On lazy representations and Sturmian graphs.*

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Abstract. In this paper we establish a strong relationship between the set of lazy representations and the set of paths in a Sturmian graph associated with a real number α. We prove that for any non-negative integer i the unique path weighted i in the Sturmian graph associated with α represents the lazy representation of i in the Ostrowski numeration system associated with α. Moreover, we provide several properties of the representations of the natural integers in this numeration system.

Keywords: numeration systems, Sturmian graphs, continued fractions.

1 Introduction

In [6] the authors have defined a new structure, the *Sturmian graph* associated with the continued fraction expansion of a real number α. They have also proved that Sturmian graphs have a counting property. In particular, given an infinite Sturmian graph, it can "count" from 0 up to infinity, which means that, for any $i \in \mathbb{N}$, there exists in this Sturmian graph a unique path starting in the initial state having weight i. Recent results on Sturmian graphs and Sturmian words and their generalizations can be found in [2]. The counting property proved in [6] has suggested us to introduce a continued fraction expansion-based numbering, a kind of numeration system that has strong relationships with Sturmian graphs. Despite this link, the new theory that we will introduce in this paper can be described in an independent way. Anyway, we will prove that it is well describable even through Sturmian graphs. We show that there exists a relation between the set of paths in a Sturmian graph and the set of lazy representations. In particular, we prove that for any number i the unique path weighted i in the Sturmian graph associated with α represents the lazy representation in the Ostrowski numeration system associated with α.

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The paper is organized as follows. In the next section, we recall some basic notation on continued fraction expansions of a real number α and introduce Sturmian graphs. In the third section we focus on the representations of the natural integers in numeration systems defined by a basis. In the fourth section we focus on the relationship between the continued fraction expansion of a real number α and numeration systems. In the fifth section we explicit the link between a set of representations, called lazy, and Sturmian graphs. Finally, the last section contains some conclusions.

2 Continued fraction expansions, Sturmian words and Sturmian graphs

If α is a real number, we can expand α as a *simple continued fraction*

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}},$$

which is usually abbreviated as $\alpha = [a_0, a_1, a_2, a_3, \dots]$, where a_0 is some integer ($a_0 \in \mathbb{Z}$) and all the other numbers a_i are positive integers. The integers in the continued fraction expansion of a real number are called *partial quotients*. The expansion may or may not terminate. If α is irrational, this representation is infinite and unique. If α is rational, there are two possible finite representations. Indeed, it is well known that $[a_0, \dots, a_{s-1}, a_s, 1] = [a_0, \dots, a_{s-1}, a_s + 1]$. For references to continued fractions, see [4], [8, Chap. 10], [12], and [13]. In this paper, we only discuss the case where a_i is a positive integer for $i \geq 0$, i.e., α is greater than or equal to 1. Moreover, we will focus on infinite continued fraction expansions. Hence, in our paper α will be an irrational number.

Given the continued fraction expansion of α, it is possible to construct a sequence of rationals $\frac{P_i(\alpha)}{Q_i(\alpha)}$, called "convergents", that converges to α, as well as a sequence of natural numbers $U_\alpha = (l_i)_{i \geq 0}$ such that $l_{i+1} = P_i + Q_i$, for any $i \geq 0$. They are defined by the following rules, with $P_i = P_i(\alpha)$ and $Q_i = Q_i(\alpha)$

$$\begin{array}{llll} P_0 = a_0 & Q_0 = 1 & l_0 = 1 & \\ P_1 = a_1 a_0 + 1 & Q_1 = a_1 & l_1 = a_0 + 1 & \\ P_{i+1} = a_{i+1} P_i + P_{i-1} & Q_{i+1} = a_{i+1} Q_i + Q_{i-1} & l_{i+1} = a_i l_i + l_{i-1} & i \geq 1 \end{array} \tag{1}$$

Notice that for $i \geq 0$, $P_i/Q_i = [a_0, \dots, a_i]$ and that under our assumptions a_i is a positive integer for $i \geq 0$ and hence the sequence $U_\alpha = (l_i)_{i \geq 0}$ is increasing.

As well-known, simple continued fractions play a leading role in the construction of Sturmian words. Indeed, among the different definitions of Sturmian words, one is obtained by applying the *standard method*. For references on Sturmian words and their geometric representation see [5], [9], [11, Chap. 2].

In [6] authors have defined a new structure, the *Sturmian graph* associated with the continued fraction expansion of a real number, and proved that these graphs turn out to be the underlying graphs of compact directed acyclic word graphs of central Sturmian words (see [6] for further details). In this paper we want to deepen the structure of Sturmian graphs in order to establish a connection between them and a particular set of representations of non-negative integers. In order to be self-contained, we give in this paper a direct definition of the semi-normalized infinite Sturmian graph $G'(\alpha)$ associated with a real number α, that can be easily derived from definitions in [6].

Definition 1. *The* semi-normalized infinite Sturmian graph *associated with a real number* α*,* $G'(\alpha)$*, is a weighted semi-normalized infinite graph where each state has outgoing degree* 2*. Moreover if we number each state, the initial one having number* 0*, the arcs are defined in the following way.*

For any $s \geq 0$*, let* $b_s = \sum_{h=0}^{s} a_h$ *and, for any* $i \geq 0$*, let* $s(i)$ *be the smallest integer such that* $i < b_{s(i)}$*. Then the state numbered* i *has an outgoing arc weighted* $l_{s(i)}$ *to state numbered* $i+1$ *and an outgoing arc weighted* $l_{s(i)+1}$ *to state numbered* $1 + b_{s(i)}$*.*

Example 1. Let us consider the irrational number $\alpha = [1, 1, 2, 1, 2, 1, 2, \ldots] = \sqrt{3}$. Sequences $(l_i)_{i\geq 0}$, $(s(i))_{i\geq 0}$, $(b_{s(i)})_{i\geq 0}$ associated with this sequence are described in the following table.

i	0	1	2	3	4	5	6	7	$\cdots$
a_i	1	1	2	1	2	1	2	1	$\cdots$
b_i	1	2	4	5	7	8	10	11	$\cdots$
l_i	1	2	3	8	11	30	41	112	$\cdots$
$s(i)$	0	1	2	2	3	4	4	5	$\cdots$
$b_{s(i)}$	1	2	4	4	5	7	7	8	$\cdots$

Hence, the semi-normalized infinite Sturmian graph $G'(\alpha)$ is the following one

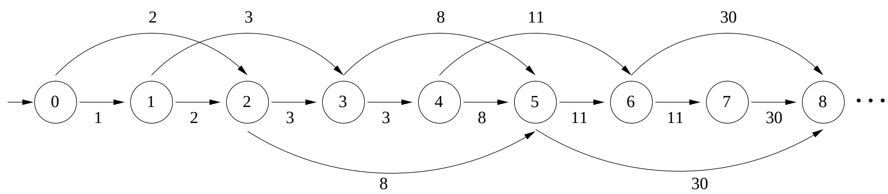

Fig. 1. The semi-normalized infinite Sturmian graph $G'([1, 1, 2, 1, 2, 1, 2, \ldots])$.

Remark 1. Notice that the graphs of Definition 1 are exactly the semi-normalized infinite Sturmian graphs defined in [6]. Indeed, given a real number α, both definitions lead to the same graph $G'(\alpha)$ that is the limit graph of the sequence $G'(\frac{P_n}{Q_n})$, $n \in \mathbb{N}$, where $\frac{P_n}{Q_n}$ is the sequence of convergents of α and $G'(\frac{P_n}{Q_n})$ is defined in [6, Remark 6].

Let us analyze the structure of the semi-normalized infinite Sturmian graphs $G'(\alpha)$. We start with a proposition that characterizes the weights of arcs ingoing a given state.

Proposition 1. *For any $h \geq 1$, every arc ingoing the states in the set $S_h = \{b_{h-1}+1, b_{h-1}+2, \ldots, b_h\}$ has weight l_h and they are the unique arcs having this weight. If $h = 0$ then arcs ingoing the states in the set $S_0 = \{1, \ldots, b_h\}$ have weight l_h and they are the unique arcs having this weight.*
Remark 2. For any $h \geq 0$ the cardinality of the set S_h is obviously a_h.

Next propositions, as well as being interesting in themselves, are important as they will also be useful in the final section where we will show the strong relationship between a particular set of representations of non-negative integers, called *lazy representations*, and the Sturmian graphs.

Proposition 2. *Let us consider an arc (i, j). If there exists a number $h \geq 0$ such that i is smaller than $b_{h-1}+1$ and j is greater than b_h then $i = b_{h-1}$ and $j = b_h + 1$.*

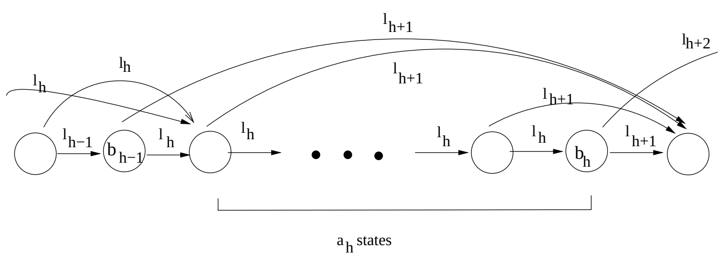

Fig. 2. The local structure of $G'(\alpha)$.

Proposition 3. *For any $h \geq 0$ there exist no paths in the graph having more than a_h arcs weighted by l_h. Moreover, if a path from the initial state reaches state b_h then it contains exactly a_h arcs weighted by l_h.*

Definition 2. *Let us suppose we have an infinite graph G where the states are labeled by the integers greater than or equal to 0. Graph G can be weighted or non-weighted. We say that G is eventually periodic if there exist integer $p > 0$, called the* period, *and $\hat{n} \geq 0$ such that for any $n \geq \hat{n}$ one has that (n, j) is an arc of G if and only if $(n+p, j+p)$ is an arc of G.*

Remark 3. Def. 2 states a property that leaves weights out of consideration.

Proposition 4. *$G'(\alpha)$ is eventually periodic if and only if the continued fraction expansion of α is eventually periodic.*

3 Numeration systems and lazy representations of integers

This section is devoted to the representations of non-negative integers in numeration systems defined by an increasing basis. In particular, we focus on greedy and lazy representations and we give a new algorithm for finding the lazy representation of a non-negative integer N.

The definitions we use are not the standard ones. Indeed our definitions are chosen among all possible "classical" ones because of our new point of view that let lazy representations be linked with Sturmian graphs.

Classically speaking, a numeration system is defined by a pair composed of either a *base* or a *basis*, which is an increasing sequence of numbers, and of an alphabet of digits. Standard numeration systems, such as the binary and the decimal ones, are represented in the first manner, i.e., through a base, while we are interested in the second one, i.e., through a basis. The reader may consult the survey [11, Chapt. 7]. More formally, we give the following definitions.

Definition 3. *Let $U = (u_i)_{i\geq 0}$ be a increasing sequence of integers with $u_0 = 1$, the* basis. *A U-representation of a non-negative integer N is a word $d_k \cdots d_0$ where the digits d_i, $0 \leq i \leq k$, are integers, such that $N = \sum_{i=0}^{k} d_i u_i$.*

Set $c_i = \lceil \frac{u_{i+1}}{u_i} \rceil - 1$. The U-representation $d_k \cdots d_0$ is said to be legal *if for any i, $0 \leq i \leq k$, one has that $0 \leq d_i \leq c_i$. The set $A = \{c \in \mathbb{N} \mid \exists i, 0 \leq c \leq c_i\}$ is the* canonical alphabet. *By convention, the representation of 0 is ϵ.*

Even if it is a very natural concept, the definition of legal U-representation is new. The concept of legal U-representation represents an essential requisite of our theory, because it allows us to link lazy representations with Sturmian graphs. In the following we will prove several properties of this representations.

Example 2. Let $F = (F_n)_{n\geq 0} = 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ be the sequence of Fibonacci numbers obtained inductively in the following way: $F_0 = 1, F_1 = 2, F_{n+1} = F_n + F_{n-1}, n \geq 1$. The canonical alphabet is equal to $A = \{0, 1\}$. It is the well-known Fibonacci numeration system. An F-representation of the number 31 is 1010010. Another representation is 1001110. By definition, every F-representation of a non-negative integer N over $A = \{0, 1\}$ is legal.

Among all possible U-representations of a given non-negative integer N, one is known as the *greedy* (or *normal*) U-representation of N. It is the largest one in the lexicographic order.

Definition 4. *A* greedy *(or* normal*) U-representation of a given non-negative integer N is the word $d_k \cdots d_0$, where the most significant digit $d_k > 0$ and $d_j \geq 0$ for $0 \leq j < k$, and satisfying for each i, $0 \leq i \leq k$, $d_i u_i + \cdots + d_0 u_0 < u_{i+1}$.*

Now we consider another peculiar U-representation, linked to greedy representations, that is called the *lazy U-representation* of a natural number N.

Definition 5. *A word $e_k \cdots e_0$, with the most significant digit $e_k > 0$, is the* lazy U-representation *of a natural number N if it is the smallest legal U-representation of N in the radix order.*

Example 3. If we consider the Fibonacci numeration system of previous examples, the lazy F-representation of the number 31 is 111110.

Definition 6. *Let $w = d_k \cdots d_0$ be a U-representation. Denote $\underline{d_i} = c_i - d_i$, and by extension, $\underline{w} = \underline{d_k} \cdots \underline{d_0}$ the complement of w.*

By using the previous definition, we can link the greedy and lazy U-representations of a natural number N. For $k \geq 0$ set $C_k = \sum_{i=0}^{k} c_i u_i$.

Proposition 5. *A U-representation w of a number N, $u_k \leq N < u_{k+1}$, is greedy if and only if its complement $\underline{w}$ is the lazy U-representation of the number $N' = C_k - N$, up to eliminating all the initial zeros.*

Proposition 5 allows us to characterize the lazy U-representation of N.

Corollary 1. *A U-representation $e_k \cdots e_0$ of a number N is lazy if and only if for each i, $0 \leq i \leq k$, $e_i u_i + \cdots + e_0 u_0 > C_i - u_{i+1}$.*

Let us denote by m_i the greatest in the radix order of greedy U-representations of length i. Clearly m_i is the greedy representation of the integer $u_i - 1$. Recall that $m_0 = \varepsilon$. Denote by $M(U) = \{m_i \mid i \geq 0\}$.

Proposition 6.
1. *A U-representation $w = d_k \cdots d_0$ of a natural number N is greedy if and only if for any i, $0 \leq i \leq k$, $d_i \cdots d_0 \leq m_{i+1}$ (in the radix order).*
2. *A U-representation $w = d_k \cdots d_0$ of a natural number N is lazy if and only if for any i, $0 \leq i \leq k$, $\underline{d_i} \cdots \underline{d_0} \leq m_{i+1}$ (in the radix order).*

A direct consequence of Proposition 6 is the following result.

Corollary 2. *For each $i \geq 0$ the number $u_i - 1$ has a unique legal U-representation.*

Now we are ready to give an algorithm computing lazy U-representations.
Lazy algorithm:
Let $k = k(N)$ be the integer such that $C_{k-1} < N \leq C_k$. This ensures that the length of the lazy U-representation of N is $k + 1$.
Compute a U-representation $d_k \cdots d_0$ of $N' = C_k - N$ by the following algorithm: let $d_k = q(N', u_k)$ and $r_k = r(N', u_k)$, and, for $i = k-1, \ldots, 0$, $d_i = q(r_{i+1}, u_i)$ and $r_i = r(r_{i+1}, u_i)$. Then $d_k \cdots d_0$ is a greedy U-representation of N' with possibly initial zeros.
By Proposition 5, the lazy U-representation of N is $\underline{d_k} \cdots \underline{d_0}$.

Denote by $\mathsf{Greedy}(U)$ and by $\mathsf{Lazy}(U)$ the sets of greedy and lazy U-representations of the non-negative integers. The regularity of the set $\mathsf{Greedy}(U)$ has been extensively studied. The following result is in [7, Prop. 2.3.51, Prop. 2.6.4].

Proposition 7. *The set $\mathsf{Greedy}(U)$ is regular if and only if the set $M(U)$ of greatest U-representations in the radix order is regular.*

Then by Proposition 6 Item 2 follows the following result.

Proposition 8. *The set $\mathsf{Lazy}(U)$ is regular if and only if the set $\mathsf{Greedy}(U)$ is regular if and only if the set $M(U)$ of greatest U-representations in the radix order is regular.*

4 Ostrowski numeration system and lazy representations

In this section we are interested in the relationship between the continued fraction expansion of a real number α and numeration systems. Let us go into details by first recalling a numeration system, originally due to Ostrowski, which is based on continued fractions, see [1, p. 106] and [3]. This numeration system, called Ostrowski numeration system, can be viewed as a generalization of the Fibonacci numeration system.

Definition 7. *The sequence $(Q_i(\alpha))_{i\geq 0}$, defined in (1), of the denominators of the convergents of the infinite simple continued fraction of the irrational $\alpha = [a_0, a_1, a_2, \ldots] > 0$ forms the basis of the* Ostrowski numeration system based on *α.*

Proposition 9. *Let $\alpha = [a_0, a_1, a_2, \ldots] > 0$. The sequence $U_\alpha = (l_i)_{i\geq 0}$ associated in (1) with α is identical to the sequence $(Q_i)_{i\geq 0} = (Q_i(\beta))_{i\geq 0}$ defined in (1) for the number $\beta = [b_0, b_1, b_2, b_3, \ldots] = [0, a_0 + 1, a_1, a_2, \ldots]$.*

It is easy to verify that there exists a relation between β and α.

Proposition 10. *If α is greater than or equal to 1 then $\beta = \frac{1}{\alpha+1}$.*

In what follows, $\alpha = [a_0, a_1, \ldots]$ is greater than 1 and thus the sequence $U_\alpha = (l_i)_{i\geq 0}$ is increasing. In view of Definition 7 and Proposition 9, the Ostrowski numeration system based on $\beta = \frac{1}{\alpha+1}$ and the numeration system with basis U_α, in the sense of Definition 3, coincide. So we call the numeration system with basis U_α the *Ostrowski numeration system associated with α*.

By Definition 3, a U_α-representation $d_k \cdots d_0$ is legal if $d_i \leq a_i$, for any i such that $0 \leq i \leq k$, since in this case $c_i = a_i = \lceil \frac{l_{i+1}}{l_i} \rceil - 1$, thus the canonical alphabet is $A = \{a \in \mathbb{N} \mid \exists i, 0 \leq a \leq a_i\}$.

The Ostrowski numeration system associated with the golden ratio φ is the Fibonacci numeration system defined in Example 2. It is folklore that a U_φ-representation of an integer is greedy if and only if it does not contain any factor of the form 11, and is lazy if and only if it does not contain any factor of the form 00. We now extend this property to U_α-representations for any $\alpha > 1$. The greedy case is classical, see [1].

Proposition 11. *1. A U_α-representation $w = d_k \cdots d_0$ is greedy if and only if it contains no factor $d_i d_{i-1}$, $1 \leq i \leq k$, with $d_i = a_i$ and $d_{i-1} > 0$.*

2. A U_α-representation $w = d_k \cdots d_0$ is lazy if and only if it contains no factor $d_i d_{i-1}$, $1 \leq i \leq k$, with $d_i = 0$ and $d_{i-1} < a_{i-1}$.

Now we give a characterization of m_{i+1}, the greatest in the radix order of greedy U_α-representations of length $i + 1$.

Lemma 1. *For any $i \geq 0$, $m_{i+1} = a_i 0 a_{i-2} 0 \cdots a_2 0 a_0$ if i is even, and $m_{i+1} = a_i 0 a_{i-2} 0 \cdots a_1 0$ if i is odd.*

The sequence $(a_i)_{i\geq 0}$ is *eventually periodic* if there exist integers $m \geq 0$ and $p \geq 1$ such that $a_{i+p} = a_i$ for $i \geq m$. It is a classical result that the sequence $(a_i)_{i\geq 0}$ is eventually periodic if and only if α is a quadratic irrational.

The regularity of the set of the greedy U_α-representations has been already studied. In particular, it is proved in Shallit [14] and Loraud [10] that the set of greedy expansions in the Ostrowski numeration system associated with $\alpha > 1$ is regular if and only if the sequence $(a_i)_{i\geq 0}$ is eventually periodic.

Lemma 2. *The set $M(U_\alpha)$ is regular if and only if the sequence $(a_i)_{i\geq 0}$ is eventually periodic.*

Next Proposition follows from Proposition 8 and Lemma 2.

Proposition 12. *The sets of greedy expansions and of lazy expansions in the Ostrowski numeration system associated with $\alpha > 1$ are regular if and only if the sequence $(a_i)_{i\geq 0}$ is eventually periodic if and only if α is a quadratic irrational.*

5 Sturmian graphs and lazy representations

The goal of this section is to establish a deep connection between the set of lazy representations and the set of paths in a well defined *Sturmian graph*.

We start by recalling a classical definition.

Definition 8. *Let G be a weighted graph, the weight of a path in G is the sum of the weights of all arcs in the path.*

Epifanio *et al.* [6] have proved several properties on finite and infinite Sturmian graphs. Among them, an important result regards a *counting property*. Concerning infinite Directed Acyclic Graphs (DAGs), this property can be stated in the following way.

Definition 9. *An infinite semi-normalized weighted DAG G' has the $(h, +\infty)$-*counting property*, or, in short, counts from h to $+\infty$, if any non-empty path starting in the initial state has weight in the range $h \cdots +\infty$ and for any i, $i \geq h$, there exists a unique path that starts in the initial state and has weight i.*

Starting from this definition, at the end of the proof of Proposition 35 in [6] it has been proved the following result concerning the counting property of infinite Sturmian graphs, that is very useful in the next.

Theorem 1. *For any positive irrational α, $G'(\alpha)$ can count from 0 up to infinity.*

Moreover, we can prove the following result that characterizes the path weights of states in $G'(\alpha)$.

Proposition 13. *For any state $i > 1$ in $G'(\alpha)$ let b_s be the maximum non-negative integer such that $i > b_s$ (cf. Def. 1). The maximum weight of the paths from the initial state ending in i is $g(i) = \sum_{j=0}^{s} a_j l_j + (i - b_s) l_{s+1}$. The only paths from the initial state ending in state $i = 0$ ($i = 1$ resp.) have weights 0 (1 resp.).*

Corollary 3. *For any state i in $G'(\alpha)$, all paths from the initial state ending in i are weighted N, where N is such that $g(i-1)+1 \le N \le g(i)$ and, conversely, if N is such that $g(i-1)+1 \le N \le g(i)$ then N is the weight of a path ending in i.*

Example 4. Let us consider the irrational number $\alpha = [1, 1, 2, 1, 2, 1, 2, \ldots]$ and its semi-normalized infinite Sturmian graph $G'(\alpha)$ represented in Figure 1. The following table represents, for any state i in $G'(\alpha)$, values b_s and s of Proposition 13, as well as the minimum, min_i, and the maximum, max_i, non-negative integers among the weights of all paths from the initial state ingoing in i.

i	0	1	2	3	4	5	6	7	$\cdots$
b_s			1	2	2	4	5	5	$\cdots$
s			0	1	1	2	3	3	$\cdots$
min_i	0	1	2	4	7	10	18	29	$\cdots$
max_i	0	1	3	6	9	17	28	39	$\cdots$

Before coming to the main result of the paper, we define the correspondence between paths and representations.

Definition 10. *Let $G'(\alpha)$ be the semi-normalized Sturmian graph associated to α and U_α be the Ostrowsky numeration system associated with α. For any $N \in \mathbb{N}$, we say that the U_α representation $d_k \cdots d_0$ of N* corresponds *to the unique path weighted N in $G'(\alpha)$ if, for any $i \ge 0$, d_i represents the number of consecutive arcs labeled Q_i in the path.*

Theorem 2. *Let N be a non-negative integer. A U_α-representation of N is lazy if and only if it corresponds to the unique path weighted N in the semi-normalized Sturmian graph $G'(\alpha)$.*

More precisely, the importance of this result lies on the fact that for any number i, we can connect the unique path weighted i in the Sturmian graph associated with α and the lazy representation of i in the Ostrowski numeration system associated with α.

Example 5. For $\alpha = [1, 1, 2, 1, 2, 1, 2, \ldots]$ the lazy U_α-representation of 7 is equal to 201. Recall that in this case $U_\alpha = \{1, 2, 3, 8, 11, 30, 41, 112, \ldots\}$. Then 201 gives the decomposition $7 = 1 + 2 \cdot 3$.

On the other hand, given the graph $G'(\alpha)$ of Figure 1, there exists a unique path starting from 0 with weight 7. It is labelled 1, 3, 3, that exactly corresponds to the lazy U_α-representation of $7 = 1 + 2 \cdot 3$.

Next corollary is an immediate consequence of previous theorem, of Proposition 4 and of Proposition 12.

Corollary 4. *The set of all the U_α-representations that correspond to paths in $G'(\alpha)$ is regular if and only if $G'(\alpha)$ is eventually periodic.*

6 Conclusions

This paper contains a neat and natural theory on lazy representations in Ostrowski numeration system based on the continued fraction expansion of a real number α. This theory provides a natural understanding of the Sturmian graph associated with α, even better the study of Sturmian graphs gave us the idea to formalize and to develop this theory. Indeed, the set of lazy representations is naturally linked with the set of paths in the Sturmian graph associated with α. It would be interesting to deepen this theory in order to prove other properties of the representations in the Ostrowski numeration system based on the continued fraction expansion of a real number α through the Sturmian graph associated with α. Moreover, it would be nice to find algorithms for performing the elementary arithmetic operations.

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