Ch. Frougny, Representation of numbers and finite automata. *Math. Systems Theory* **25** (1992), 37–60.
7. Ch. Frougny, On the sequentiality of the successor function. *Inform. and Computation* **139** (1997), 17–38.
8. Ch. Frougny, On-line digit set conversion in real base. *Theor. Comp. Sci.* **292** (2002), 221–235.

9. Ch. Frougny, On-line odometers, manuscript.
10. Ch. Frougny and J. Sakarovitch, Synchronisation déterministe des automates à délai borné. *Theor. Comp. Sci.* **191** (1998), 61–77.
11. Ch. Frougny and B. Solomyak, Finite beta-expansions. *Ergodic Theory & Dynamical Systems* **12** (1992), 713–723.
12. P. Grabner, P. Liardet, R. Tichy, Odometers and systems of numeration. *Acta Arithmetica* **LXXX.2** (1995), 103–123.
13. S. Ito and Y. Sano, Substitutions, atomic surfaces, and periodic beta expansions. *Analytic Number Theory*, C. Jia and K. Matsumoto eds., Developments in Mathematics, Volume 6, 2002.
14. D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, 1995.
15. M. Lothaire, *Algebraic Combinatorics on Words*, Cambridge University Press, 2002.
16. J.-M. Muller, Some characterizations of functions computable in on-line arithmetic. *I.E.E.E. Trans. on Computers*, **43** (1994), 752–755.
17. W. Parry, On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.* **11** (1960), 401–416.
18. A. Rényi, Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar.* **8** (1957), 477–493.
19. K. Schmidt, Algebraic coding of expansive group automorphisms and two-sided beta-shifts. *Monatsh. Math.* **129** (2000), 37–61.
20. N. Sidorov and A. Vershik, Ergodic properties of the Erdős measure, the entropy of the golden shift, and related problems. *Monatsh. Math.* **126** (1998), 215–261.