k-block parallel addition versus 1-block parallel addition in non-standard numeration systems

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Abstract

Parallel addition in integer base is used for speeding up multiplication and division algorithms. k-block parallel addition has been introduced by Kornerup in [14]: instead of manipulating single digits, one works with blocks of fixed length k. The aim of this paper is to investigate how such notion influences the relationship between the base and the cardinality of the alphabet allowing block parallel addition. In this paper, we mainly focus on a certain class of real bases — the so-called Parry numbers. We give lower bounds on the cardinality of alphabets of non-negative integer digits allowing block parallel addition. By considering quadratic Pisot bases, we are able to show that these bounds cannot be improved in general and we give explicit parallel algorithms for addition in these cases. We also consider the d-bonacci base, which satisfies the equation $X^d = X^{d-1} + X^{d-2} + \cdots + X + 1$. If in a base being a d-bonacci number 1-block parallel addition is possible on an alphabet $\mathcal{A}$, then $\#\mathcal{A} \geqslant d+1$; on the other hand, there exists a $k \in \mathbb{N}$ such that k-block parallel addition in this base is possible on the alphabet $\{0,1,2\}$, which cannot be reduced. In particular, addition in the Tribonacci base is 14-block parallel on alphabet $\{0,1,2\}$.

Keywords: Numeration system, addition, parallel algorithm.

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1. Introduction

This work is a continuation of our two papers [9] and [10] devoted to the study of parallel addition. Suppose that two numbers x and y are given by their expansion $x = \bullet x_1 x_2 \cdots$ and $y = \bullet y_1 y_2 \cdots$ in a given base β, and the digits x_j's and y_j's are elements of a digit set $\mathcal{A}$. A parallel algorithm to compute their sum $z = x + y = \bullet z_1 z_2 \cdots$ with $z_j \in \mathcal{A}$ exists when each digit z_j can be determined by the examining a window of fixed length around the digit $(x_j + y_j)$. This avoids carry propagation.

Parallel addition has received a lot of attention, because the complexity of the addition of two numbers becomes constant, and so it is used for internal addition in multiplication

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and division algorithms, see [8] for instance.

A parallel algorithm for addition has been given by Avizienis [2] in 1961; there, numbers are represented in base $\beta = 10$ with digits from the set $\mathcal{A} = \{-6, -5, \ldots, 5, 6\}$. This algorithm has been generalized to any integer base $\beta \geqslant 3$. The case $\beta = 2$ and alphabet $\mathcal{A} = \{-1, 0, 1\}$ has been elaborated by Chow and Robertson [7] in 1978. It is known that the cardinality of an alphabet allowing parallel addition in integer base $\beta \geqslant 2$ must be at least equal to $\beta + 1$.

We consider non-standard numeration systems, where the base is a real or complex number β such that $|\beta| > 1$, and the digit set $\mathcal{A}$ is a finite alphabet of contiguous integer digits containing 0. If parallel addition in base β is possible on $\mathcal{A}$, then β must be an algebraic number.

In [9], we have shown that if β is an algebraic number, $|\beta| > 1$, such that all its conjugates in modulus differ from 1, then there exists a digit set $\mathcal{A} \subset \mathbb{Z}$ such that addition on $\mathcal{A}$ can be performed in parallel. The proof gives a method for finding a suitable alphabet $\mathcal{A}$ and provides an algorithm — a generalization of Avizienis' algorithm — for parallel addition on this alphabet. But the obtained digit set $\mathcal{A}$ is in general quite large, so in [10] we have given lower bounds on the cardinality of minimal alphabets (of contiguous integers containing 0) allowing parallel addition for a given base β.

In [14] Kornerup has proposed a more general concept of parallel addition. Instead of manipulating single digits, one works with blocks of fixed length k. So, in this terminology, the "classical" parallel addition is just k-block parallel addition with $k = 1$, and all the results recalled above actually concern 1-block addition.

The aim of this article is to investigate how Kornerup's generalization influences the relationship between the base and the alphabet for block parallel addition, in the hope of reducing the size of the alphabet. For instance, consider the Penney numeration system with the complex base $\beta = \imath - 1$, see [19]. We know from [10] that 1-block parallel addition in base $\imath - 1$ requires an alphabet of cardinality at least 5, whereas Herreros in [13] gives an algorithm for 4-block parallel addition on the alphabet $\mathcal{A} = \{-1, 0, 1\}$.

The paper is organized as follows. Definitions and previous results are recalled in Section 2. In Section 3 we show that for an algebraic base with a conjugate of modulus 1, block parallel addition is never possible, Theorem 3.1.

Then we consider bases $\beta > 1$ whose Rényi expansion of unity $d_\beta(1) = t_1 t_2 t_3 \cdots$ is eventually periodic, i.e., β is a so called *Parry number*. Assuming that the coefficients t_i's satisfy certain conditions and that block parallel addition is possible in base β on alphabet $\mathcal{A} = \{0, 1, \ldots, M\}$, we deduce two lower bounds on M, see Theorem 3.5 and Theorem 3.12.

A *Pisot number* is an algebraic integer larger than 1 such that all its Galois conjugates have modulus smaller than 1. It is known that Pisot numbers are Parry numbers, see the survey [11] for instance. By considering quadratic Pisot bases, we are able to show that the two previously mentioned (lower) bounds for Parry numbers cannot be improved in general. We give explicit (1-block) parallel algorithms for addition in these two cases (simple quadratic Parry numbers, and non-simple quadratic Parry numbers).

The main result of Section 4 is Theorem 4.1, which implies that there are many bases for which Kornerup's concept of block parallel addition reduces substantially the size of the alphabet.

A number $\beta > 1$ is said to *satisfy the (PF) Property* if the sum of any two positive numbers with finite greedy β-expansion in base β has its greedy β-expansion finite as well. We deduce that if $\beta > 1$ satisfies the (PF) Property, then there exists a $k \in \mathbb{N}$ such that k-block parallel addition is possible on the alphabet $\mathcal{A} = \{0, 1, \ldots, 2\lfloor\beta\rfloor\}$.

We then consider a class of well studied Pisot numbers, that generalize the golden mean $\frac{1+\sqrt{5}}{2}$. Let d be in $\mathbb{N}$, $d \geqslant 2$. The real root $\beta > 1$ of the equation $X^d = X^{d-1} + X^{d-2} + \cdots + X + 1$ is said to be the *d-bonacci number*. These numbers satisfy the (PF) Property. If, in base a d-bonacci number, 1-block parallel addition is possible on the alphabet $\mathcal{A}$, then $\#\mathcal{A} \geqslant d+1$. Moreover, there exists some $k \in \mathbb{N}$ such that k-block parallel addition is possible on the alphabet $\mathcal{A} = \{0, 1, 2\}$, and this alphabet cannot be further reduced. In particular, addition in the Tribonacci base is 14-block parallel on $\mathcal{A} = \{0, 1, 2\}$.

Part of our results concerns only non-negative alphabets. The reason is simple. For non-negative alphabet a strong tool — namely the greedy expansions of numbers — can be applied when proving theorems. That is why we recall some properties of the greedy expansions in Section 2.1.

2. Preliminaries

2.1. Numeration systems

For a detailed presentation of these topics, the reader may consult [11].

A *positional numeration system* $(\beta, \mathcal{A})$ within the complex field $\mathbb{C}$ is defined by a *base* β, which is a complex number such that $|\beta| > 1$, and a *digit set* $\mathcal{A}$ usually called the *alphabet*, which is a subset of $\mathbb{C}$. In what follows, $\mathcal{A}$ is finite and contains 0. If a complex number x can be expressed in the form $\sum_{-\infty \leqslant j \leqslant n} x_j \beta^j$ with coefficients x_j in $\mathcal{A}$, we call the sequence $(x_j)_{-\infty \leqslant j \leqslant n}$ a *$(\beta, \mathcal{A})$-representation* of x and note $x = x_n x_{n-1} \cdots x_0 {\scriptstyle\bullet} x_{-1} x_{-2} \cdots$. If a *$(\beta, \mathcal{A})$-representation* of x has only finitely many non-zero entries, we say that it is *finite* and the trailing zeroes are omitted.

In analogy with the classical algorithms for arithmetical operations, we work only on the set of numbers with *finite* representations, i.e., on the set

$$\mathrm{Fin}_{\mathcal{A}}(\beta) = \Big\{ \sum_{j \in I} x_j \beta^j \mid I \subset \mathbb{Z}, \quad I \ \text{ finite}, \quad x_j \in \mathcal{A} \Big\}.$$

Such a finite sequence $(x_j)_{j \in I}$ of elements of $\mathcal{A}$ is identified with a bi-infinite string $(x_j)_{j \in \mathbb{Z}}$ in $\mathcal{A}^{\mathbb{Z}}$, where only a finite number of digits x_j have non-zero values.

The best-understood case is the one of representations of real numbers in a base $\beta > 1$, the so-called *greedy expansions*, introduced by Rényi [20]. Every number $x \in [0, 1]$ can be given a β-expansion by the following *greedy algorithm*:

$$r_0 := x; \ \text{ for } j \geqslant 1 \text{ put } x_j := \lfloor \beta r_{j-1} \rfloor \text{ and } r_j := \beta r_{j-1} - x_j.$$

Then $x = \sum_{j \geqslant 1} x_j \beta^{-j}$, and the digits x_j are elements of the so-called *canonical alphabet* $\mathcal{C}_\beta = \{j \in \mathbb{Z} \,|\, 0 \leqslant j < \beta\}$. For $x \in [0, 1)$, the sequence $(x_j)_{j \geqslant 1}$ is said to be the *Rényi expansion* or the *β-greedy expansion* of x.

The greedy algorithm applied to the number 1 gives the β-expansion of 1, denoted by $d_\beta(1) = (t_j)_{j \geqslant 1}$, which plays a special role in this theory. We define also the *quasi-greedy expansion* $d^*_\beta(1) = (t^*_j)_{j \geqslant 1}$ by: if $d_\beta(1) = t_1 \cdots t_m$ is finite, then $d^*_\beta(1) = (t_1 \cdots t_{m-1}(t_m - 1))^\omega$, otherwise $d^*_\beta(1) = d_\beta(1)$. A number $\beta > 1$ such that $d_\beta(1)$ is eventually periodic, that is to say, of the form $t_1 \cdots t_m (t_{m+1} \cdots t_{m+p})^\omega$ is called a *Parry number.* If $d_\beta(1)$ is finite, $d_\beta(1) = t_1 \cdots t_m$, then β is a *simple* Parry number.

Some numbers have more than one $(\beta, \mathcal{C}_\beta)$-representation. The greedy expansion of x is lexicographically the greatest among all $(\beta, \mathcal{C}_\beta)$-representations of x.

A sequence $(x_j)_{j \geqslant 1}$ is said to be *β-admissible* if it is the greedy expansion of some $x \in [0, 1)$. Let us stress that not all sequences over the alphabet $\mathcal{C}_\beta$ are β-admissible. Parry in [18] used the *quasi-greedy expansion* $d^*_\beta(1) = (t^*_j)_{j \geqslant 1}$ for characterization of β-admissible sequences: Let $s = (s_j)_{j \geqslant 1} = s_1 s_2 s_3 \cdots$ be an infinite sequence of non-negative integers. The sequence s is β-admissible if and only if for all $i \geqslant 1$ the inequality $s_i s_{i+1} \cdots \prec_{lex} d^*_\beta(1)$ holds in the lexicographic order.

A $(\beta, \mathcal{C}_\beta)$-representation $x_n x_{n-1} \ldots x_0 \bullet x_{-1} x_{-2} \cdots$ of a number $x \geqslant 1$ is called the β-greedy expansion of x, if the sequence $x_n x_{n-1} \ldots x_0 x_{-1} x_{-2} \cdots$ is β-admissible.

Some real bases introduced in [12] have a property which is interesting in connection with parallel addition. A number $\beta > 1$ is said to *satisfy the (PF) Property* if the sum of any two positive numbers with finite greedy β-expansions in base β has a greedy β-expansion which is finite as well, that is to say, every element of $\mathbb{N}[\beta^{-1}] \cap [0, 1)$ has a finite greedy β-expansion. A number $\beta > 1$ is said to *satisfy the (F) Property* if every element of $\mathbb{Z}[\beta^{-1}] \cap [0, 1)$ has a finite greedy β-expansion. Of course, the (F) Property implies the (PF) Property.

If $\beta > 1$ has the (PF) Property, then β is a Pisot number, but there exist also Pisot numbers not satisfying the (PF) Property.

In [12], two classes of Pisot numbers with the (PF) Property are presented:

- β has the (F) Property, and thus the (PF) Property as well, if $d_\beta(1) = t_1 t_2 \cdots t_m$ and $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_m \geqslant 1$.

- β has the (PF) Property if $d_\beta(1) = t_1 t_2 \cdots t_m t^\omega$ and $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_m > t \geqslant 1$.

In fact, Akiyama in [1] shows that if β has the (PF) Property but not the (F) Property, then necessarily $d_\beta(1) = t_1 t_2 \cdots t_m t^\omega$ and $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_m > t \geqslant 1$. Let us note that every quadratic Pisot number satisfies the (PF) Property.

2.2. Parallel addition

Let us first formalize the notion of parallel addition as it is considered in most works concentrated on this topic, including our recent papers.

Definition 2.1. A function $\varphi : \mathcal{A}^\mathbb{Z} \to \mathcal{B}^\mathbb{Z}$ is said to be *p-local* if there exist two non-negative integers r and t satisfying $p = r + t + 1$, and a function $\Phi : \mathcal{A}^p \to \mathcal{B}$ such that, for any $u = (u_j)_{j\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ and its image $v = \varphi(u) = (v_j)_{j\in\mathbb{Z}} \in \mathcal{B}^{\mathbb{Z}}$, we have $v_j = \Phi(u_{j+t} \cdots u_{j-r})$ for every j in $\mathbb{Z}$.

This means that the image of u by φ is obtained through a sliding window of length p. The parameter r is called the *memory* and the parameter t is called the *anticipation* of the function φ. Such functions, restricted to finite sequences, are computable by a parallel algorithm in constant time.

Definition 2.2. Given a base β with $|\beta| > 1$ and two alphabets $\mathcal{A}$ and $\mathcal{B}$ of contiguous integers containing 0, a *digit set conversion* in base β from $\mathcal{A}$ to $\mathcal{B}$ is a function $\varphi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ such that

1. for any $u = (u_j)_{j\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ with a finite number of non-zero digits, the image $v = (v_j)_{j\in\mathbb{Z}} = \varphi(u) \in \mathcal{B}^{\mathbb{Z}}$ has only a finite number of non-zero digits as well, and
2. $\sum_{j\in\mathbb{Z}} v_j\beta^j = \sum_{j\in\mathbb{Z}} u_j\beta^j$.

Such a conversion is said to be *computable in parallel* if it is a p-local function for some $p \in \mathbb{N}$.

Thus, addition in $\mathrm{Fin}_{\mathcal{A}}(\beta)$ is computable in parallel if there exists a digit set conversion in base β from $\mathcal{A} + \mathcal{A}$ to $\mathcal{A}$ which is computable in parallel.

Let us stress that all alphabets we use are composed of contiguous integers and contain 0. This restriction already forces the base β to be an algebraic number. In [9] we give a sufficient condition on β to allow parallel addition:

Theorem 2.3 ([9])**.** *Let β be an algebraic number such that $|\beta| > 1$ and all its conjugates in modulus differ from 1. Then there exists an alphabet $\mathcal{A}$ of contiguous integers containing 0 such that addition in $\mathrm{Fin}_{\mathcal{A}}(\beta)$ can be performed in parallel.*

The proof of the previous theorem gives a method for finding a suitable alphabet $\mathcal{A}$ and provides an algorithm for parallel addition on this alphabet. But, in general, the alphabet $\mathcal{A}$ obtained in this way is quite large. An exaggerated size of the alphabet does not allow to compare numbers by means of the lexicographic order on their $(\beta, \mathcal{A})$-representations. For instance, in base $\beta = 2$ and alphabet $\mathcal{A} = \{0, 1, 2\}$, we have $02 \prec_{lex} 10$ in the lexicographic order, but $x = 0{\bullet}02 \not< y = 0{\bullet}10$.

Therefore, in [10], we have studied the cardinality of minimal alphabets allowing parallel addition for a given base β. In particular, we have found the following lower bounds:

Theorem 2.4 ([10])**.** *Let β, with $|\beta| > 1$, be an algebraic integer of degree d with minimal polynomial f. Let $\mathcal{A}$ be an alphabet of contiguous integers containing 0 and 1. If addition in $\mathrm{Fin}_{\mathcal{A}}(\beta)$ is computable in parallel then $\#\mathcal{A} \geqslant |f(1)|$. If, moreover, β is a positive real number, $\beta > 1$, then $\#\mathcal{A} \geqslant |f(1)| + 2$.*

In [14], Kornerup suggested a more general concept of parallel addition. Instead of manipulating single digits, one works with blocks of digits with fixed block length k. For the precise description of the Kornerup's idea, we introduce the notation

$$\mathcal{A}_{(k)} = \{a_0 + a_1\beta + \cdots + a_{k-1}\beta^{k-1} \mid a_i \in \mathcal{A}\},$$

where $\mathcal{A}$ is an alphabet and k a positive integer. Clearly, $\mathcal{A}_{(1)} = \mathcal{A}$.

Definition 2.5. Given a base β with $|\beta| > 1$ and two alphabets $\mathcal{A}$ and $\mathcal{B}$ of contiguous integers containing 0, a *digit set conversion* in base β from $\mathcal{A}$ to $\mathcal{B}$ is said to be *block parallel computable* if there exists some $k \in \mathbb{N}$ such that the digit set conversion in base β^k from $\mathcal{A}_{(k)}$ to $\mathcal{B}_{(k)}$ is computable in parallel. When the specification of k is needed, we say *k-block parallel computable.*

In this terminology, the original parallel addition is 1-block parallel addition, and the results just recalled concern 1-block parallel addition.

Remark 2.6. Suppose that the base is an integer β with $|\beta| \geqslant 2$. It is known that 1-block parallel addition is possible on an alphabet of cardinality $\#\mathcal{A} = \beta + 1$ (see [17] and [10]). But k-block parallel addition on an alphabet $\mathcal{A}$ is just 1-block parallel addition in integer base β^k on $\mathcal{A}_{(k)}$. Thus k-block parallel addition in integer base β can only be possible on an alphabet $\mathcal{A}$ such that $\#\mathcal{A}_{(k)} \geqslant \beta^k + 1$. This shows that k-block parallel addition with $k \geqslant 2$ does not allow the use of any smaller alphabet than already achieved with $k = 1$.

The bound from Theorem 2.4 on the minimal cardinality of alphabet $\mathcal{A}$ cannot be applied to block parallel addition. This fact can be demonstrated on the Penney numeration system with the complex base $\beta = \imath - 1$. The minimal polynomial of this base is $f(X) = X^2 + 2X + 2$. From Theorem 2.4 we get that 1-block parallel addition in base $\imath - 1$ requires an alphabet of cardinality at least 5, whereas Herreros in [13] gave an algorithm for 4-block parallel addition on the alphabet $\{-1, 0, 1\}$. According to our up-to-now knowledge, the base $\beta = \imath - 1$ was the only known example where the Kornerup block approach reduced the size of the needed alphabet. In Corollary 4.6, we provide new explicit examples of bases for which this phenomenon occurs. And even more such new examples can be obtained by applying Corollaries 4.4 and 4.5.

3. Necessary conditions for existence of block parallel addition

3.1. General result

In [9] we have shown that the assumption that all the algebraic conjugates of β have modulus different from 1 enables 1-block parallel addition on $\mathrm{Fin}_{\mathcal{A}}(\beta)$ for some suitable alphabet $\mathcal{A} \subset \mathbb{Z}$. The following theorem shows that this assumption is also necessary and, even more, the generalization of parallelism via working with k-blocks does not change the situation.

Theorem 3.1. *Let the base $\beta \in \mathbb{C}$, $|\beta| > 1$, be an algebraic number with a conjugate γ of modulus $|\gamma| = 1$ and let $\mathcal{A} \subset \mathbb{Z}$ be an alphabet of contiguous integers containing* 0*. Then addition on $\mathcal{A}$ cannot be block parallel computable.*

Proof. Within the proof, we denote by $\Re(x)$ the real part of a complex number x. Let us assume that there exist $k, p \in \mathbb{N}$ such that $\Phi : (\mathcal{A}_{(k)} + \mathcal{A}_{(k)})^p \to \mathcal{A}_{(k)}$ performs k-block parallel addition on $\mathcal{A}$. Denote $S := \max\Big\{\Big|\sum_{j=0}^{pk-1} a_j \gamma^j\Big| : a_j \in \mathcal{A}\Big\}$. Since there exist infinitely many $j \in \mathbb{N}$ such that $\Re(\gamma^j) > \frac{1}{2}$, one can find $N > p$ and $\varepsilon_j \in \{0,1\}$ such that $\Re\Big(\sum\limits_{j=0}^{kN-1} \varepsilon_j \gamma^j\Big) > 2S$.

Let $T := \max\{\big|\Re\big(\sum_{j=0}^{kN-1} b_j\gamma^j\big)\big| \ : \ b_j \in \mathcal{A}\}$. Then find $x = \sum\limits_{j=0}^{kN-1} x_j\beta^j$ such that $|\Re(x')| = T$, where x' denotes the image of x under the field isomorphism $\mathbb{Q}(\beta) \to \mathbb{Q}(\gamma)$. The choice of N ensures $|\Re(x')| > 2S$. Adding $x + x$ by the k-block p-local function Φ, we get

$$x + x = \sum_{j=kN}^{k(N+p)-1} z_j\beta^j + \sum_{j=0}^{kN-1} z_j\beta^j + \sum_{j=-kp}^{-1} z_j\beta^j\,, \quad \text{with } z_j \in \mathcal{A}\,.$$

For the image of $x + x$ under the field isomorphism, we have

$$2T = |\Re(x' + x')| \leqslant |\gamma^{kN}|S + |\Re(x')| + |\gamma^{-kp}|S = 2S + |\Re(x')| = 2S + T < 2T,$$

which is a contradiction. $\square$

As a corollary of Theorems 2.3 and 3.1, we obtain the following result:

Theorem 3.2. *Let β be in $\mathbb{C}$, $|\beta| > 1$. Then there exists an alphabet $\mathcal{A}$ of contiguous integers containing 0 such that addition on $\mathcal{A}$ is block parallel computable if and only if β is an algebraic number with no conjugate of modulus 1.*

If it is the case, then there also exists an alphabet on which addition is 1-block parallel computable.

3.2. Positive real bases

Since the integer base case has been resolved in Remark 2.6, in the following we suppose that β is not an integer.

For positive bases β belonging to some classes of Parry numbers we deduce a lower bound on the size of the alphabet $\mathcal{A} \subset \mathbb{N}$ allowing block parallel addition. For a non-negative alphabet we utilize the well known properties of the greedy representations, which are in the lexicographic order the greatest ones among all representations.

At first we state a simple observation we will use in our later considerations.

Lemma 3.3. *Let $\beta > 1$ be a base and let $\mathcal{A} = \{0, 1, \ldots, M\}$ with $M \geqslant 1$ be an alphabet. Let $z = g_0{\bullet}g_1g_2\cdots$ be a $(\beta, \mathcal{A})$-representation of z and $n \geqslant 0$ be an integer such that for all $i \in \mathbb{N}$, $0 \leqslant i \leqslant n$ the inequality*

$$1{\bullet}g_{i+1}g_{i+2}g_{i+3}\cdots \geqslant 0{\bullet}M^\omega \tag{1}$$

holds true. Then any finite lexicographically smaller $(\beta, \mathcal{A})$-representation of z coincides with the original representation on the first $n + 1$ digits, i.e., it has the form $z = g_0{\bullet}g_1g_2\cdots g_nz_{n+1}z_{n+2}\cdots$.

Proof. Let $z = z_0{\bullet}z_1z_2\cdots z_nz_{n+1}z_{n+2}\cdots$ be a finite lexicographically smaller representation of z and i be the minimal index for which $z_i < g_i$. Then

$$0{\bullet}M^\omega > 0{\bullet}z_{i+1}z_{i+2}\cdots = (g_i - z_i){\bullet}g_{i+1}g_{i+2}\cdots \geqslant 1{\bullet}g_{i+1}g_{i+2}\cdots.$$

Since for $i \leqslant n$ the opposite inequality (1) holds, necessarily $i \geqslant n+1$. The choice of i implies that $z_j = g_j$ for all $j = 0,1,2,\ldots,n$, as was to show. □

For the quasi-greedy expansion $d^*_\beta(1) = t^*_1t^*_2t^*_3\cdots$ we denote

$$T_i = 0{\bullet}t^*_it^*_{i+1}t^*_{i+2}\cdots \quad \text{for} \quad i \geqslant 1, i \in \mathbb{N}.$$

Let us point out some properties which follow directly from the definition of T_i and will be used in the sequel.

1. $T_1 = 1$ and $0 < T_i \leqslant 1$ for any $i \in \mathbb{N}, i \geqslant 1$. If $\beta \notin \mathbb{N}$, then $T_2 < T_1 = 1$.
2. $\beta T_i = t^*_i + T_{i+1}$ for any $i \in \mathbb{N}, i \geqslant 1$.
3. A base $\beta > 1$ is a Parry number if and only if the set $\{T_i \,|\, i \in \mathbb{N}, i \geqslant 1\}$ is finite.
4. Let $\beta \notin \mathbb{N}$ be a Parry number and j be the smallest index such that $T_j = T_i$ for some $i < j$.
   If $i = 1$ then β is a simple Parry number. In this case, as usually, we denote $j = m$. We have
   $$d_\beta(1) = t_1t_2\cdots t_m0^\omega \quad \text{and} \quad d^*_\beta(1) = \big(t_1t_2\cdots t_{m-1}(t_m-1)\big)^\omega.$$
   If $i > 1$ then β is non-simple Parry. In this case, as is the custom, we denote $i = m$ and $j - i = p$. We have
   $$d_\beta(1) = d^*_\beta(1) = t_1t_2\cdots t_m(t_{m+1}\cdots t_{m+p})^\omega.$$

In the remaining part of this section, β is a Parry number. By *Per* we denote the periodic part of the quasi-greedy expansion $d^*_\beta(1)$, i.e.

$$Per = \begin{cases} t^*_{m+1}t^*_{m+2}\cdots t^*_{m+p} = t_{m+1}t_{m+2}\cdots t_{m+p} & \text{if } \; d^*_\beta(1) = d_\beta(1)\,; \\[2ex] t^*_1t^*_2\cdots t^*_m = t_1t_2\cdots t_{m-1}(t_m-1) & \text{if } \; d^*_\beta(1) \neq d_\beta(1)\,. \end{cases}$$

3.2.1. Non-simple Parry numbers

The main goal of this subsection is to find a good lower bound on the cardinality of the alphabet $\mathcal{A} \subset \mathbb{N}$ allowing parallel addition in base β. Such a bound is deduced in Theorem 3.5. Then we concentrate on non-simple quadratic bases β to demonstrate that in general the bound cannot be improved.

Proposition 3.4. Let β be a non-simple Parry number with $d_\beta(1) = t_1t_2t_3\cdots$ satisfying $t_1 > t_j$ for all $j \geqslant 2$. If block parallel addition in base β can be performed on alphabet $\mathcal{A} = \{0,1,\ldots,M\}$, then

$$0{\bullet}(M - t_1 + 1)^\omega \geqslant 1 - \max\{T_j \,|\, j \geqslant 2\}.$$

Proof. Let us assume the contrary, i.e.

$$0{\scriptstyle\bullet}M^\omega < 0{\scriptstyle\bullet}(t_1-1)^\omega + 1 - 0{\scriptstyle\bullet}t_jt_{j+1}t_{j+2}\cdots \quad \text{for all} \quad j = 2,3,4,\dots . \tag{2}$$

Since the set $\{T_j \mid j \geqslant 2\}$ is finite, there exists $h \in \mathbb{N}$ such that

$$0{\scriptstyle\bullet}M^\omega < 0{\scriptstyle\bullet}(t_1-1)^h + 1 - 0{\scriptstyle\bullet}t_jt_{j+1}t_{j+2}\cdots t_{j+h} \quad \text{for all } j = 2,3,4,\dots . \tag{3}$$

Consider $y = 0{\scriptstyle\bullet}(t_1-1)^n$ with $n > h$.

Statement 1: *Any representation of y in base β on $\mathcal{A}$ has the form*

$$y = 0{\scriptstyle\bullet}(t_1-1)^{n-h}y_{n-h+1}y_{n-h+2}\cdots .$$

Proof. The representation $0{\scriptstyle\bullet}(t_1-1)^n$ is greedy. According to Lemma 3.3, it is enough to verify that $1{\scriptstyle\bullet}(t_1-1)^{n-i} > 0{\scriptstyle\bullet}M^\omega$ for $i = 0,1,\dots n-h$. Thanks to (3), it is satisfied. □

Consider $z = 0{\scriptstyle\bullet}(M+1)(t_1-1)^{n-1}t_1$ with $n > h$.

Statement 2: *The greedy representation of z is*

$$1{\scriptstyle\bullet}(M+1-t_1)(t_1-1-t_2)(t_1-1-t_3)\cdots(t_1-1-t_n)z_{n+1}z_{n+2}\dots ,$$

where $z_{n+1}z_{n+2}\dots$ is the greedy representation of the number $0{\scriptstyle\bullet}t_1 - 0{\scriptstyle\bullet}t_{n+1}t_{n+2}\cdots$.

Proof. Because of the assumption $t_1 > t_j$, every digit in $(M+1-t_1)(t_1-1-t_2)(t_1-1-t_3)\cdots(t_1-1-t_n)$ is non-negative. According to (2), the fraction $\frac{M-t_1+1}{\beta-1} = 0{\scriptstyle\bullet}(M-t_1+1)^\omega$ is smaller than 1. Consequently, $M-t_1+1 < \beta-1 < t_1$. It means that every digit in $(M+1-t_1)(t_1-1-t_2)(t_1-1-t_3)\cdots(t_1-1-t_n)$ is strictly smaller than t_1. This already implies that $1{\scriptstyle\bullet}(M+1-t_1)(t_1-1-t_2)(t_1-1-t_3)\cdots(t_1-1-t_n)z_{n+1}z_{n+2}\dots$, is the greedy representation of a number. It is easy to check that it is the number z. □

Statement 3: *Any finite non-greedy representation of z and the greedy representation of z have a common prefix of length $n-h$.*

Proof. We again use Lemma 3.3. Let us check the assumption of the lemma for $i = 0,1,\dots,n-h$.

For $i = 0$, we have to check that $z \geqslant 0{\scriptstyle\bullet}M^\omega$. Since also $z = 0{\scriptstyle\bullet}(M+1)(t_1-1)^{n-1}t_1$, we have to verify $1{\scriptstyle\bullet}(t_1-1)^{n-1}t_1 \geqslant 0{\scriptstyle\bullet}M^\omega$. It follows from (3).

If $1 \leqslant i \leqslant n-h$, we have to check $1{\scriptstyle\bullet}(t_1-1-t_i)(t_1-1-t_{i+1})\cdots(t_1-1-t_n) \geqslant 0{\scriptstyle\bullet}M^\omega$. This inequality is a consequence of (3) as well. □

Now we can deduce the desired contradiction to the assumption of the existence of a k-block s-local function Φ performing parallel addition on the alphabet $\{0,1,\dots M\}$, where M satisfies (2). Statement 1 implies that necessarily $\Phi((t_1-1)^{ks}) = (t_1-1)^k$. This fact contradicts to Statement 2 and Statement 3. □

Theorem 3.5. *Let β be a non-simple Parry number with $d_\beta(1) = t_1t_2t_3\cdots$ satisfying $t_1 > t_j$ for all $j \geqslant 2$. If block parallel addition in base β can be performed on alphabet $\mathcal{A} = \{0, 1, \dots, M\}$, then*

$$M \geqslant 2t_1 - t - 1, \quad \textit{where} \quad t = \max\{t_j \,|\, j \geqslant 2\}\,.$$

Proof. Let ℓ be the index such that $T_\ell = \max\{T_j \,|\, j \geqslant 2\}$. Clearly $t = t_\ell$ and $T_i > 0$ for all $i \geqslant 1$. According to Proposition 3.4, we have $\frac{M-t_1+1}{\beta-1} \geqslant 1 - T_\ell$ or equivalently,

$$M - t_1 + 1 \geqslant (\beta - 1)(1 - T_\ell)\,. \tag{4}$$

We use twice – for $i = 1$ and for $i = \ell$ – the relation $\beta T_i = t_i + T_{i+1}$ and we rewrite the right side of the above inequality:

$$(\beta - 1)(1 - T_\ell) = t_1 - 1 + T_2 - t_\ell - T_{\ell+1} + T_\ell \geqslant t_1 - 1 + T_2 - t_\ell > t_1 - 1 - t_\ell\,.$$

This together with (4) gives $M - t_1 + 1 > t_1 - 1 - t_\ell$. ☐

We illustrate on the larger root β of the equation $X^2 = aX - b$, where $a, b \in \mathbb{N}, a \geqslant b + 2, b \geqslant 1$ that our bound on the cardinality of alphabet in Theorem 3.5 is sharp. The Rényi expansion of unity is $d_\beta(1) = (a-1)(a-b-1)^\omega$.

We show that the smallest possible alphabet $\mathcal{A} = \{0, \dots, a+b-2\}$ and the smallest possible size $k = 1$ of the block enable parallel addition by a k-block local function.

Proposition 3.6. Let $d_\beta(1) = (a-1)(a-b-1)^\omega$, where $a \geqslant b+2$, $b \geqslant 1$, be the Rényi expansion of 1 in base β. Parallel addition in base β is possible on the alphabet $\mathcal{A} = \{0, \dots, a+b-2\}$, namely by means of a 1-block local function.

By Proposition 18 in [10], it is enough to show that the greatest digit elimination from $\{0, \dots, a+b-1\}$ to $\{0, \dots, a+b-2\} = \mathcal{A}$ can be done in parallel:

Algorithm *GDE*($\beta^2 = a\beta - b$): Base $\beta > 1$ satisfying $\beta^2 = a\beta - b$, with $a \geqslant b+2$, $b \geqslant 1$, parallel conversion (greatest digit elimination) from $\{0, \dots, a+b-1\}$ to $\{0, \dots, a+b-2\} = \mathcal{A}$.

Input: a finite sequence of digits (z_j) from $\{0, \dots, a+b-1\}$, with $z = \sum_j z_j\beta^j$.
Output: a finite sequence of digits (x_j) from $\{0, \dots, a+b-2\}$, with $z = \sum_j x_j\beta^j$.

`for each` j `in parallel do`

$$1. \quad \texttt{case} \left\{ \begin{array}{l} z_j = a+b-1 \\ a-1 \leqslant z_j \leqslant a+b-2 \;\; \texttt{and} \; \big(z_{j+1} \geqslant a-1 \;\; \texttt{or} \; z_{j-1} \geqslant a-1\big) \\ z_j = a-2 \;\; \texttt{and} \; z_{j+1} = a+b-1 \;\; \texttt{and} \; z_{j-1} = a+b-1 \\ z_j = a-2 \;\; \texttt{and} \; z_{j+1} = a+b-1 \;\; \texttt{and} \; z_{j-1} \geqslant a-1 \;\; \texttt{and} \; z_{j-2} \geqslant a-1 \\ z_j = a-2 \;\; \texttt{and} \; z_{j-1} = a+b-1 \;\; \texttt{and} \; z_{j+1} \geqslant a-1 \;\; \texttt{and} \; z_{j+2} \geqslant a-1 \\ z_j = a-2 \;\; \texttt{and} \; z_{j\pm1} \geqslant a-1 \;\; \texttt{and} \; z_{j\pm2} \geqslant a-1 \end{array} \right\}$$

```
      then q_j := 1
    else q_j := 0
```

2. $x_j := z_j - aq_j + bq_{j+1} + q_{j-1}$

Proof. Let us denote $w_j := z_j - aq_j$; and remind that $q_j \in \{0,1\}$ for any j, and thus $bq_{j+1} + q_{j-1} \in \{0, 1, b, b+1\}$.

- If $z_j \in \{0, \dots, a-3\}$, then $x_j = z_j + bq_{j+1} + q_{j-1} \in \{0, \dots, a+b-2\} = \mathcal{A}$.
- If $z_j = a+b-1$, then $w_j = b-1$, thus $0 \leqslant x_j \leqslant 2b \leqslant a+b-2$ as $a \geqslant b+2$. Therefore $x_j \in \mathcal{A}$.
- When $a-1 \leqslant z_j \leqslant a+b-2$, and $z_{j-1} \geqslant a-1$ or $z_{j+1} \geqslant a-1$, then $-1 \leqslant w_j \leqslant b-2$ and $q_{j+1} + q_{j-1} \in \{1,2\}$. Thus $x_j \in \{0, \dots, 2b-1\} \subset \mathcal{A}$.
- When $a-1 \leqslant z_j \leqslant a+b-2$ and both its neighbours $z_{j\pm1} < a-1$, then $w_j = z_j$ and $q_{j+1} = q_{j-1} = 0$. Thus $x_j \in \mathcal{A}$.
- If $z_j = a-2$ and $q_j = 1$, then necessarily $q_{j\pm1} = 1$. Since $w_j = -2$, we get $x_j = b-1 \in \mathcal{A}$.
- If $z_j = a-2$ and $q_j = 0$, then $w_j = a-2$, and $q_{j-1} + q_{j+1} \in \{0,1\}$. Therefore, the resulting $x_j \in \{a-2, a-1, a+b-2\} \subset \mathcal{A}$.

Lastly, it is obvious that a string of zeroes is not converted into a string of non-zeroes by this algorithm, so all the necessary conditions of parallel addition are fulfilled. ☐

The previous algorithm acts on alphabet $\mathcal{A} \subset \mathbb{N}$. Looking for the letters $h \in \mathcal{A}$ such that the algorithm keeps unchanged the constant sequences $(h)_{j\in\mathbb{Z}}$ allows us to modify the alphabet of the algorithm:

Definition 3.7. Let $\mathcal{A}$ and $\mathcal{B}$ be two alphabets containing 0 such that $\mathcal{A} \cup \mathcal{B} \subset \mathbb{Z}[\beta]$. Let $\varphi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ be a s-local function realized by the function $\Phi : \mathcal{A}^p \to \mathcal{B}$. The *letter* h in $\mathcal{A}$ is said to be *fixed by* φ if $\varphi((h)_{j\in\mathbb{Z}}) = (h)_{j\in\mathbb{Z}}$, or, equivalently, $\Phi(h^s) = h$.

Proposition 3.8. Let β satisfy $\beta^2 = a\beta - b$, with $a \geqslant b+2$, $b \geqslant 1$. Parallel addition in base β is possible on any alphabet of cardinality $a+b-1$ of the form $\mathcal{A} = \{-d, \dots, a+b-2-d\}$ for $b \leqslant d \leqslant a-2$.

Proof. Every letter h, $0 \leqslant h \leqslant a-2$, is fixed by the above algorithm. So for $b \leqslant d \leqslant a-2$, both letters d and $a+b-2-d$ are fixed, and, by Corollary 24 in [10], parallel addition is possible on any alphabet of the form $\mathcal{A} = \{-d, \dots, a+b-2-d\}$ with $b \leqslant d \leqslant a-2$. ☐

It is an open question to prove that in base β satisfying $\beta^2 = a\beta - b$, with $a \geqslant b+2$, $b \geqslant 2$, parallel addition is not possible on alphabets of positive and negative contiguous integer digits not containing $\{-b, \dots, 0, \dots, b\}$, as it is the case in rational base $\beta = a/b$ when $b \geqslant 2$, see [10].

3.2.2. Parry numbers β with $d_\beta(1) \neq t_1 \cdots t_m t_{m+1}^\omega$, $t_{m+1} \neq 0$

For Parry numbers specified in the title of this subsection, we deduce in Theorem 3.12 a lower bound on the cardinality of the alphabet $\mathcal{A} \subset \mathbb{N}$ allowing parallel addition in base β. Proof of Theorem 3.12 is rather technical and we split it into several auxiliary claims. Then we illustrate on quadratic simple Parry bases that in general our bound is the best possible.

Lemma 3.9. *Let β be a Parry number and $\mathcal{A} = \{0, 1, \ldots, M\}$ be an alphabet where $M \in \mathbb{N}$ and*

$$0{\bullet}M^\omega \leqslant 1 + \min\{T_i \mid i \geqslant 2\}.$$

Then there exists a constant $h \in \mathbb{N}$ such that for any integer $n > h$ and any y satisfying $1 \leqslant y < 1 + \frac{1}{\beta^n}$ the following implication holds true: If $0{\bullet}y_1y_2\cdots y_n$ is a finite representation of y on $\mathcal{A}$, then the string $y_1y_2\cdots y_{n-h}$ is a prefix of $d^*_\beta(1)$ or β is simple Parry with $d_\beta(1) = t_1t_2\cdots t_m0^\omega$ and $y_1y_2\cdots y_{n-h}$ is a prefix of a string $(t_1^*t_2^*\cdots t_m^*)^j\, t_1^*t_2^*\cdots t_{m-1}^*(t_m^*+1)0^\omega$ for some $j \in \mathbb{N}$.

Proof. Denote $T_\ell = \min\{T_i \mid i \geqslant 2\}$. Consider $y = 0{\bullet}y_1y_2y_3\cdots y_n0^\omega$ such that $1 \leqslant y < 1 + \frac{1}{\beta^n}$. Let i be the smallest index, $1 \leqslant i \leqslant n$ such that $y_i \neq t_i^*$. The equality of strings $y_1y_2y_3\cdots y_n0^\omega$ and $d^*_\beta(1)$ is impossible, as $d^*_\beta(1)$ has infinitely many non-zero entries. We will discuss two cases: $y_1y_2y_3\cdots y_n0^\omega \prec d^*_\beta(1)$ and $y_1y_2y_3\cdots y_n0^\omega \succ d^*_\beta(1)$.

1) Let $y_1y_2y_3\cdots y_n0^\omega \prec d^*_\beta(1)$. Then $y = 0{\bullet}t_1^*\cdots t_{i-1}^*y_iy_{i+1}\cdots$ with $y_i \leqslant t_i^* - 1$ and

$$y < 0{\bullet}t_1^*\cdots t_{i-1}^*(t_i^*-1)M^\omega < 0{\bullet}t_1^*\cdots t_{i-1}^* + \frac{t_i^*-1}{\beta^i} + \frac{1}{\beta^i}(1+T_\ell) \leqslant 0{\bullet}t_1^*\cdots t_{i-1}^*t_i^*t_{i+1}^*\cdots = 1$$

– a contradiction with the assumption $y \geqslant 1$.

2) Let $y_1y_2y_3\cdots y_n0^\omega \succ d^*_\beta(1)$. Then $y = 0{\bullet}t_1^*\cdots t_{i-1}^*y_iy_{i+1}\cdots$ with $y_i = t_i^* + c$ where $c \in \mathbb{N}, c \geqslant 1$. Denote $\mu = \max\{T_j \mid j \geqslant 2 \text{ and } T_j < 1\}$. Set h to be the smallest integer such that $\beta^h(1-\mu) > 1$. Then

$$y = 0{\bullet}t_1^*\cdots t_{i-1}^*t_i^* + \frac{1}{\beta^i}(c + 0{\bullet}y_{i+1}y_{i+2}\ldots y_n) = 1 + \frac{1}{\beta^i}(c - T_{i+1}) + \frac{1}{\beta^i}(0{\bullet}y_{i+1}y_{i+2}\cdots y_n) \tag{5}$$

As $T_{i+1} \leqslant 1$, for $c \geqslant 2$, we have $y \geqslant 1 + \frac{1}{\beta^i}$. The assumptions $1 + \frac{1}{\beta^n} > y$ forces $i \geqslant n$. Hence it suffices to consider $c = 1$.

If $T_{i+1} < 1$ then $y \geqslant 1 + \frac{1}{\beta^i}(1-\mu) > 1 + \frac{1}{\beta^{i+h}}$. The assumption $1 + \frac{1}{\beta^n} > y$ implies $i > n - h$ as we want to show.

It remains to discuss the case $T_{i+1} = 1$. This means that β is simple Parry and $i = 0 \mod m$. The representation of y has the form $0{\bullet}\,(t_1^*t_2^*\cdots t_m^*)^j\, t_1^*t_2^*\cdots t_{m-1}^*(t_m^*+1)y_{i+1}y_{i+2}\cdots y_n$ for some $j \in \mathbb{N}$. To finish the proof we need to show that $y_k = 0$ for all $k \in \mathbb{N}, i < k \leqslant n - h$. Let $K \geqslant i + 1$ be the minimal index for which $y_K \geqslant 1$. Using (5) we deduce $y \geqslant 1 + \frac{y_K}{\beta^K}$. This inequality together with the assumption $1 + \frac{1}{\beta^n} > y$ implies $K > n$.

$\square$

Proposition 3.10. *Let β be a Parry number and let the shortest period of the quasi-greedy expansion $d^*_\beta(1)$ be longer than 1. If block parallel addition can be performed on alphabet $\mathcal{A} = \{0, 1, \ldots, M\}$, then*

$$M \geqslant (\beta - 1)\big(1 + \min\{T_i \mid i \geqslant 2\}\big)\,.$$

Proof. Let $\Phi : (\mathcal{A}_{(k)} + \mathcal{A}_{(k)})^s \to \mathcal{A}_{(k)}$ be the function performing k-block parallel addition on the alphabet $\mathcal{A} = \{0, 1, \ldots, M\}$. Let us suppose that the proposition does not hold. It means

$$0\bullet M^\omega < 1\bullet t^*_i t^*_{i+1} t^*_{i+2} \cdots \quad \text{for all } i = 2, 3, \ldots\,. \tag{6}$$

Since the set $\{T_i \,|\, i \geqslant 1\}$ is finite and $T_1 = 1 \geqslant T_i$ for any $i = 2, 3, \ldots$ there exists $H \in \mathbb{N}$ such that

$$0\bullet M^\omega < 1\bullet t^*_i t^*_{i+1} \cdots t^*_{i+H} \quad \text{for all} \;\; i = 1, 2, 3, \ldots\,. \tag{7}$$

Let us fix $n \in \mathbb{N}$ such that $n > H$ and $n > h$, where h is given by statement of Lemma 3.9. Consider the two numbers

$$z = 0\bullet t^*_1 t^*_2 t^*_3 \cdots t^*_n \qquad \text{and} \qquad y = 0\bullet(M + 1)t^*_2 t^*_3 \cdots t^*_{n-1} t^*_n (t^*_{n+1} + 1)\,.$$

If n is sufficiently large, then the above representations of y and z contain many repetitions of the string Per.

Statement 1: *Any finite representation of z in base β on $\mathcal{A} = \{0, 1, \ldots, M\}$ has the form $z = 0\bullet t^*_1 t^*_2 t^*_3 \cdots t^*_{n-s-1} z_{n-s} z_{n-s+1} \cdots$*

Proof. The representation $0\bullet t^*_1 t^*_2 t^*_3 \cdots t^*_n$ is the greedy representation of z. Thanks to (7), the indices $i = 1, 2, \ldots, n - s$ satisfy

$$0\bullet M^\omega < 1\bullet t^*_i t^*_{i+1} \cdots t^*_{i+s} \leqslant 1\bullet t^*_i t^*_{i+1} \cdots t^*_n\,.$$

Statement 1 follows by Lemma 3.3. $\square$

Statement 2: *The greedy representation of y has the form*

$$y = 1\bullet(M + 1 - t^*_1)0^n y_{n+2} y_{n+3} \cdots\,,$$

where $0\bullet y_{n+2} y_{n+3} \cdots$ is the greedy representation of the number $1 - T_{n+2}$.

Proof. It is easy to check that $1\bullet(M+1-t^*_1)0^n y_{n+2} y_{n+3} \cdots$ represents the number y. The inequality (6) implies $M \leqslant (1+T_i)(\beta-1) \leqslant 2(\beta-1) < 2t^*_1$. It gives $M - t^*_1 + 1 \leqslant t^*_1$ and thus the string $1(M + 1 - t^*_1)0^n y_{n+2} y_{n+3} \cdots$ fulfils the Parry condition. $\square$

Statement 3: *Any finite non-greedy representation of y in base β on $\{0, 1, \ldots, M\}$ has*

- *either the form $1\bullet(M + 1 - t^*_1)0^n \tilde{y}_{n+2} \tilde{y}_{n+3} \cdots$,*
- *or the form $1\bullet(M - t^*_1)x_1 x_2 x_3 \ldots$, where $x_1 x_2 x_3 \ldots x_{n+1-h}$ is a prefix of $d^*_\beta(1)$ or β is simple Parry and $x_1 x_2 x_3 \ldots x_{n+1-h}$ is a prefix of a string $(t^*_1 t^*_2 \cdots t^*_m)^j\, t^*_1 t^*_2 \cdots t^*_{m-1}(t^*_m + 1)0^\omega$ for some $j \in \mathbb{N}$.*

Proof. Any non-greedy representation of y is lexicographically smaller than the greedy one.

If a representation of y has the form $1\bullet(M+1-t_1^*)\tilde{y}_2\tilde{y}_3\cdots$ then necessarily $\tilde{y}_2 = \tilde{y}_3 = \cdots = \tilde{y}_{n+1} = 0$.

If a representation of y has the form $1\bullet(M-t_1^*)x_1x_2x_3\ldots$, due to Statement 2, $0\bullet x_1x_2x_3\ldots$ is a representation of the number $1+\frac{1-T_{n+2}}{\beta^n}$. Applying Lemma 3.9 we get Statement 3.

To finish the proof, we have to verify that no representation of y starts with $0\bullet\cdots$ and no representation of y starts with $1\bullet x\cdots$ where $x < M-t_1^*$. Both these facts follows from (6), in particular from $0\bullet M^\omega < 1+T_2$. $\square$

Let us now complete the proof of Proposition 3.10. By $|Per|$ we denote the length of the period Per and by q the length of the preperiod of $d_\beta^*(1)$. For all sufficiently large $n \in \mathbb{N}$, according to Statement 1, the sequence $0\bullet t_1^*t_2^*t_3^*\cdots t_n^*$ has to be rewritten by the local function Φ into the sequence with a long common prefix with. It means that the word Per must be rewritten by Φ into the same word Per and its occurrences start at the same positions (namely $q+i|Per|$ for $i \in \mathbb{N}$) after the fractional point in the original string as well as in the string rewritten by the function Φ. In particular, Per is not rewritten into $0^{|Per|}$.

Consider now the sequence $0\bullet(M+1)t_2^*t_3^*\cdots t_n^*(t_{n+1}^*+1)$. In this string the periodic part Per starts at the positions $q+mi$ for $i \in \mathbb{N}$, $i \geqslant 1$.

According to Statement 3, the periodic string Per has to be rewritten either as the string $0^{|Per|}$ or as the string Per. In the latter case, the string Per starts at the positions $1+q+i|Per|$. Since Per is not a power of a single letter, no such local function Φ can exist. $\square$

For almost all Parry bases β, the lower bound on M from the previous proposition shall serve us for deducing a good estimate on the cardinality of an alphabet allowing block parallel addition, see Theorem 3.12. The only exceptions are bases with $d_\beta(1) = t_1\cdots t_m t_{m+1}^\omega$, where $t_{m+1} \neq 0$ and $d_\beta(1) = t_1t_20^\omega$. In the former case, Proposition 3.10 gives no bound at all. In the latter case, $1+T_2 = 1+\frac{t_2}{\beta} = 0\bullet(t_1+t_2-1)^\omega$ and thus Proposition 3.10 gives the inequality $t_1+t_2-1 \leqslant M$ which is not the optimal one as shown in the next proposition.

Proposition 3.11. Let β be a Parry number with $d_\beta(1) = t_1t_20^\omega$. If block parallel addition is performable on the alphabet $\{0,1,\ldots,M\}$ then $M \geqslant t_1+t_2$.

Proof. We proceed by contradiction. Let there exist k and s in $\mathbb{N}$ such that k-block parallel addition be performable by an s-local function $\Phi : \mathcal{A}_{(k)}^s \mapsto \mathcal{A}_{(k)}$, where $\mathcal{A} = \{0,1,\ldots,M\}$ with $M = t_1+t_2-1$. Set $y = 0\bullet(t_1+t_2-1)^n(t_1+t_2)t_2$. It easy to see that if β differs from the golden ratio, the representation $y = 1\bullet t_20^\omega$ is greedy. If β is the golden ratio then $y = 10\bullet0^\omega$ is the greedy representation of y.

The digit $\sum_{j=0}^{k-1} M\beta^j$ is the biggest element and 0 is the smallest element of $\mathcal{A}_{(k)}$. According to Claims 13 and 14 in [10], the function Φ assigns to the string containing only repetitions of the biggest digit neither the biggest digit nor the smallest digit 0, i.e., $\Phi(M^{sk}) \neq M^k$ and $\Phi(M^{sk}) \neq 0^k$. In particular, the string representing y must be rewritten as a non-greedy $(\beta, \mathcal{A})$-representation.

Since $2 > 0{\bullet}M^\omega = 1 + \frac{t_2}{\beta} = y$ any finite non-greedy $(\beta, \mathcal{A})$-representations of y has the form $y = 1{\bullet}(t_2-1)y_2y_3\ldots y_N$ for some $N \in \mathbb{N}$. Simultaneously, $y = 1{\bullet}t_2 0^\omega$. It means that $1{\bullet} = 0{\bullet}y_2y_3\ldots y_N$. By Lemma 3.9 we get that $y_2y_3\ldots y_N$ is a prefix of $\big(t_1(t_2-1)\big)^j t_1t_2 0^\omega$ for some $j \in \mathbb{N}$. The representation of y gained by the block parallel algorithm has the form $y = 1{\bullet}(t_2-1)t_1(t_2-1)t_1(t_2-1)t_1\cdots$. In particular, the length k of blocks must be even and $\Phi(M^{ks}) = \big((t_2-1)t_1\big)^{\frac{k}{2}}$.

On the other hand, if we set $z = \beta y = (t_1+t_2-1){\bullet}(t_1+t_2-1)^{n-1}t_2$, the same consideration leads to the only possible form of z after applying parallel addition algorithm, namely $z = 1(t_2-1){\bullet}t_1(t_2-1)t_1(t_2-1)t_1\cdots$. Consequently, $\Phi(M^{ks}) = \big(t_1(t_2-1)\big)^{\frac{k}{2}}$ – a contradiction. $\square$

Theorem 3.12. *Let β be a Parry number and $d_\beta(1) \neq t_1t_2\cdots t_m t_{m+1}^\omega$, where $t_{m+1} \neq 0$. Let us denote*

$$t = \begin{cases} \min\{t_2, t_3, \ldots, t_m\} & \text{if } d_\beta(1) = t_1t_2\cdots t_m 0^\omega\,, \\ \min\{t_2, t_3, \ldots, t_{m+p}\} & \text{if } d_\beta(1) = t_1t_2\cdots t_m (t_{m+1}\cdots t_{m+p})^\omega\,. \end{cases}$$

If block parallel addition can be performed on alphabet $\mathcal{A} = \{0, 1, \ldots, M\}$, then $M \geqslant t_1 + t$.

Proof. We assume $d_\beta(1) \neq t_1t_2 0^\omega$, as the case $d_\beta(1) = t_1t_2 0^\omega$ is treated in Proposition 3.11. If $t = 0$, then the bound $M \geqslant t_1$ is trivial (otherwise the set $\mathrm{Fin}_{\mathcal{A}}(\beta)$ is not closed under addition). Let us suppose that $t \geqslant 1$. Let ℓ be the smallest index where $\min\limits_{j\geqslant 2} T_j$ is reached. Using Proposition 3.10, we have $M \geqslant (\beta-1)(1+T_\ell)$. Clearly $T_\ell < 1$ and $t_\ell^* \leqslant t_j^*$ for all $j \geqslant 2$. Let us realize that

$$(\beta-1)(1+T_\ell) = \beta - 1 + \beta T_\ell - T_\ell = t_1 - 1 + T_2 + t_\ell^* + T_{\ell+1} - T_\ell\,. \tag{8}$$

If $\ell = 2$ then $T_3 \neq 1$ (otherwise $d_\beta(1) = t_1t_2 0^\omega$). In this case, $t_\ell^* = t_2^* = t_2 = t$ and by (8) we get

$$M \geqslant (\beta-1)(1+T_\ell) \geqslant t_1 - 1 + t_2 + T_{\ell+1} > t_1 - 1 + t\,.$$

If $T_2 > T_\ell$ and $T_{\ell+1} \neq 1$, then $t_\ell^* = t_\ell = t$ and

$$M \geqslant (\beta-1)(1+T_\ell) > t_1 - 1 + t_\ell = t_1 - 1 + t\,.$$

If $T_2 > T_\ell$ and $T_{\ell+1} = 1$, then β is a simple Parry number, $\ell = m$ and $t_\ell^* = t_m^* = t_m - 1$. The inequality $T_i \neq T_j$ for $1 \leqslant j < i \leqslant m$, gives $T_m < T_j$. Thus

$T_m = 0{\bullet}t_m^*\big(t_1^*\cdots t_m^*\big)^\omega < T_j = 0{\bullet}t_j^*\cdots t_m^*\big(t_1^*\cdots t_m^*\big)^\omega$ for $1 < j < m$ implies $t_m^* < t_j^* = t_j$ and therefore $t = t_m \leqslant t_j$. Again by (8), we have

$$M \geqslant (\beta-1)(1+T_\ell) > t_1 - 1 + t_m - 1 + T_{\ell+1} = t_1 - 1 + t\,.$$

All cases lead to the inequality $M > t_1 - 1 + t$. As M is an integer, we can write $M \geqslant t_1 + t\,.$ $\square$

We will illustrate that the lower bound on the cardinality of the alphabet in Theorem 3.12 is sharp, i.e. can be attained, in quadratic cases. In order to do so, we exploit the positive root of the equation $X^2 = aX + b$.

Let β be the root > 1 of the polynomial $X^2 - aX - b$ with a and b integers, $a \geqslant b \geqslant 1$. Then $d_\beta(1) = ab$. We first recall the case $b = 1$.

Proposition 3.13 ([10])**.** Let β satisfy $\beta^2 = a\beta + 1$ with $a \in \mathbb{N}$, $a \geqslant 1$. Then 1-block parallel addition is possible on any alphabet of contiguous integers containing 0 with cardinality $a + 2$, and this cardinality is minimum.

We now consider the case $b \geqslant 2$. First we suppose that $a \geqslant b + 1$.

Proposition 3.14. Let β satisfy $\beta^2 = a\beta + b$, where $a \geqslant b + 1$ and $b \geqslant 2$. Then 1-block parallel addition in base β is possible on the alphabet $\mathcal{A} = \{0, \ldots, a + b\}$.

By Proposition 18 in [10], it is enough to show that the greatest digit elimination from $\{0, \ldots, a + b + 1\}$ to $\{0, \ldots, a + b\} = \mathcal{A}$ can be done in parallel:

Algorithm *GDE*($\beta^2 = a\beta + b$): Base $\beta > 1$ satisfying $\beta^2 = a\beta + b$, $a \geqslant b + 1$, $b \geqslant 2$, 1-block parallel conversion (greatest digit elimination) from $\{0, \ldots, a + b + 1\}$ to $\{0, \ldots, a + b\} = \mathcal{A}$.

Input: a finite sequence of digits (z_j) from $\{0, \ldots, a + b + 1\}$, with $z = \sum_j z_j \beta^j$.
Output: a finite sequence of digits (x_j) from $\{0, \ldots, a + b\}$, with $z = \sum_j x_j \beta^j$.

`for each` j `in parallel do`

$$
\text{1.}\quad \texttt{case} \left\{ \begin{array}{l}
z_j = a + b + 1 \ \texttt{ and } z_{j+1} \leqslant a + b \\
z_j = a + b + 1 \ \texttt{ and } z_{j+1} = a + b + 1 \texttt{ and } z_{j+2} \geqslant a \\
z_j = a + b \ \texttt{ and } z_{j+1} \leqslant a - 1 \\
z_j = a + b \ \texttt{ and } a \leqslant z_{j+1} \leqslant a + b \texttt{ and } z_{j-1} \geqslant a \\
z_j = a + b \ \texttt{ and } z_{j+1} = a + b + 1 \texttt{ and } z_{j+2} \geqslant a \texttt{ and } z_{j-1} \geqslant a \\
a + 1 \leqslant z_j \leqslant a + b - 1 \ \texttt{ and } z_{j+1} \leqslant a - 1 \\
z_j = a \ \texttt{ and } z_{j+1} \leqslant a - 1 \texttt{ and } z_{j-1} \geqslant a
\end{array} \right\} \texttt{then } q_j := 1
$$

$\quad$ `if` $\quad z_j \leqslant b - 1$ `and` $z_{j+1} \geqslant a + 1$ `then` $q_j := -1$

$\quad$ `else` $q_j := 0$

2. $x_j := z_j - aq_j - bq_{j+1} + q_{j-1}$

Proof. The formula defining the value x_j in Step 2. of the above algorithm guarantees that the new string (x_j) represents the number z as well. It is also obvious that a string of zeroes cannot be converted by the local function in this algorithm into a string of non-zeroes.

It remains to show that the new digits x_j belong to the alphabet $\mathcal{A}$. For that purpose, let us denote $w_j := z_j - aq_j$, and inspect all the possible combinations of $(z_{j+2}, z_{j+1}, z_j, z_{j-1})$ which can occur:

- $z_j = a + b + 1$

- For $z_{j+1} \leqslant a-1$, we set $q_j := 1$, and obtain $w_j = b+1$. Both q_{j+1} and q_{j-1} are from $\{-1, 0\}$, so $b \leqslant x_j \leqslant 2b+1$ and $x_j \in \mathcal{A}$, as $b+1 \leqslant a$.
- If $z_{j+1} \in \{a, \ldots, a+b\}$, or if $z_{j+1} = a+b+1$ and $z_{j+2} \geqslant a$, then q_{j+1} is limited to $\{0, 1\}$, and we set $q_j := 1$. As a result, $w_j = b+1$ and $0 \leqslant x_j \leqslant b+2$ is in $\mathcal{A}$, as $2 \leqslant b < a$.
- When $z_{j+1} = a+b+1$ and $z_{j+2} \leqslant a-1$, then $q_{j+1} = 1$. Putting $q_j := 0$, we have $w_j = a+b+1$, and the digit $x_j \in \{a, a+1, a+2\} \subset \mathcal{A}$, as $2 \leqslant b$.

• $z_j = a+b$

- For $z_{j+1} \leqslant a-1$, then $q_{j+1} \in \{-1, 0\}$, and we set $q_j := 1$, so $w_j = b$. Thus, $b-1 \leqslant x_j \leqslant 2b+1$ and $x_j \in \mathcal{A}$, since $2 \leqslant b \leqslant a-1$.
- Having $z_{j+1} \in \{a, \ldots, a+b\}$, or $z_{j+1} = a+b+1$ and $z_{j+2} \geqslant a$, implies $q_{j+1} \in \{0, 1\}$. If, at the same time, $z_{j-1} \geqslant a$, then $q_{j-1} \in \{0, 1\}$ too. By setting $q_j := 1$, we get $w_j = b$, and, finally, $0 \leqslant x_j \leqslant b+1$ so $x_j \in \mathcal{A}$.
- For $z_{j+1} \in \{a, \ldots, a+b\}$, or $z_{j+1} = a+b+1$ and $z_{j+2} \geqslant a$, we have $q_{j+1} \in \{0, 1\}$. If, at the same time, $z_{j-1} \leqslant a-1$, then $q_{j-1} \in \{-1, 0\}$. We put $q_j := 0$ and obtain $w_j = a+b$. As a result, $a-1 \leqslant x_j \leqslant a+b$.
- If $z_{j+1} = a+b+1$ and $z_{j+2} \leqslant a-1$, then $q_{j+1} = 1$. With $q_j := 0$, we proceed via $w_j = a+b$ to $x_j \in \{a-1, a, a+1\} \subset \mathcal{A}$, as $1 < b$.

• $z_j \in \{a+1, \ldots, a+b-1\}$

- When $z_{j+1} \leqslant a-1$, we have $q_{j+1} \in \{-1, 0\}$ and $q_j := 1$. Consequently, $w_j \in \{1, \ldots, b-1\}$, and $0 \leqslant x_j \leqslant 2b$ thus x_j is in $\mathcal{A}$, as $b < a$.
- If $z_{j+1} \geqslant a$, then $q_{j+1} \in \{0, 1\}$. By putting $q_j := 0$, we have $w_j \in \{a+1, \ldots, a+b-1\}$, so $a-b \leqslant x_j \leqslant a+b$.

• $z_j = a$

- For $z_{j+1} \leqslant a-1$, we have $q_{j+1} \in \{-1, 0\}$, and $z_{j-1} \geqslant a$ implies $q_{j-1} \in \{0, 1\}$. Setting $q_j := 1$ results in $w_j = 0$, and, finally, $0 \leqslant x_j \leqslant b+1$ thus $x_j \in \mathcal{A}$.
- When both $z_{j+1} \leqslant a-1$ and $z_{j-1} \leqslant a-1$, then $q_{j+1} \in \{-1, 0\}$ and $q_{j-1} = 0$. With $q_j := 0$, we proceed via $w_j = a$ to $a \leqslant x_j \leqslant a+b$.
- If $z_{j+1} \geqslant a$, then both q_{j+1} and q_{j-1} are limited to $\{0, 1\}$, and we set $q_j := 0$. Thus, we obtain $w_j = a$, and, finally, $a-b \leqslant x_j \leqslant a+1$ thus $x_j \in \mathcal{A}$, as $1 < b < a$.

• $z_j \in \{b, \ldots, a-1\}$

- Since here we have $q_{j-1} \in \{0, 1\}$ for any choice of z_{j-1}, we can keep $q_j := 0$ and $w_j \in \{b, \ldots, a-1\}$. Finally, we obtain $0 \leqslant x_j \leqslant a+b$.

• $z_j \in \{0, \ldots, b-1\}$

– When $z_{j+1} \leqslant a$, we have $q_{j+1} \in \{-1, 0\}$. As $q_{j-1} \in \{0, 1\}$, we keep $q_j := 0$ and $w_j \in \{0, \dots, b-1\}$. Then $0 \leqslant x_j \leqslant 2b$ thus $x_j \in \mathcal{A}$, as $b < a$.
– If $z_{j+1} \geqslant a+1$, then $q_{j+1} \in \{0, 1\}$. Also $q_{j-1} \in \{0, 1\}$, and we set $q_j := -1$. Consequently, $w_j \in \{a, \dots, a+b-1\}$, so $a - b \leqslant x_j \leqslant a + b$.

Therefore, the algorithm performs a correct digit set conversion from $\{0, \dots, a+b+1\}$ to $\{0, \dots, a+b\} = \mathcal{A}$. $\square$

The previous algorithm acts on alphabet $\mathcal{A} \subset \mathbb{N}$. Looking for the letters $h \in \mathcal{A} = \{0, \dots, a+b\}$ such that the algorithm keeps unchanged the constant sequences $(h)_{j \in \mathbb{Z}}$ allows us again to modify the alphabet of the algorithm:

Proposition 3.15. Let $\beta > 1$ satisfy $\beta^2 = a\beta + b$, with $a \geqslant b+1$, $b \geqslant 2$. Then 1-block parallel addition in base β is possible on any alphabet of cardinality $a+b+1$ of contiguous integers containing 0.

Proof. Every letter h, $0 \leqslant h \leqslant a+b-1$, is fixed by the Algorithm *GDE*($\beta^2 = a\beta+b$) above. So, for any $d = 1, \dots, a+b-1$, both letters d and $a+b-d$ are fixed by the algorithm, and, by Corollary 24 in [10], 1-block parallel addition is possible on any alphabet of the form $\{-d, \dots, a+b-d\} = \mathcal{A}$, with $d \in \{0, \dots, a+b\}$. $\square$

For $b \geqslant 2$ and $a = b$, the lower bound on the cardinality of the alphabet $\mathcal{A}$ from Theorem 3.12 is attained as well. It follows from Corollary 4.4, where the existence of k-block parallel addition for this case is guaranteed on the alphabet $\mathcal{A} = \{0, 1, \dots, 2a\}$. Besides, it is believed that also here 1-block parallel addition should be possible on any alphabet of the minimal cardinality $\#\mathcal{A} = 2a+1$, but the algorithm is a lot more complicated than for the case of $a \geqslant b+1$, and it still remains an open task to construct it.

So we finally gather all the cases.

Theorem 3.16. *Let β satisfy $\beta^2 = a\beta + b$, where $a \geqslant b$ and $b \geqslant 1$. Then block parallel addition in base β is possible on alphabet $\mathcal{A} = \{0, \dots, a+b\}$.*

Let us now consider a class of well studied Pisot numbers, generalizing the (quadratic) golden mean:

Definition 3.17. Let $d \in \mathbb{N}, d \geqslant 2$. The real root $\beta > 1$ of the equation $X^d = X^{d-1} + X^{d-2} + \cdots + X + 1$ is said to be the *d-bonacci number*. Specifically, the 2-bonacci number (the golden mean) is called the *Fibonacci number*, and the 3-bonacci number is called the *Tribonacci number*.

Using Theorem 3.12 and the simple fact that $d_\beta(1) = 1^d$ for any d-bonacci number β, we get the following result:

Corollary 3.18. Let β be the d-bonacci number, $d \geqslant 2$. There exists no k-block p-local function performing parallel addition in base β on the alphabet $\mathcal{A} = \{0, 1\}$.

Remark 3.19. In the case when β is a non-simple Parry number with the period of $d_\beta(1)$ longer than 1, one can apply two different lower bounds on cardinality of the alphabet $\mathcal{A} = \{0, 1, \dots, M\}$ allowing parallel addition, namely the bound from Theorem 3.5 and the one from Theorem 3.12.

For example, consider base β with $d_\beta(1) = t_1(t_2t_3)^\omega$ with $t_1 > t_2 \geqslant t_3$. By Theorem 3.5 we get $M \geqslant 2t_1 - t_2 - 1$ and by Theorem 3.12 we get $M \geqslant t_1 + t_3$.

4. Upper bounds on minimal alphabet allowing block parallel addition

Theorem 4.1. *Given a base β and an alphabet $\mathcal{B}$ of contiguous integers containing 0; let us suppose that there exist non-negative integers ℓ and s such that for any $x = x_n \cdots x_0\bullet$ and $y = y_n \cdots y_0\bullet$ from $\mathrm{Fin}_\mathcal{B}(\beta)$ the sum $x + y$ has a $(\beta, \mathcal{B})$-representation of the form*

$$z = x + y = z_{n+\ell} \cdots z_0\bullet z_{-1} \cdots z_{-s}\,.$$

Then there exists a k-block 3-local function performing parallel addition in base β on the alphabet $\mathcal{A} = \mathcal{B} + \mathcal{B}$, where $k = 2(\ell + s)$.

Proof. According to the assumptions, any $x = \sum_{j=0}^{k-1} x_j\beta^j$ with $x_j \in \mathcal{B}+\mathcal{B}$ can be written as $x = \sum_{j=-s}^{k+\ell-1} x'_j\beta^j$ with $x'_j \in \mathcal{B}$. And thus any $z = \sum_{j=0}^{k-1} z_j\beta^j$ with $z_j \in \mathcal{A}+\mathcal{A}$ can be written as

$$z = \sum_{j=-2s}^{k+2\ell-1} z'_j\beta^j \quad \text{with} \quad z'_j \in \mathcal{B}\,.$$

It means that for any $u \in \mathcal{A}_{(k)} + \mathcal{A}_{(k)}$ there exist

$$L(u) \in \mathcal{B}_{(2\ell)}, \quad C(u) \in \mathcal{B}_{(k)}, \quad \text{and} \quad S(u) \in \mathcal{B}_{(2s)}$$

such that

$$u = L(u)\beta^k + C(u) + S(u)\beta^{-2s}\,. \tag{9}$$

It may happen that for $u \in \mathcal{A}_{(k)} + \mathcal{A}_{(k)}$ there exist several triples $L(u)$, $C(u)$, $S(u)$ with the required property. But for any u, we fix just one triple. We can set

$$L(u) = S(u) = 0 \quad \text{and} \quad C(u) = u \quad \text{for any} \quad u \in \mathcal{B}_{(k)}. \tag{10}$$

In particular, we set $L(0) = C(0) = S(0) = 0$.
Let us define a 3-local function Φ with domain $(\mathcal{A}_{(k)} + \mathcal{A}_{(k)})^3$ by

$$\Phi(f, g, h) = L(h) + C(g) + S(f)\beta^{2\ell}\,. \tag{11}$$

As $k = 2(\ell + s)$, $\mathcal{B}_{(k)} = \mathcal{B}_{(2\ell)} + \mathcal{B}_{(2s)}\beta^{2\ell}$, and the function Φ maps $(\mathcal{A}_{(k)} + \mathcal{A}_{(k)})^3$ to $\mathcal{B}_{(k)} + \mathcal{B}_{(k)} = \mathcal{A}_{(k)}$.

Let $\cdots u_2u_1u_0u_{-1}u_{-2}\cdots$ be a sequence with finitely many non-zero $u_j \in \mathcal{A}_{(k)} + \mathcal{A}_{(k)}$. We show that

$$\sum_{j\in\mathbb{Z}} u_j\beta^{jk} = \sum_{j\in\mathbb{Z}} v_j\beta^{jk}\,, \text{ where } v_j = \Phi(u_{j+1}\, u_j\, u_{j-1}).$$

Indeed, by (9) and (11), we have

$$\sum_{j\in\mathbb{Z}} u_j\beta^{jk} = \sum_{j\in\mathbb{Z}} L(u_j)\beta^{k(j+1)} + \sum_{j\in\mathbb{Z}} C(u_j)\beta^{kj} + \sum_{j\in\mathbb{Z}} S(u_j)\beta^{kj-2s} =$$

$$= \sum_{j\in\mathbb{Z}} L(u_{j-1})\beta^{kj} + \sum_{j\in\mathbb{Z}} C(u_j)\beta^{kj} + \beta^{2\ell}\sum_{j\in\mathbb{Z}} S(u_{j+1})\beta^{kj} = \sum_{j\in\mathbb{Z}} \Phi\big(u_{j+1}\,u_j\,u_{j-1}\big)\beta^{kj}.$$

Our choice $L(0) = C(0) = S(0) = 0$ guarantees that the sequence $\cdots v_2v_1v_0v_{-1}v_{-2}\cdots$ has only finitely many non-zero elements as well. Therefore, Φ is the desired k-block 3-local function performing parallel addition in base β on the alphabet $\mathcal{A} = \mathcal{B} + \mathcal{B}$. $\square$

Remark 4.2. From equations (10) and (11) in the previous proof we see that $\Phi(u, u, u) = u$ for any $u \in \mathcal{B}_{(k)}$. It means that the infinite constant sequence $(u)_{j\in\mathbb{Z}}$ is fixed by the corresponding parallel algorithm for any $u \in \mathcal{B}_{(k)}$.

Proposition 4.3. Let $\beta > 1$ be a number with the (PF) Property. Then there exists $k \in \mathbb{N}$ such that k-block parallel addition in base β is possible on the alphabet $\mathcal{A} = \{0, 1, \ldots, 2\lfloor\beta\rfloor\}$, and also on the alphabet $\mathcal{A} = \{-\lfloor\beta\rfloor, \ldots, -1, 0, 1, \ldots, \lfloor\beta\rfloor\}$.

Proof. Let $d_\beta(1) = t_1t_2\cdots$ be the Rényi expansion of unity in base β; obviously, $t_1 = \lfloor\beta\rfloor$. We apply the previous Theorem 4.1 to $\mathcal{B} = \{0, 1, \ldots, \lfloor\beta\rfloor\}$. In [5], the numbers x for which the greedy expansion in base β has a form $x_nx_{n-1}\cdots x_1x_0{\bullet}$ were called β-integers. The set of β-integers is usually denoted $\mathbb{Z}_\beta$. Using the Parry lexicographical condition, we can write formally

$$\mathbb{Z}_\beta = \Big\{\sum_{j=0}^{n} x_j\beta^j \mid x_j \in \mathcal{B} \ \text{ and } \quad x_jx_{j-1}\cdots x_1x_0 \prec t_1t_2t_3\cdots \quad \text{for any } j = 0, 1, \ldots, n\Big\}.$$

Let us denote by

$$\mathcal{B}[\beta] = \Big\{\sum_{j=0}^{n} x_j\beta^j \mid x_j \in \mathcal{B}\Big\}.$$

Clearly, $\mathbb{Z}_\beta \subset \mathcal{B}[\beta]$, but, in general, the opposite inclusion does not hold. Nevertheless, for a given base β with the (PF) Property, there exists a constant $h \in \mathbb{N}$ such that any $x \in \mathcal{B}[\beta]$ can be written as a sum of at most h elements from $\mathbb{Z}_\beta$:

- If $t_1 > 1$, then $h = 2$, since any coefficient $x_j \in \mathcal{B}$ can be written as $x_j = x'_j + x''_j$, where $x'_j, x''_j < t_1$. Thus $\sum_{j=0}^{n} x_j\beta^j = \sum_{j=0}^{n} x'_j\beta^j + \sum_{j=0}^{n} x''_j\beta^j$ and coefficients in both sums on the right side satisfy the Parry condition.

- If $t_1 = 1$, then $t_i \in \{0, 1\}$ for all $i \geqslant 2$ and $\mathcal{B} = \{0, 1\}$. We can take as h the minimal integer $h \geqslant 2$ such that $t_h \neq 0$. This choice of h guarantees that $d_\beta(1) = t_10^{h-2}t_h\cdots$ and that any $(\beta, \mathcal{B})$-representation $z_nz_{n-1}\cdots z_1z_0{\bullet}z_{-1}z_{-2}\cdots$ of a number z in which each nonzero coefficient $z_j = 1$ is followed by $h-1$ zeros $z_{j-1} = z_{j-2} = \cdots = z_{j-h+1} =$

0, is already the greedy expansion of z. Therefore, any $x = \sum_{j=0}^{n} x_j\beta^j \in \mathcal{B}[\beta]$ can be written as $x = x^{(0)} + x^{(1)} + \cdots + x^{(h-1)}$, with $x^{(c)} = \sum_{j=0}^{n} x_j^{(c)}\beta^j \in \mathbb{Z}_\beta$ defined by

$$x_j^{(c)} = \begin{cases} 0 & \text{if } j \neq c \mod h \\ x_j & \text{if } j = c \mod h\,. \end{cases}$$

Bernat studies in [3] the number of fractional digits in the greedy expansion of $x+y$ of two β-integers x and y. He shows that if β is a Perron number (i.e., an algebraic integer with all its algebraic conjugates of modulus strictly less than β) with no algebraic conjugate of modulus 1, then there exists a constant $L_\oplus \in \mathbb{N}$, such that if $x+y$ has finite greedy β-expansion, then the number of fractional digits in the greedy expansion of $x+y$ is less than or equal to $L_\oplus$. Let us stress that the value $L_\oplus$ is effectively computable when β is a Parry number. Since our base β has the (PF) Property, the greedy expansion of the sum of any two β-integers is finite, and thus we are going to apply the previous Theorem 4.1 with $s = hL_\oplus$.

In order to exploit the Theorem 4.1, we have to find also a suitable ℓ. Let ℓ be the smallest integer such that $\frac{2\lfloor\beta\rfloor}{\beta-1} < \beta^\ell$. Since for any $x \in \mathcal{B}[\beta]$ we have $x = x_n \cdots x_0\bullet \leqslant \lfloor\beta\rfloor\frac{\beta^{n+1}-1}{\beta-1}$, we can estimate $x+y = x_n\cdots x_0\bullet \;+\; y_n\cdots y_0\bullet \leqslant 2\lfloor\beta\rfloor\frac{\beta^{n+1}}{\beta-1} < \beta^{n+\ell+1}$. The inequality $z = x+y < \beta^{n+\ell+1}$ implies that at least one representation of z (namely the greedy expansion prolonged to the left by zero coefficients if needed) has the form $z = z_{n+\ell}\cdots z_0\bullet z_{-1}z_{-2}\cdots$.

Using Theorem 4.1, we have proved that parallel addition is possible on the alphabet $\mathcal{A} = \{0,1,\ldots,2\lfloor\beta\rfloor\}$. According to Remark 4.2, the sequence $(h)_{j\in\mathbb{Z}}$ is fixed by the algorithm for parallel addition for any $h \in \{0,1,\ldots,\lfloor\beta\rfloor\} = \mathcal{B}$. Therefore, due to Corollary 24 in [10], the alphabet $\mathcal{A} - \lfloor\beta\rfloor = \{-\lfloor\beta\rfloor,\ldots,0,\ldots,\lfloor\beta\rfloor\}$ allows parallel addition as well. $\square$

Combining Proposition 4.3, Theorem 3.5, and Theorem 3.12, we can derive the following conclusions:

Corollary 4.4. Let $d_\beta(1) = t_1t_2\cdots t_m$, with $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_m \geqslant 1$ be the Rényi expansion of 1 in base β. Then there exists $M \in \mathbb{N}$ such that parallel addition by a k-block local function in a non-integer base β is possible on the alphabet $\mathcal{A} = \{0,1,\ldots,M\}$ with $t_1 + t_m \leqslant M \leqslant 2t_1$.

Corollary 4.5. Let $d_\beta(1) = t_1t_2\cdots t_m t^\omega$ with $t_1 > t_2 \geqslant t_2 \geqslant \cdots \geqslant t_m > t \geqslant 1$ be the Rényi expansion of 1 in base β. Then there exists $M \in \mathbb{N}$ such that parallel addition by a k-block local function in base β is possible on the alphabet $\mathcal{A} = \{0,1,\ldots,M\}$ with $2t_1 - t_2 - 1 \leqslant M \leqslant 2t_1$.

On those bases β that are d-bonacci numbers we will demonstrate how the concept of k-block local function can substantially reduce the cardinality of alphabet which allows parallel addition:

Corollary 4.6. Let β be a d-bonacci number for some $d \in \mathbb{N}$, $d \geqslant 2$.

- If an alphabet $\mathcal{A}$ allows 1-block parallel addition in base β, then its cardinality is $\#\mathcal{A} \geqslant d+1$.
- There exists $k \in \mathbb{N}$ such that k-block parallel addition in base β is possible on the alphabets $\mathcal{A} = \{0,1,2\}$ and $\mathcal{A} = \{-1,0,1\}$, and these alphabets cannot be further reduced.

Proof. The minimal polynomial of a d-bonacci number is $f(X) = X^d - X^{d-1} - X^{d-2} - \cdots - X - 1$. Theorem 2.4 says that 1-block parallel addition is possible only on an alphabet with cardinality at least $|f(1)| + 2 = d+1$.

The Rényi expansion of unity for a d-bonacci number is $d_\beta(1) = 1^d$, and thus the d-bonacci number satisfies the (PF) Property. Since $\lfloor \beta \rfloor = 1$, due to Proposition 4.3, k-block parallel addition in base β is possible on the alphabets $\mathcal{A} = \{0,1,2\}$ and $\mathcal{A} = \{-1,0,1\}$. With respect to Corollary 3.18, this alphabet is minimal. □

Example 4.7. In [4], Bernat computes the value of $L_\oplus$ (as defined in the proof of Proposition 4.3) for the Tribonacci base, namely $L_\oplus = 5$. It is readily seen that for the Tribonacci base, the set of β-integers $\mathbb{Z}_\beta$ defined in the proof of Proposition 4.3 and the set $\mathcal{B}[\beta]$ defined ibidem coincide. Therefore the parameter s in Theorem 4.1 is equal to 5. It is easy to see that $\ell = 2$. Thus, addition in the Tribonacci base is 14-block 3-local parallel on the alphabets $\mathcal{A} = \{0,1,2\}$ or $\mathcal{A} = \{-1,0,1\}$.

Remark 4.8. Theorem 4.1 requires an alphabet $\mathcal{B}$ for which $\mathrm{Fin}_{\mathcal{B}+\mathcal{B}}(\beta) = \mathrm{Fin}_{\mathcal{B}}(\beta)$ and also it requires existence of non-negative integers s and ℓ as defined above. They control the number of additional positions by which the sum of two elements from $\mathrm{Fin}_{\mathcal{B}}(\beta)$ is prolonged to the right and to the left, respectively. To satisfy the assumptions of Theorem 4.1 we suppose in Proposition 4.3 that the base β has the (PF) Property. Although this property is too restrictive we decided to use it as we did not find any other published result which allow us determine s and ℓ. We expect that the (PF) Property can by replaced by a more suitable assumption.

Remark 4.9. This paper deals mainly with positive bases β. However, Theorem 4.1 can be applied to complex bases as well. One such class of bases defines the so-called *Canonical Number Systems* (CNS), see [15] and [16].

An algebraic number β and the alphabet $\mathcal{B} = \{0,1,\ldots,|N(\beta)|-1\}$, where $N(\beta)$ denotes the norm of β over $\mathbb{Q}$, form a Canonical Number System, if any element x of the ring of integers $\mathbb{Z}[\beta]$ has a unique representation in the form $x = \sum_{k=0}^{n} x_k \beta^k$, where $x_k \in \mathcal{B}$ and $x_n \neq 0$.

In particular, it means that the sum of two elements of $\mathbb{Z}[\beta]$ has also a finite representation in the form $\sum_{k=0}^{m} x_k \beta^k$, where $x_k \in \mathcal{B}$ and $x_m \neq 0$, and thus in Theorem 4.1 we can set $s = 0$. It can be proved that in a CNS the constant ℓ required in that theorem also exists. We can conclude that, in a CNS, block parallel addition is possible on the alphabet $\mathcal{A} = \{0,1,\ldots,2|N(\beta)|-2\}$ or on the alphabet $\mathcal{A} = \{-|N(\beta)|+1,\ldots,0,\ldots,|N(\beta)|-1\}$.

More specifically for the Penney numeration system, the base $\beta = \imath - 1$ has norm $N(\beta) = 2$, and together with the alphabet $\mathcal{B} = \{0,1\}$ forms a CNS. Therefore, due to Theorem 4.1, block parallel addition in the Penney numeration system is possible not only on the alphabet $\mathcal{A} = \{-1, 0, 1\}$ (as shown by Herreros in [13]), but also on the alphabet $\mathcal{A} = \{0, 1, 2\}$.

Analogously to the previous remark, the assumption that β defines a CNS is too restrictive as the set $\mathrm{Fin}_{\mathcal{B}}(\beta)$ can be closed under addition without being a CNS. This phenomenon is studied in [1].

5. Comments and open questions

When designing the algorithms for (block) parallel addition in a given base β, we need to take into consideration three core parameters:

1) the cardinality $\#\mathcal{A}$ of the used alphabet $\mathcal{A}$,

2) the width p of the sliding window, i.e., the number p appearing in the definition of the p-local function Φ, and

3) the length k of the blocks in which we group the digits of the $(\beta, \mathcal{A})$-representations for k-block parallel addition.

There are mathematical reasons (for example comparison of numbers) and even more technical reasons to minimize all these three parameters. But intuitively, the smaller is one of the parameters, the bigger have to be the other ones. The question of which relationship binds the values $\#\mathcal{A}$, p, and k is far from being answered.

In that respect, we are able to list just several isolated observations made for specific bases:

- In [9], we studied 1-block parallel addition, i.e., k was fixed to 1. For base β being the Fibonacci number (i.e. the golden mean $\frac{1+\sqrt{5}}{2}$), we gave a parallel algorithm for addition on the alphabet $\mathcal{A} = \{-3, \ldots, 0, \ldots, 3\}$ by a 13-local function. On the other hand, for the same base, we have also described an algorithm for parallel addition on the minimal alphabet $\mathcal{A} = \{-1, 0, 1\}$, where the corresponding function Φ is 21-local.

- The d-bonacci bases illustrate that if we do not care about the length k of the blocks, the alphabet can be substantially reduced, namely to $\mathcal{A} = \{0, 1, 2\}$, see Corollary 4.6. But the price for that is rather high; already for the Tribonacci base our algorithm requires blocks of length $k = 14$, see Example 4.7.

- If we fix in the Penney numeration system the value $k = 1$, an alphabet of cardinality 5 is necessary for parallel addition. Herreros in [13] provided an algorithm for parallel addition in the Penney base $\beta = \imath - 1$ on the alphabet $\mathcal{A} = \{-1, 0, 1\}$, but his algorithm uses $k = 4$. This value is not optimal; we have found (not yet published) that $k = 2$ is enough to perform parallel addition on the alphabet $\mathcal{A} = \{-1, 0, 1\}$.

Besides the width p of the sliding window as such, there is another characteristic which is desired for the algorithms performing parallel addition, namely to be *neighbour-free.*

This property has to do with the way how one determines the value q_j within the first step of the algorithm. It is in fact the key task of the algorithm. Once having the correct set of the values q_j after the first step, one only deducts the q_j-multiple of an appropriate form of a representation of zero, and the task is finished.

Being neighbour-free means that the value q_j depends only on the digit on the j-th position of the processed string, irrespective of its neighbours. Note that this is something else than being 1-local! On the other hand, an algorithm of parallel addition which is not neighbour-free, is called *neighbour-sensitive*, see the discussion in [9].

For integer bases, as explained in Remark 2.6, the concept of k-block parallel addition with $k \geqslant 2$ is not interesting from the point of view of the minimality of the cardinality of the alphabet. However, grouping of digits into k-blocks can improve the parallel algorithm in another way, namely with respect to the neighbour-free property.

For instance, in base $\beta = 2$, 1-block parallel addition is doable on the minimal alphabet $\mathcal{A} = \{-1, 0, 1\}$ by the neighbour-sensitive algorithm of Chow and Robertson [7]. But 2-block addition here means just addition in base $\beta^2 = 4$ on alphabet $\mathcal{A}_{(2)} = \{-3, \dots, 0, \dots, 3\}$, and is performable by the simpler algorithm of Avizienis [2], which is neighbour-free.

The most common reason why to work in a numeration system with an algebraic base β, instead of a system with base 2 or 10, consists in the requirement to perform precise computations in the algebraic field $\mathbb{Q}(\beta)$. If the base β is not "nice enough", we can choose another base γ such that $\mathbb{Q}(\beta) = \mathbb{Q}(\gamma)$ and then work in the numeration system with the base γ. The question is which base in $\mathbb{Q}(\beta)$ is "nice enough" and how to find it effectively.

- Certainly, the "beauty" of the Pisot bases is not questionable. Cheng and Zhu in [6] described an algorithm for finding a Pisot number which generates the whole algebraic field $\mathbb{Q}(\gamma)$.

- From another point of view, a base allowing parallel addition on a binary alphabet would be "beautiful" as well; but there is no example of such a base known yet. May it exist?

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[1] S. Akiyama, Positive fineteness of number systems, *Number Theory: Tradition and Modernization*, ed. by W. Zhang and Y. Tanigawa, DEVM 15, Springer (2006) 1–10.

[2] A. Avizienis, Signed-digit number representations for fast parallel arithmetic, *IRE Trans. Electron. Comput.* **10** (1961) 389–400.

[3] J. Bernat, Arithmetics in β-numeration, *Discrete Math. Theor. Comput. Sci.* **9** (2007) 85–106.

[4] J. Bernat, Computation of $L_\oplus$ for several cubic Pisot numbers, *Discrete Math. Theor. Comput. Sci.* **9** (2007) 175–193.

[5] C. Burdík, Ch. Frougny, J.-P. Gazeau, and R. Krejcar, Beta-integers as natural counting systems for quasicrystals, *J. of Physics A: Math. Gen.* **31** (1998) 6449–6472.

[6] Q. Cheng, J. Zhu, On certain computations of Pisot numbers, *Information Processing Letters* **113** (2013) 271–275.

[7] C.Y. Chow, J.E. Robertson, Logical design of a redundant binary adder, *Proc. 4th IEEE Symposium on Computer Arithmetic* (1978) 109–115.

[8] M. D. Ercegovac and T. Lang, *Digital Arithmetic*, Morgan Kaufmann, 2004.

[9] Ch. Frougny, E. Pelantová, and M. Svobodová, Parallel addition in non-standard numeration systems, *Theor. Comput. Sci.* **412** (2011) 5714–5727.

[10] Ch. Frougny, E. Pelantová, and M. Svobodová, Minimal Digit Sets for Parallel Addition in Non-Standard Numeration Systems, *Journal of Integer Sequences* **16** (2013), Article 13.2.17.

[11] Ch. Frougny and J. Sakarovitch, Number representation and finite automata, *Combinatorics, Automata and Number Theory*, Encyclopedia of Mathematics and its Applications, Vol. 135, Cambridge University Press, 2010, Chapter 2, pp. 34–107.

[12] Ch. Frougny and B. Solomyak, Finite beta-expansions, *Ergodic Theory & Dynamical Systems* **12** (1992) 713–723.

[13] Y. Herreros, Contribution à l'arithmétique des ordinateurs, Ph. D. dissertation, Institut National Polytechnique de Grenoble, 1991.

[14] P. Kornerup, Necessary and sufficient conditions for parallel, constant time conversion and addition, *Proc. 14th IEEE Symposium on Computer Arithmetic*, (1999) 152–155.

[15] B. Kovács, Canonical number systems in algebraic number field, *Acta Math. Acad. Sci. Hung.* **37** (4) (1981) 405–407.

[16] B. Kovács and A. Pethő, Number systems in integral domains, especially in orders of algebraic number fields, *Acta Sci. Math Szeged* **55** (1991) 287–299.

[17] B. Parhami, On the implementation of arithmetic support functions for generalized signed-digit number systems, *IEEE Trans. Computers* **42** (1993) 379–384.

[18] W. Parry, On the β-expansions of real numbers, *Acta Math. Acad. Sci. Hungar.* **11** (1960) 401–416.

[19] W. Penney, A 'binary' system for complex numbers, *J. ACM* **12** (1965) 247–248.

[20] A. Rényi, Representations for real numbers and their ergodic properties, *Acta Math. Acad. Sci. Hungar.* **8** (1957) 477–493.