Conversion between two multiplicatively dependent linear numeration systems

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Abstract. We consider two linear numeration systems, with characteristic polynomial equal to the minimal polynomial of two Pisot numbers β and γ respectively, such that β and γ are multiplicatively dependent. It is shown that the conversion between one system and the other one is computable by a finite automaton.

1 Introduction

This work is about the conversion of integers represented in two different numeration systems, linked in a certain sense. Recall that the conversion between base 4 and base 2 is computable by a finite automaton, but that conversion between base 3 and base 2 is not. More generally, two numbers $p > 1$ and $q > 1$ are said to be *multiplicatively dependent* if there exist positive integers k and ℓ such that $p^k = q^\ell$. A set of natural integers is said to be *p-recognizable* if the set of representations in base p of its elements is recognizable by a finite automaton. Bűchi has shown that the set $\{q^n \mid n \geq 0\}$ is p-recognizable only if p and q are multiplicatively dependent integers [5]. In contrast, the famous theorem of Cobham [7] states that the only sets of natural integers that are both p- and q-recognizable, when p and q are two multiplicatively independent integers > 1, are unions of arithmetic progressions, and thus are k-recognizable for any integer $k > 1$. Several generalizations of Cobham's Theorem have been given, see [18, 10, 6, 8]. In particular this result has been extended by Bès [4] to non-standard numeration systems.

The most popular non-standard numeration system is probably the Fibonacci numeration system. Recall that every non-negative integer can be represented as a sum of Fibonacci numbers, which can be chosen non-consecutive (see Section 2.2). It is also possible to represent an integer as a sum of Lucas numbers. Since Fibonacci and Lucas numbers satisfy the same recurrence relation, the question of the conversion between Lucas representations and Fibonacci representations is very natural.

A *linear numeration system* is defined by an increasing sequence of integers satisfying a linear recurrence relation. The generalization of the Cobham's Theorem by Bès [4] is the following: let two linear numeration systems such that their characteristic polynomials are the minimal polynomials of two multiplicatively independent Pisot numbers[1]; the only sets of natural integers such that their representations in these two systems are recognizable by a finite automaton are unions of arithmetic progressions.

From the result of Bès follows that the conversion between two linear numeration systems U and Y linked to two multiplicatively independent Pisot numbers cannot be realized by a finite automaton. In this paper, we prove that the conversion between two linear numeration systems U and Y such that their characteristic polynomials are the minimal polynomials of two multiplicatively *dependent* Pisot numbers is computable by a finite automaton. This implies that a set of integers which is U-recognizable is then also Y-recognizable. Note that in [6] it is shown that if U and V are two linear numeration systems with the same characteristic polynomial which is the minimal polynomial of a Pisot number, then a U-recognizable set is also V-recognizable.

The paper is organized as follows. First we recall several results which will be of use in this paper. In particular, the normalization in a linear numeration system consists in converting a representation on a "big" alphabet onto the so-called *normal* representation, obtained by a greedy algorithm. Here the system U is fixed. It is shown in [15] that, basically, when the sequence U is linked to a Pisot number, like the Fibonacci numbers are linked to the golden mean, then normalization is computable by a finite automaton on any alphabet of digits. In the present work we first construct a finite automaton realizing the conversion from Lucas representations to Fibonacci representations. Then we consider two sequences of integers U and V. If the elements of V can be linearly expressed (with rational coefficients) in those of U, and if the normalization in the system U is computable by a finite automaton, then so it is for the conversion from V-representations to U-representations. From this result we deduce that if U and V have for characteristic polynomial the same minimal polynomial of a Pisot number, with different initial conditions, then the

[1] A Pisot number is an algebraic integer such that its algebraic conjugates are strictly less than 1 in modulus. The golden mean and the natural integers are Pisot numbers.

conversion from V-representations to U-representations is computable by a finite automaton.

Next we introduce two different linear numeration systems associated with a Pisot number β of degree m. The first one, U_β, is defined from the point of view of the symbolic dynamical system defined by β. We call it *Fibonacci-like*, because when β is equal to the golden mean, it is the Fibonacci numeration system. The second one, V_β, is defined from the algebraic properties of β. More precisely, for $n \geq 1$, the n-th term of V_β is $v_n = \beta^n + \beta_2^n + \cdots + \beta_m^n$, where $\beta_2, \ldots, \beta_m$ are the algebraic conjugates of β. We call it *Lucas-like*, because when β is equal to the golden mean, it is the Lucas numeration system. The conversion from V_β to U_β (or any sequence with characteristic polynomial equal to the minimal polynomial of β) is shown to be computable by a finite automaton.

Then we consider two linear numeration systems, U and Y, such that their characteristic polynomial is equal to the minimal polynomial of a Pisot number β, or γ respectively, where β and γ are multiplicatively dependent. Then the conversion from Y to U is shown to be computable by a finite automaton.

In the appendix we give some connections between the Lucas-like numeration system V_β and the base β numeration system for the case where β is a quadratic Pisot unit. Note that in [14] we have proved that the conversion from U_β-representations to folded β-representations is computable by a finite automaton, and in [16], that this is possible only if β is a quadratic Pisot unit.

2 Preliminaries

Words. An *alphabet* A is a finite set. A finite sequence of elements of A is called a *word*, and the set of words on A is the free monoid A^*. The *empty word* is denoted by ε. The set of infinite sequences or infinite words on A is denoted by $A^{\mathbb{N}}$. Let v be a non-empty word of A^*, denote by v^n the concatenation of v to itself n times, and by v^ω the infinite concatenation $vvv\cdots$. An infinite word of the form uv^ω is said to be *eventually periodic*. A *factor* of a (finite or infinite) word w is a finite word f such that $w = ufv$.

***U*-representations.** The definitions recalled below and related results can be found in the survey [13]. We consider a generalization of the usual notion of numeration system, which yields a representation of the natural numbers. The base is replaced by an infinite sequence of integers. The basic example is the well-known Fibonacci numeration system.

Let $U = (u_n)_{n\geq 0}$ be a strictly increasing sequence of integers with $u_0 = 1$. A *U-representation* of a non-negative integer N is a finite sequence of integers $(d_i)_{k\geq i\geq 0}$ such that $N = \sum_{i=0}^{k} d_i u_i$. Such a representation will be written $(N)_U = d_k \cdots d_0$, most significant digit first.

Among all possible U-representations of a given non-negative integer N one is distinguished and called the *normal U-representation* of N; it is also called the *greedy* representation, since it can be obtained by the following greedy algorithm [11]: given integers m and p let us denote by $q(m,p)$ and $r(m,p)$ the quotient and the remainder of the Euclidean division of m by p. Let $k \geq 0$ such that $u_k \leq N < u_{k+1}$ and let $d_k = q(N, u_k)$ and $r_k = r(N, u_k)$, and, for $i = k-1, \ldots, 0$, $d_i = q(r_{i+1}, u_i)$ and $r_i = r(r_{i+1}, u_i)$. Then $N = d_k u_k + \cdots + d_0 u_0$. The normal U-representation of N is denoted by $\langle N\rangle_U$. The normal U-representation of 0 is the empty word ε. The set of greedy or normal U-representations of all the non-negative integers is denoted by $G(U)$. In this work, we consider only the case where the sequence U is linearly recurrent. Then the numeration system associated with U is said to be a *linear numeration system*. The digits of a normal U-representation are contained in a *canonical* finite alphabet A_U associated with U.

Let D be a finite alphabet of integers. The *normalization* in the system U on D^* is the partial function $\nu_{U,D^*} : D^* \to A_U^*$ that maps a word $d_k \cdots d_0$ of D^* such that $N = \sum_{i=0}^{k} d_i u_i$ is non-negative onto the normal U-representation of N.

Let U and V be two sequences of integers, and let D be a finite alphabet of integers. The *conversion* from the numeration system V to the numeration system U on D^* is the partial function $\chi : D^* \to A_U^*$ that maps a V-representation $d_k \cdots d_0$ in D^* of a non-negative integer $N = \sum_{i=0}^{k} d_i v_i$ onto the normal U-representation of N. In fact the alphabet D plays no peculiar role, and we will simply speak of the conversion from V to U.

Beta-expansions. We now consider numeration systems where the base is a real number $\beta > 1$. Representations of real numbers in such systems were introduced by Rényi [17] under the name of *beta-expansions.* Let the base $\beta > 1$ be a real number. First let x be a real number in the interval $[0, 1]$. A *representation in base β* of x is an infinite sequence of integers $(x_i)_{i\geq 1}$ such that $x = \sum_{i\geq 1} x_i \beta^{-i}$. A particular beta-representation, called the *beta-expansion*, can be computed by the "greedy algorithm" : denote by $\lfloor y \rfloor$ and $\{y\}$ the integer part and the fractional part of a number y. Set $r_0 = x$ and let for $i \geq 1$, $x_i = \lfloor \beta r_{i-1} \rfloor$, $r_i = \{\beta r_{i-1}\}$. Then $x = \sum_{i \geq 1} x_i \beta^{-i}$, where the x_i's are elements of the canonical alphabet $A_\beta = \{0, \ldots, \lfloor \beta \rfloor\}$ if β is not an integer, or $A_\beta = \{0, \ldots, \beta - 1\}$ if β is an integer. The beta-expansion of x is denoted by $d_\beta(x)$. Let D_β be the set of β-expansions of numbers of $[0, 1[$. The closure of D_β is called the *β-shift* S_β. It is a symbolic dynamical system, that is to say a closed shift invariant subset of $A_\beta^{\mathbb{N}}$.

Let D be a finite alphabet of integers. The *normalization* in base β on $D^{\mathbb{N}}$ is the partial function $\nu_{\beta, D^{\mathbb{N}}} : D^{\mathbb{N}} \to A_\beta^{\mathbb{N}}$ that maps a word $(x_i)_{i \geq 1}$ of $D^{\mathbb{N}}$ such that $x = \sum_{i \geq 1} x_i \beta^{-i} \in [0, 1[$ onto the β-expansion of x.

Secondly, we consider a real number x greater than 1. There exists $k \in \mathbb{N}$ such that $\beta^k \leq x < \beta^{k+1}$. Hence $0 \leq x/\beta^{k+1} < 1$, thus it is enough to represent numbers from the interval $[0, 1]$, since by shifting we will get the representation of any positive real number. A β-representation of an $x = \sum_{k \leq i \leq -\infty} x_i \beta^i$ will be denoted by $(x)_\beta = x_k \cdots x_0 . x_{-1} x_{-2} \cdots$.

If a representation ends in infinitely many zeros, like $v0^\omega$, the ending zeros are omitted and the representation is said to be *finite*.

A *Pisot number* is an algebraic integer such that its algebraic conjugates are strictly less than 1 in modulus. It is known that if β is a Pisot number then $d_\beta(1)$ is finite or infinite eventually periodic [2].

Automata. We refer the reader to [9]. An *automaton over A*, $\mathcal{A} = (Q, A, E, I, T)$, is a directed graph labelled by elements of A. The set of vertices, traditionally called *states*, is denoted by Q, $I \subset Q$ is the set of *initial* states, $T \subset Q$ is the set of *terminal* states and $E \subset Q \times A \times Q$ is the set of labelled *edges*. If $(p, a, q) \in E$, we denote $p \xrightarrow{a} q$. The automaton is *finite* if Q is finite. A subset H of A^* is said to be *recognizable by a finite automaton* if there exists a finite automaton $\mathcal{A}$ such that H is equal to the set of labels of paths starting in an initial state and ending in a terminal state. A *2-tape automaton* with input alphabet A and output alphabet B is an automaton over the non-free monoid $A^* \times B^*$: $\mathcal{A} = (Q, A^* \times B^*, E, I, T)$ is a directed graph the edges of which are labelled by elements of $A^* \times B^*$. The automaton is finite if Q and E are finite. The finite 2-tape automata are also known as *transducers*. A relation R of $A^* \times B^*$ is said to be *computable by a finite automaton* if there exists a finite 2-tape automaton $\mathcal{A}$ such that R is equal to the set of labels of paths starting in an initial state and ending in a terminal state. A function is computable by a finite automaton if its graph is computable by a finite 2-tape automaton. These definitions extend to relations (and functions) of infinite words as follows: a relation R of infinite words is computable by a finite automaton if there exists a finite 2-tape automaton such that R is equal to the set of labels of infinite paths starting in an initial state and going infinitely often through a terminal state. Recall that the set of relations computable by a finite automaton is closed under composition and inverse.

Previous results. In this work we will make use of the following results. Let U be a linearly recurrent sequence of integers such that its characteristic polynomial is exactly the minimal polynomial of a Pisot number. Then the set $G(U)$ of normal U-representations of non-negative integers is recognizable by a finite automaton, and, for every alphabet of positive or negative integers D, normalization ν_{U,D^*} is computable by a finite automaton [15]. Normalization in base β, when β is a Pisot number, is computable by a finite automaton on any alphabet D [12]. Addition and multiplication by a fixed positive integer constant are particular cases of normalization, and thus are computable by a finite automaton, in the system U and in base β.

3 Fibonacci and Lucas

Let us recall that the *Fibonacci numeration system* is defined by the sequence F of Fibonacci numbers

$$F = \{1, 2, 3, 5, 8, 13, \ldots\}.$$

The canonical alphabet is $A_F = \{0, 1\}$ and the set of normal representations is equal to $G(F) = 1\{0,1\}^* \setminus \{0,1\}^*11\{0,1\}^* \cup \varepsilon$. Words containing a factor 11 are forbidden.

The *Lucas numeration system* is defined by the sequence L of Lucas numbers

$$L = \{1, 3, 4, 7, 11, 18, \ldots\}.$$

The canonical alphabet is $A_L = \{0, 1, 2\}$ and the set of normal representations is equal to $G(L) = G(F) \cup (G(F) \setminus \varepsilon)\{02\} \cup \{2\}$. We give in Table 1 below the normal Fibonacci and Lucas representations of the first natural integers.

The Fibonacci and the Lucas sequences both have for characteristic polynomial

$$P(X) = X^2 - X - 1.$$

N	Fibonacci	Lucas
1	1	1
2	10	2
3	100	10
4	101	100
5	1000	101
6	1001	102
7	1010	1000
8	10000	1001
9	10001	1002
10	10010	1010
11	10100	10000

Table 1. Normal Fibonacci and Lucas representations of the 11 first integers

The root > 1 of P is denoted by φ, the golden mean, and its algebraic conjugate by φ'. Since $\varphi + \varphi' = 1$, for coherence of notations with the general case, we denote $F = (F_n)_{n\geq 0}$ and $L = (L_n)_{n\geq 1}$. Recall that for every $n \geq 1$, $L_n = \varphi^n + {\varphi'}^n$. The associated dynamical system is the golden mean shift, which is the set of bi-infinite sequences on $\{0,1\}$ having no factor 11.

Although the following result is a consequence of the more general one below (Theorem 1), we give here a direct construction.

Proposition 1. *The conversion from a Lucas representation of an integer to the normal Fibonacci representation of that integer is computable by a finite automaton.*

Proof. First, for every $n \geq 3$, we get $L_n = F_{n-1} + F_{n-3}$. Take N a positive integer and a L-representation $(N)_L = d_k \cdots d_1$, where the d_i's are in an alphabet $B \supseteq \{0,1,2\}$, and $k \geq 4$. Then $N = d_k L_k + \cdots + d_1 L_1$, thus $N = d_k F_{k-1} + d_{k-1} F_{k-2} + (d_{k-2} + d_k) F_{k-3} + \cdots + (d_3 + d_5) F_2 + (d_2 + d_4) F_1 + (d_1 + d_2 + d_3) F_0$, hence the word $d_k d_{k-1} (d_{k-2} + d_k) \cdots (d_3 + d_5)(d_2 + d_4)(d_1 + d_2 + d_3)$ is a Fibonacci representation of N on a certain finite alphabet of digits D.

The conversion from a word of the form $d_k \cdots d_1$ in B^*, where $k \geq 4$, onto a word of the form $d_k d_{k-1} (d_{k-2} + d_k) \cdots (d_3 + d_5)(d_2 + d_4)(d_1 + d_2 + d_3)$ on D^* is computable by a finite automaton $\mathcal{A} = (Q, B \times C, E, \{\varepsilon\}, \{t\})$: the set of states is $Q = \{\varepsilon\} \cup B \cup (B \times B) \cup \{t\}$ where $\{t\}$ is the unique terminal state. The initial state is ε. For each d in B, there is an edge $\varepsilon \stackrel{d/d}{\to} d$. For each d and c in B, there is an edge $d \stackrel{c/c}{\to} (d,c)$. For each $(d,c) \in B \times B$ and a in B, there is an edge $(d,c) \stackrel{a/a+d}{\to} (c,a)$. For each

$(d, c) \in B \times B$ and a in B, there is a terminal edge $(d, c) \overset{a/a+c+d}{\rightarrow} t$. Words of length less than 4 are handled directly.

Then it is enough to normalize in the Fibonacci system on D^*, and it is known that this is realizable by a finite automaton, see Section 2.6. $\square$

On Figure 1 we give an automaton realizing the conversion from normal Lucas representations to Fibonacci representations on $\{0, 1, 2\}^*(\{\varepsilon\} \cup \{3\})$. States of the form (d, c) are denoted by dc. Note that this automaton is not deterministic on inputs. Since we are dealing with *normal* Lucas representations, the automaton has less states than the one constructed in the proof of Proposition 1 above. To decrease the complexity of the drawing, we introduce more than one terminal state. Terminal states are indicated by an outgoing arrow. The result must be normalized afterwards.

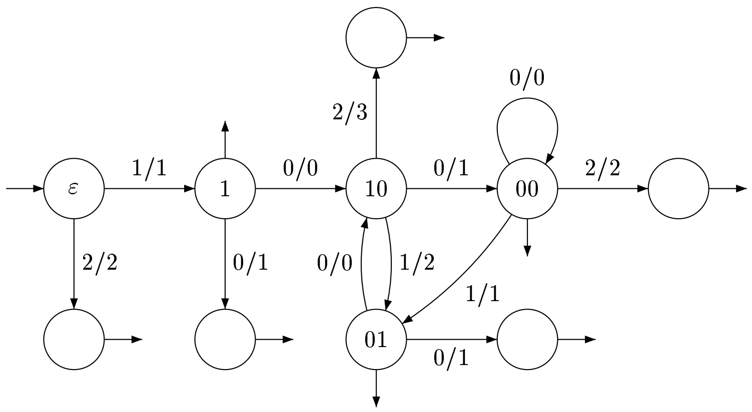

Fig. 1. Conversion from normal Lucas representations to Fibonacci representations

4 A technical result

We now consider two linearly recurrent sequences $U = (u_n)_{n \geq 0}$ and $V = (v_n)_{n \geq 0}$ of positive integers. The result below is the generalization of Proposition 1.

Proposition 2. *If there exist r rational constants λ_i's for $1 \leq i \leq r$ and $K \geq 0$ such that for every $n \geq K$, $v_n = \lambda_1 u_{n+r-1} + \cdots + \lambda_r u_n$, and if* *the normalization in the system U is computable by a finite automaton on any alphabet, then the conversion from a V-representation of an integer to the normal U-representation of that integer is computable by a finite automaton.*

Proof. One can assume that the λ_i's are all of the form p_i/q where the p_i's belong to $\mathbb{Z}$ and q belongs to $\mathbb{N}$, $q \neq 0$. Let N be a positive integer and consider a V-representation $(N)_V = b_j \cdots b_0$, where the b_i's are in an alphabet of digits $B \supseteq A_V$. Then $qN = b_j q v_j + \cdots + b_0 q v_0$. Since for $n \geq K$, $qv_n = p_1 u_{n+r-1} + \cdots + p_r u_n$, and $v_0, v_1, \ldots, v_{K-1}$ can be expressed in the system U, we get that qN is of the form $qN = d_{j+r-1} u_{j+r-1} + \cdots + d_0 u_0$ where the d_i's are elements of a finite alphabet of digits $D \supset A_U$. By assumption, ν_{U,D^*} is computable by a finite automaton. It remains to show that the function which maps $\nu_{U,D^*}(d_{j+r-1} \cdots d_0) =< qN >_U$ onto $< N >_U$ is computable by a finite automaton, and this is due to the fact that it is the inverse of the multiplication by the natural q, which is computable by a finite automaton in the system U, see Section 2.6. □

5 Common characteristic polynomial

The Fibonacci and the Lucas numeration systems are examples of different numeration systems having the same characteristic polynomial, but different initial conditions.

Theorem 1. *Let P be the minimal polynomial of a Pisot number of degree m. Let U and V be two sequences with common characteristic polynomial P and different initial conditions. The conversion from a V-representation of a positive integer to the normal U-representation of that integer is computable by a finite automaton.*

Proof. Since the polynomial P is the minimal polynomial of a Pisot number, normalization in the system U is computable by a finite automaton on any alphabet (see Section 2.6). On the other hand, the family $\{u_n, u_{n+1}, \ldots, u_{n+m-1} \mid n \geq 0\}$ is free, because the annihilator polynomial is the minimal polynomial. Since U and V have the same characteristic polynomial, it is known from standard results of linear algebra that there exist rational constants λ_i such that, for each $n \geq 0$, $v_n = \lambda_1 u_{n+m-1} + \cdots + \lambda_m u_n$. The result follows then from Proposition 2. □

6 Two numeration systems associated with a Pisot number

Let β be a Pisot number. We define two numeration systems associated with β.

Fibonacci-like numeration system. Recall that the β-expansion of 1 is finite or eventually periodic [2].

First suppose that the β-expansion of 1 is finite, $d_\beta(1) = t_1 \cdots t_N$. A linear recurrent sequence $U_\beta = (u_n)_{n\geq 0}$ is canonically associated with β as follows

$$u_n = t_1 u_{n-1} + \cdots + t_N u_{n-N} \text{ for } n \geq N$$

$$u_0 = 1, \text{ and for } 1 \leq i \leq N-1, \ u_i = t_1 u_{i-1} + \cdots + t_i u_0 + 1.$$

The characteristic polynomial of U_β is thus

$$K(X) = X^N - t_1 X^{N-1} - \cdots - t_N.$$

Suppose now that the β-expansion of 1 is infinite eventually periodic,

$$d_\beta(1) = t_1 \cdots t_N (t_{N+1} \cdots t_{N+p})^\omega$$

with N and p minimal. The sequence $U_\beta = (u_n)_{n\geq 0}$ is the following one

$$u_n = t_1 u_{n-1} + \cdots + t_{N+p} u_{n-N-p} + u_{n-p} - t_1 u_{n-p-1} - \cdots - t_N u_{n-N-p}$$

for $n \geq N+p$,

$$u_0 = 1, \text{ and for } 1 \leq i \leq N+p-1, u_i = t_1 u_{i-1} + \cdots + t_i u_0 + 1.$$

The characteristic polynomial of U_β is now

$$K(X) = X^{N+p} - \sum_{i=1}^{N+p} t_i X^{N+p-i} - X^N + \sum_{i=1}^{N} t_i X^{N-i}.$$

Note that in general $K(X)$ may be reducible. Since $K(X)$ is defined from the β-expansion of 1, we will say that it is the *beta-polynomial* of β.

The system U_β is said to be the *canonical* numeration system associated with β. In [3] it is shown that the set of normal representations of the integers $G(U_\beta)$ is exactly the set of finite factors of the β-shift S_β. The numeration system U_β is the natural one from the point of view of symbolic dynamical systems.

Lucas-like numeration system. Now we introduce another linear recurrent sequence $V_\beta = (v_n)_{n\geq 0}$ associated with β a Pisot number of degree m as follows. Denote by $\beta_1 = \beta, \beta_2, \ldots, \beta_m$ the roots of the minimal polynomial $P(X) = X^m - a_1 X^{m-1} - \cdots a_m$ of β. Set

$$v_0 = 1, \text{ and for } n \geq 1, \; v_n = \beta_1^n + \cdots + \beta_m^n.$$

Then the characteristic polynomial of V_β is equal to $P(X)$.

As an example let us take $\beta = \varphi$ the golden mean. Then U_φ is the set of Fibonacci numbers, and V_φ is the set of Lucas numbers (for $n \geq 1$). If β is an integer, then the two systems U_β and V_β are the same, the standard β-ary numeration system.

Proposition 3. *Let β be a Pisot number such that its beta-polynomial $K(X)$ is equal to its minimal polynomial. Let U be any linear sequence with characteristic polynomial equal to $K(X)$ (in particular U_β). The conversion from the linear numeration system V_β to the linear numeration system U (and conversely) is computable by a finite automaton.*

Proof. It comes from the fact that U and V_β have the same characteristic polynomial, which is the minimal polynomial of a Pisot number. Thus normalization in both systems is computable by a finite automaton on any alphabet, and the result follows by Theorem 1. $\square$

7 Multiplicatively dependent numeration systems

First recall that if β is a Pisot number of degree m then, for any positive integer k, β^k is a Pisot number of degree m (see [1]). Two Pisot numbers β and γ will be said to be *multiplicatively dependent* if there exist two positive integers k and ℓ such that $\beta^k = \gamma^\ell$. Then β and γ have the same degree m.

Theorem 2. *Let β and γ be two multiplicatively dependent Pisot numbers. Let U and Y be two linear sequences with characteristic polynomial equal to the minimal polynomial of β and γ respectively. Then the conversion from the Y-numeration system to the U-numeration system is computable by a finite automaton.*

Proof. Set $\delta = \beta^k = \gamma^\ell$. As above, let $V_\beta = (v_n)_{n\geq 0}$ with $v_0 = 1$ and $v_n = \beta_1^n + \cdots + \beta_m^n$ for $n \geq 1$. The conjugates of δ are of the form $\delta_i = \beta_i^k$, for $2 \leq i \leq m$. Set $W = (w_n)_{n\geq 0}$ with $w_n = \delta_1^n + \cdots +$

δ_m^n for $n \geq 1$. Then W is the Lucas-like numeration system associated with δ. Now, for $n \geq 1$, $w_n = v_{kn}$. Thus any W-representation of an integer N of the form $(N)_W = d_k \cdots d_0$ gives a V_β-representation $(N)_{V_\beta} = d_k 0^{k-1} d_{k-1} 0^{k-1} \cdots d_1 0^{k-1} d_0$, and thus the conversion from W-representations to Lucas-like V_β-representations is computable by a finite automaton. The same is true for the conversion from W-representations to V_γ-representations. By Proposition 3 the conversion from Y to V_γ, and that from V_β to U are computable by a finite automaton, and the result follows. $\square$

A set S of natural integers is said to be *U-recognizable* if the set $\{< n >_U \mid n \in S\}$ is recognizable by a finite automaton.

Corollary 1. *Let β and γ be two multiplicatively dependent Pisot numbers. Let U and Y be two linear sequences with characteristic polynomial equal to the minimal polynomial of β and γ respectively. Then a set which is U-recognizable is Y-recognizable as well.*

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Appendix: Quadratic Pisot units

Here we are interested only in the case where β is a quadratic Pisot unit, that is to say the root > 1 of the polynomial $P(X) = X^2 - aX - 1$, with $a \geq 1$, or of the polynomial $P(X) = X^2 - aX + 1$, with $a \geq 3$. We denote the conjugate of β by β', $|\beta'| < 1$. In that case there are nice properties relating the numeration in the systems U_β and V_β and in base β. It is known that, when β is a quadratic Pisot unit, every positive integer has a finite β-expansion [15], the conversion from U_β-representations to β-representations folded around the radix point is computable by a finite automaton [14], and this property is characteristic of quadratic Pisot units [16].

As an example, we give in Table 2 the φ-expansions of the first integers.

N	φ-expansions
1	1.
2	10.01
3	100.01
4	101.01
5	1000.1001
6	1010.0001
7	10000.0001
8	10001.0001
9	10010.0101
10	10100.0101
11	10101.0101

Table 2. φ-expansions of the 11 first integers

We now make the link with the Lucas-like numeration V_β.

Case $\beta^2 = a\beta + 1$. Then $d_\beta(1) = a1$.

First suppose that $a \geq 2$. The following result is a simple consequence of the fact that for $n \geq 1$, $v_n = \beta^n + \beta'^n$ and that $\beta' = -\beta^{-1}$. The signed digit $-d$ is denoted by $\bar{d}$.

Lemma 1. *Let B be a finite alphabet of digits containing A_{V_β}. If $(N)_{V_\beta} = d_k \cdots d_0$, with $d_i \in B$, then $(N)_\beta = d_k \cdots d_0 {\boldsymbol .} \bar{d}_1 d_2 \bar{d}_3 \cdots (-1)^k d_k$.*

Note that the digits in $(N)_\beta$ are elements of the alphabet $\tilde{B} = \{d, \bar{d} \mid d \in B\}$. Then the β-expansion of N is obtained by using the normalization $\nu_{\beta,\tilde{B}^{\mathbb{N}}}$ (which is computable by a finite automaton).

Now we treat the case $a = 1$. The connection between Lucas representations and representations in base the golden mean φ is the following one.

Lemma 2. *Let B be a finite alphabet of digits containing A_L. If $(N)_L = d_k \cdots d_1$, with $d_i \in B$, then $(N)_\varphi = d_k \cdots d_1 0 {\boldsymbol .} \bar{d}_1 d_2 \cdots (-1)^k d_k$.*

As above, the φ-expansion of N is obtained by using the normalization $\nu_{\varphi,\tilde{B}^{\mathbb{N}}}$.

Case $\beta^2 = a\beta - 1$. Then $d_\beta(1) = (a-1)(a-2)^\omega$.

The following lemma is just a consequence of the fact that for $n \geq 1$, $v_n = \beta^n + \beta'^n$ and that $\beta' = \beta^{-1}$.

Lemma 3. *Let B be a finite alphabet of digits containing A_{V_β}. If $(N)_{V_\beta} = d_k \cdots d_0$, with $d_i \in B$, then $(N)_\beta = d_k \cdots d_0 {\boldsymbol .} d_1 \cdots d_k$.*

Proposition 4. *If $d_k \cdots d_0$ is the normal V_β-representation of N then $d_k \cdots d_0 {\boldsymbol .} d_1 \cdots d_k$ is the β-expansion of N.*

Proof. Note that $G(V_\beta) = \{w \in G(U_\beta) \mid w \neq w'(a-1)(a-2)^n,\ n \geq 1\}$. Now, it is enough to show that if $w = d_k \cdots d_0$ is in $G(V_\beta)$, then $d_k \cdots d_1 d_0 d_1 \cdots d_k$ contains no factor in $I = \{(a-1)(a-2)^n(a-1) \mid n \geq 0\}$. First, w has no factor in I since $G(V_\beta) \subset G(U_\beta)$. Second, $d_0 d_1 \cdots d_k$ has no factor in I either, because I is symmetrical. Third, suppose that $g = d_k \cdots d_1 d_0 d_1 \cdots d_k$ is of the form $g = g'(a-1)(a-2)^j(a-2)^{n-j}(a-1)g''$, with $w = g'(a-1)(a-2)^j$. Then $w \notin G(V_\beta)$, a contradiction. $\square$