Beta-Integers as Natural Counting Systems for Quasicrystals

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Abstract. Recently, discrete sets of numbers, the β-integers $\mathbb{Z}_\beta$, have been proposed as numbering tools in quasicrystalline studies. Indeed, there exists a unique numeration system based on the irrational $\beta > 1$ in which the β-integers are all real numbers with no fractional part. These β-integers appear as being quite appropriate to describing some quasilattices relevant to quasicrystallography when precisely β is equal to $\frac{1+\sqrt{5}}{2}$ (golden mean τ), to $1+\sqrt{2}$, or to $2+\sqrt{3}$, i.e. when β is one of the self-similarity ratios observed in quasicrystalline structures. As a matter of fact, β-integers are natural candidates for coordinating quasicrystalline nodes, and also the Bragg peaks beyond a given intensity in corresponding diffraction patterns: they could play the same role as ordinary integers do in crystallography.

In this paper, we prove interesting algebraic properties of the sets $\mathbb{Z}_\beta$ when β is a "quadratic unit PV number", a class of algebraic integers which includes the quasicrystallographic cases. We completely characterize their respective Meyer additive and multiplicative properties

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F, \quad \mathbb{Z}_\beta \, \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + G,$$

where F and G are finite sets, and also their respective Galois conjugate sets $\mathbb{Z}'_\beta$. These properties allow one to develop a notion of a quasiring $\mathbb{Z}_\beta$. We hope in this way initiate a sort of algebraic quasicrystallography in which we can understand quasilattices which be "module on a quasiring" in $\mathbb{R}^d$: $\Lambda_\beta = \sum_i \mathbb{Z}_\beta \mathbf{e}_i$. We give also some two-dimensional examples with $\beta = \tau$.

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1. Introduction

Studies on physical and mathematical properties of deterministic aperiodic structures have been very intensive in the last years, strongly motivated by the experimental discovery of quasicrystals (see [18, 19, 36]).

In this context, quasilattices can be defined as mathematical discrete sets supporting atomic sites in quasiperiodic material structures like quasicrystals. They play the same role as the lattices do for crystals. Various interesting definitions of quasilattices have been proposed in the past, dating back to 1984 with the discovery of the first quasicrystalline alloy. Most of these definitions are of geometrical nature, sticking to crystalline lattice theory through the celebrated Cut and Project method (see [33]), or issued from involved packing construction in real space like the generalized dual method [24, 34]. More "algebraic" approaches were initiated, by several authors (see for instance [1, 2]). Recent school or workshop proceedings give a good account of this original interactive field mixing number theory, lattices and experimental physics (see in Refs. [26] and [28] for instance).

It has also been acknowledged that most of the algebraic and functional approaches to quasilattices, e.g. the Cut and Project method and involved Fourier analysis, should mention pioneer results obtained more than 25 years ago by Y. Meyer [25, 26]. The notion of a quasilattice $\Lambda \subset \mathbb{R}^d$ proposed by Meyer rests upon the idea that the quasilattice should be "almost" closed under subtraction

$$\Lambda - \Lambda \subset \Lambda + F, \tag{1}$$

where F is some finite set. For most of such quasilattices Λ should follow from (1) well-controlled properties for their diffraction spectrum, but it is not true in general (as Lagarias has shown in [22, 23]).

On the other hand, one can deal with lattice internal laws within an equivalence class of quasilattices which differ from each other by the addition of finite sets. For the definition and application of a Meyer set in the problem of finite generation of quasilattices, we refer to [28]. Some authors make use of the name "quasicrystal" to designate the structures they build in $\mathbb{R}^2$ and $\mathbb{R}^3$. We here prefer the generic term of quasilattice, since certain real material structures are called quasicrystal.

An interesting algebraic definition of quasilattices [5, 27] has been introduced more than five years ago by Moody and Patera, and their possible symmetry groups and semi-groups have been investigated [3, 4, 31]. More recently, one of us [12, 13, 14, 15, 16] suggested to study algebraic models of quasilattices based on countable sets of numbers, denoted by $\mathbb{Z}_\beta$, and named *β-integers*, where β is some real number. These quasilattices Λ_β are scaling invariant under dilation by $\beta > 1$, and $\mathbb{Z}_\beta$ is precisely the counting system with origin, i.e. the numerical frame, in which we should think about structural properties of Λ_β, exactly like the first crystallographers did with lattices and ordinary integers.

As a matter of fact, these sets $\mathbb{Z}_\beta$ are natural candidates for coordinating quasicrystalline nodes in 1, 2 or 3 dimensions, and also the Bragg peaks in related diffraction patterns [9, 17]. In the observed cases :

$$\beta = \tau = \frac{1+\sqrt{5}}{2} = 2\cos\frac{\pi}{5} \qquad \text{(penta or decagonal quasilattices)} \tag{2}$$

$$\beta = \gamma = 1+\sqrt{2} = 1+2\cos\frac{\pi}{4} \quad \text{(octogonal case)} \tag{3}$$

$$\beta = \delta = 2+\sqrt{3} = 2+2\cos\frac{\pi}{12} \quad \text{(dodecagonal case).} \tag{4}$$

Generically, $\mathbb{Z}_\beta$ is obtained by means of a finite algorithm where β is a *Pisot-Vijayaraghavan number* or more simply Pisot number. i.e. an algebraic integer $\beta > 1$, which is solution to the irreducible polynomial of the form

$$X^m = a_{m-1}X^{m-1} + \cdots + a_1 X + a_0, \quad a_i \in \mathbb{Z} \tag{5}$$

such that all other solutions $\beta^{(i)}$ of (5) (Galois conjugates of β) have modulus strictly smaller than 1,

$$\beta^{(0)} = \beta\ ,\ |\beta^{(i)}| < 1, \quad i = 1, 2, \cdots, m-1.$$

Therefore $d-$dimensional discrete sets of the form

$$\Lambda \stackrel{\text{def}}{=} \sum_{i=1}^{d} \mathbb{Z}_\beta e_i, \quad \text{where } \{e_i\} \text{ is a basis in } \mathbb{R}^d, \quad d = 1, 2, 3 \tag{6}$$

can advantageously play the role of "grid frame" or ("millimeter paper" if $d = 2$) for labelling quasicrystalline atomic sites in real space, exactly as integer lattices ($\mathbb{Z}-$modules) are appropriate to real crystalline structures. Indeed, it seems that most of quasilattices obtained by Cut and Project [20] or by algebraic "filtering" [27] within the dense $\mathbb{Z}[\beta]$-module are supported by sets of the type Λ. In Figures 1 and 2 we give simple demonstrative examples of such labelling properties.

Those β-integer quasilattices are neither translationally nor rotationally invariant of course, although they contain rotationally invariant subsets obtained through cut and project method. Moreover they still display nice algebraic and geometrical features, which straightforwardly generalize the ones for lattices. We already mentioned their similarity property under scaling by β:

$$\beta\Lambda \subset \Lambda,$$

which is due to

$$\beta\,\mathbb{Z}_\beta \subset \mathbb{Z}_\beta.$$

We shall eventually see how a concept of a quasiring and modules on a quasiring emerges from the study of additive and multiplicative properties of $\mathbb{Z}_\beta$:

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + X, \tag{7}$$

$$\mathbb{Z}_\beta \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + Y, \tag{8}$$

where X and Y are to be determined in a non-ambiguous way. The aim of this paper is to present some interesting properties of β-integers and to give (partial) answer to (7) and (8). Motivations were already present in [14]. We here give complete proofs of some results claimed in [14] and we extend them substantially to more general cases. Of course, we are mainly concerned by the three quasicrystallographic cases (2), (3) and (4), but we shall also give original results for general quadratic unit Pisot numbers, namely those β which are solution to

$$x^2 = ax + 1 \quad a \in \mathbb{Z},\ a \geq 1\ , \tag{9}$$

$$x^2 = ax - 1 \quad a \in \mathbb{Z},\ a \geq 3\ . \tag{10}$$

In the next section, we shall present the basic definitions concerning $\mathbb{Z}_\beta$ on one hand, and concerning β-quasilattices and Meyer sets in $\mathbb{R}^d$ on the other hand. These definitions will be followed by a first result about the Meyer property of $\mathbb{Z}_\beta$ when β is Pisot.

Section 3 is devoted to the ubiquitous golden mean $\beta = \tau$. Indeed icosahedral or decagonal quasicrystals are among the most stable quasicrystalline phases and the irrational τ is the simplest Pisot number in many aspects. It is the reason why it deserves a specific and pedagogical treatment on its own. As a matter of fact, we prove the following:

$$\begin{aligned} \mathbb{Z}_\tau + \mathbb{Z}_\tau &\subset \frac{\mathbb{Z}_\tau}{\tau^2} \subset \mathbb{Z}_\tau + \{0, \pm\frac{1}{\tau}, \pm\frac{1}{\tau^2}\}, \\ \mathbb{Z}_\tau \mathbb{Z}_\tau &\subset \frac{\mathbb{Z}_\tau}{\tau^2} \subset \mathbb{Z}_\tau + \{0, \pm\frac{1}{\tau}, \pm\frac{1}{\tau^2}\}. \end{aligned}$$

In Section 4, we give precise inclusions for general quadratic unit Pisot β of the type (9-10), and these results are also new. In Section 5, we discuss the notion of quasiring structure which emerges from our results. In particular we show how the existence of a quasiaddition as an internal law $\dot{+}$ for $\mathbb{Z}_\beta$ allows one to generate the whole set in an inductive way, starting from the "seed" $\{-1, 1\}$. Finally in Section 6, we shall consider τ-quasilattices in the plane, of the form (6), in order to give a pedagogical insight of the importance of $\mathbb{Z}_\beta$ in quasicrystalline studies.

2. Delaunay-Meyer sets, β-expansions and β-integers

A Delaunay set in space is typically a mathematical model for the set of atomic sites in large material structures. It fills the space in a not too dense and not too discrete manner. More precisely we define (see [26]):

Definition 2.1. A subset Λ of $\mathbb{R}^d$ is a *Delaunay set* if there exist two radii $R_2 > R_1 > 0$ such that each ball with radius R_1, whatever be its location, shall contain at most one point from Λ while each ball with radius R_2, whatever be its location, shall contain at least one point from Λ.

A Meyer quasilattice is a Delaunay set which is endowed with arithmetic properties: it is closed under subtraction modulo a finite set. More precisely we define:

Definition 2.2. A *Meyer quasilattice* Λ is a Delaunay set in $\mathbb{R}^d$ such that

$$\Lambda - \Lambda \subset \Lambda + F, \tag{11}$$

where F is a finite set in $\mathbb{R}^d$.

In the sequel we shall mainly deal with sets symmetrical with respect to space inversion $\Lambda \rightarrow -\Lambda$. Such a Meyer quasilattice then obeys $\Lambda + \Lambda \subset \Lambda + F$ and it becomes possible to provide Λ with a quasiaddition law when F is given a nonambiguous characterization. Indeed, if we have $x, y \in \Lambda$, $x + y = \eta + f$ where $\eta \in \Lambda, f \in F$, then we define $x \dot{+} y = \eta$. This internal law is commutative but not associative. Its natural framework is the class of equivalent Meyer sets defined in the following way.

Definition 2.3. Two subsets Λ and Λ' of $\mathbb{R}^d$ are *equivalent modulo finite sets* if there exist two finite sets F and F' such that

$$\Lambda \subset \Lambda' + F' \ \text{ and } \ \Lambda' \subset \Lambda + F$$

and we then write $\Lambda \ \sim \ \Lambda'$.

It is clear that $\Lambda + \Lambda \subset (\Lambda + F) \sim \Lambda$ if Λ is a Meyer symmetrical set. If one imposes a Meyer quasilattice Λ to be scaling invariant under dilation by $\beta > 1$:

$$\beta\Lambda \subset \Lambda, \tag{12}$$

then β has to be an algebraic integer. More precisely Meyer has proved the following assertion [25].

Theorem 2.1. If Λ is a Meyer quasilattice, if $\beta > 1$ is a real number, and if (12) holds true, then β is either a Pisot number or is a Salem number. Conversely, for each dimension d and each Pisot or Salem number β, there exists a Meyer quasilattice Λ in $\mathbb{R}^d$ such that $\beta\Lambda \subset \Lambda$.

We recall that a *Salem number* $\beta = \beta^{(0)}$ is an algebraic integer $\beta > 1$ such that all algebraic conjugates $\beta^{(i)}$, $i > 0$, lie within the closed unit disk and at least one of them lies on the unit circle.

Our construction of (possible) quasilattices has something to do with this remarkable connection between a class of algebraic integers and self-similarity of discrete subsets in $\mathbb{R}^d$. For any $\beta > 1$, there exist countable canonical sets in $\mathbb{R}$ which are β-scaling invariant. Their construction rests upon notions first introduced by A. Rényi [32], and later developed in [7].

Let β be non-integer and $\beta > 1$. The *β-expansion* of an arbitrary positive real number x is the series $(\xi_l)_{-\infty \leq l \leq j}$ such that

$$x = \sum_{i=-\infty}^{j} \xi_i \beta^i$$

where j is the highest integer such that

$$\beta^j \leq x < \beta^{j+1},$$

the positive integers ξ_l assume their values in the alphabet

$$\{0, 1, 2, \ldots, [\beta]\}, \quad ([\beta] \text{ is the integer part of } \beta)$$

and are computed by using the so-called *greedy algorithm*. One recursively defines

$$\begin{aligned} &\xi_j = [x/\beta^j], \qquad r_j = \{x/\beta^j\} \ \text{(the fractional part of } \ x/\beta^j), \\ &\text{and for } \ l < j, \quad \xi_l = [\beta r_{l+1}], \qquad r_l = \{\beta r_{l+1}\} \ldots, \\ &\text{finally, if } j < 0, \text{ we put } \xi_0 = \xi_{-1} = \cdots = \xi_{j+1} = 0. \end{aligned}$$

For short we also write

$$x = \xi_j \xi_{j-1} \xi_{j-2} \cdots \xi_0 . \xi_{-1} \xi_{-2} \xi_{-3} \cdots \ , \ (\text{e.g. } 2 = 10.01 \text{ when } \beta = \tau).$$

The highest power of β appearing in the β-expansion of x will be called the *β-degree* of x and will be denoted by $\deg_\beta(x)$, so we have $j = \deg_\beta(x)$. When a β-expansion ends in infinitely many zeroes, it is said to be *finite*, and the ending zeroes are omitted.

We will denote by

$$int(x) = \xi_j \beta^j + \cdots + \xi_0$$

the "integral part" of x and by

$$frac(x) = \xi_{-1} \beta^{-1} + \xi_{-2} \beta^{-2} + \cdots$$

the "fractional" part of x.

The set of real numbers which have a zero fractional part in their β-expansion is named set of *β-integers* and is denoted by

$$\mathbb{Z}_\beta \stackrel{\text{def}}{=} \{\pm(\xi_j \beta^j + \xi_{j-1} \beta^{j-1} + \cdots + \xi_1 \beta + \xi_0)\} = \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+),$$

where $\mathbb{Z}_\beta^+$ designates the set of non-negative β-integers.

Some configurations $\xi_j \xi_{j-1} \cdots \xi_l \cdots$ in the above definition are not possible. What is allowed and what is forbidden in the set of β-expansions is completely determined by what is called the *Rényi β-expansion of 1*:

$$\begin{aligned} d(1,\beta) &= t_1 \beta^{-1} + t_2 \beta^{-2} + \cdots \\ &= 0.t_1 t_2 \cdots t_l \cdots, \end{aligned}$$

where $t_l \in \{0, 1, \ldots, [\beta]\}$. This expansion is reminiscent of the identity $1 = 0.99\ldots 9\ldots$ in the decimal system. It is obtained by the following process:

$$t_1 = [\beta], \;\; r_1 = \{\beta\}, \cdots, t_l = [\beta r_{l-1}], \; r_l = \{\beta r_{l-1}\} \cdots$$

or, equivalently,

$$t_l = [\beta T_\beta^{l-1}(1)],$$

where

$$T_\beta(x) = \beta x (\bmod 1).$$

In this context, note that the greedy-algorithm coefficient ξ_l of a real number $x \in [0,1)$ is also equal to

$$\xi_{-l} = [\beta T_\beta^{l-1}(x)].$$

We then have the β-expansion rule [29]:

Proposition 2.2. No infinite sequence of positive integers is present in any β-expansion if itself and all its (one-sided) shifted are lexicographically larger or equal to:

$$t_1 t_2 \ldots \quad \textit{if the latter is infinite,}$$

and to:

$$(t_1 t_2 \ldots t_{m-1}(t_m - 1))^\omega \quad \textit{if} \quad d(1,\beta) = 0.t_1 t_2 \ldots t_{m-1} t_m \;\; \textit{is finite.}$$

$(\quad)^\omega$ *means that the word within* $(\quad)$ *is indefinitely repeated.*

Therefore, once $d(1,\beta)$ is known, it becomes possible (in principle, but it may turn out to be unpracticable!) to build up $\mathbb{Z}_\beta$ by following the lexicographical order of the allowed sequences.

The countable set $\mathbb{Z}_\beta$ is naturally self-similar and symmetrical with respect to the origin:

$$\beta \mathbb{Z}_\beta \subset \mathbb{Z}_\beta \;, \;\; \mathbb{Z}_\beta = -\mathbb{Z}_\beta.$$

It tiles the line with a finite or infinite number of intervals separating two nearest neighbours $x_i < x_{i+1}$ with lengths of the tiles $l_i = x_{i+1} - x_i$. Now we are concerned by sets $\mathbb{Z}_\beta$ that be Delaunay and possibly Meyer. This is at a certain extent assured for the two following important results.

Theorem 2.3. ([7]) *Suppose β is a Pisot number. Then the Rényi β-expansion of 1 is eventually periodic,*

$$d(1, \beta) = 0.t_1 t_2 \ldots t_m (t_{m+1} \ldots t_{m+p})^\omega.$$

When β is a Pisot number, then it follows that $\mathbb{Z}_\beta$ is a self-similar tiling of the line with a finite set of different tiles. The lengths of the tiles are $\{T_\beta^i(1),\ 0 \le i \le m+p-1\}$ (see [37]). More precisely the lengths assume their values in the set

$$\begin{gathered} 1, \beta - t_1, \beta^2 - t_1\beta - t_2, \ldots, \\ \beta^{m+p-1} - t_1\beta^{m+p-2} - \cdots - t_{m+p-1}. \end{gathered}$$

Hence, if β is Pisot, then $\mathbb{Z}_\beta$ is Delaunay, and the radii R_1 and R_2 are given by

$$R_1 \ = \ \min_i T_\beta^i(1) - \epsilon \ , \ \text{where } \epsilon > 0 \text{ is a suitable small number,}$$

and

$$R_2 \ = \ \max_i T_\beta^i(1) = 1.$$

Our aim now is to prove that, when β is a Pisot number, then the set $\mathbb{Z}_\beta$ of β-integers is a Meyer set. We will denote by $\mathbb{Z}[\beta]$ the following extension ring

$$\mathbb{Z}[\beta] = \{m + n\beta;\ m, n \in \mathbb{Z}\}.$$

Lemma 2.1 *Let $R \ge 0$ and let*

$$F_R = \{frac(z) \ \mid \ z = a_k\beta^k + \cdots + a_1\beta + a_0,\ a_i \in \mathbb{Z},\ |a_i| \le R\}.$$

If β is a Pisot number, then F_R is a finite subset of $\mathbb{Z}[\beta]$.

Proof. Let $z = \sum_{-\infty \le i \le j} z_i\beta^i$ be the β-expansion of z. We have $frac(z) = \sum_{i \ge 1} z_{-i}\beta^{-i} = z - int(z) = \sum_{i=0}^k a_i\beta^i - \sum_{i=0}^j z_i\beta^i$. Since $0 \le z_i \le [\beta]$ and $|a_i| \le R$, $frac(z)$ is a polynomial from $\mathbb{Z}[\beta]$, the coefficients of which are bounded by $R + [\beta]$. The lemma is then a consequence of the fact that $frac(z) \in [0, 1[$ and of the following classical result. $\square$

Lemma 2.2. (see [35] Lemma 6.6) *Let β be a Pisot number and let $R \ge 0$. Then the set F_R is discrete.*

Theorem 2.4. Let $\beta > 1$ be a Pisot number. Then the set $\mathbb{Z}_\beta$ of β-integers is a Meyer set.

Proof. First consider the sum of two elements $x = x_k\beta^k + \cdots + x_0$ and $y = y_l\beta^l + \cdots + y_0$ from $\mathbb{Z}_\beta^+$. Then $z = x + y$ is of the form $z = a_j\beta^j + \cdots + a_0$, with $0 \le a_i \le 2[\beta]$. By Lemma 2.1., $F_{2[\beta]} = \{f(z); z \in \mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+\}$ is finite. Thus

$$\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta^+ + F_{2[\beta]},$$

Now, suppose that $x \ge y$, and let $z = x - y = a_m\beta^m + \cdots + a_0$, $-[\beta] \le a_i \le [\beta]$. As above one gets

$$\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta \pm F_{[\beta]}.$$

□

Finally, let us mention the following result giving more precisions about the set $\mathbb{Z}_\beta + \mathbb{Z}_\beta$:

Theorem 2.5. ([11]) *Let β be a Pisot number. There exists $L = L(\beta)$ having the following properties. Let $x, y \in \mathbb{Z}_\beta$. If $x + y$ (resp. $x - y$) has a finite β-expansion, then*

$$x + y \ \ (resp.\ x - y) \ \in \ \mathbb{Z}_\beta/\beta^L.$$

3. Algebraic properties of the set of τ-integers $\mathbb{Z}_\tau$

The golden mean τ can be considered as the simplest Pisot number. It is the smallest one among the totally real (i.e. whose all conjugates are real) Pisot numbers (see [6]). Because τ and $-\frac{1}{\tau} = \tau'$ are solutions to

$$X^2 = X + 1,$$

the Rényi expansion of 1 reads:

$$d(1, \tau) = 0.11\ .$$

This means that no τ-expansion sequence $\xi_j \cdots \xi_0 . \xi_{-1} \cdots \xi_l \cdots$ displays two adjacent 1. More precisely, any positive real number has a τ-expansion

$$x = \sum_{l=-\infty}^{j} \xi_l \tau^l, \quad \text{with } \ \xi_l \in \{0, 1\} \ \text{ and } \ \xi_l\xi_{l+1} = 0.$$

The subset of $\mathbb{R}$ defined by

$$\mathbb{Z}_\tau \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid \sum_{i=0}^{j} \xi_i \tau^i \text{ is the } \tau\text{-expansion of } |x|\}$$

is the set of τ-integers. Positive τ-integers are thus represented by a finite string of $0'$s and $1'$s, with condition that no run of two adjacent $1'$s occur. It is well known that this corresponds to the representation of natural numbers in the Fibonacci numeration system where the Fibonacci numbers are defined by

$$f_{n+2} \stackrel{\text{def}}{=} f_{n+1} + f_n, \; f_1 \stackrel{\text{def}}{=} 2, \; f_0 \stackrel{\text{def}}{=} 1.$$

So there is an explicit bijection between integers and the τ-integers (see for instance [8]),

$$n = \sum_{i=0}^{j_n} \xi_i f_i \longrightarrow x_n = \sum_{i=0}^{j_n} \xi_i \tau^i. \tag{13}$$

Note that properties of Fibonacci representations of natural numbers have been investigated by many people (see [21]). Any τ-integer is an element of the algebraic ring

$$\mathbb{Z}[\tau] = \{m + n\tau \mid m, n \in \mathbb{Z}\}.$$

The latter is actually identical to the set of real numbers which have a finite τ-expansion (see [11]).

An interesting question then arises. For what values of m and n is the combination $m + n\tau$ a τ-integer? We have plotted on Figure 3 the lattice points (m, n) in $\mathbb{Z}^2$ such that $m + n\tau \in \mathbb{Z}_\tau$. They are clearly all the lattice points lying within the bands defined by

$$\tau x - \tau^2 < y < \tau x + \tau$$

in the first quadrant, and by

$$\tau x - \tau < y < \tau x + \tau^2$$

in the opposite sign quadrant. Note that the band width is $\tau^3/\sqrt{1+\tau^2}$. This "inverse" Cut and Project method (see [20]) leads to the following definition of the positive and negative τ-integers, denoted respectively by $\mathbb{Z}_\tau^+$ and $\mathbb{Z}_\tau^-$

$$\mathbb{Z}_\tau^+ = \{m + n\tau \; ; \; m, n \in \mathbb{Z}, \; m, n \geq 0, \; -1 < m - \frac{n}{\tau} < \tau\} \tag{14}$$

$$\mathbb{Z}_\tau^- = \{m + n\tau \; ; \; m, n \in \mathbb{Z}, \; m, n \leq 0, \; -\tau < m - \frac{n}{\tau} < 1\}. \tag{15}$$

The algebraic meaning of (14) and (15) involves the standard ring automorphism of $\mathbb{Z}[\tau]$,

$$x = m + n\tau \quad \longrightarrow \quad x' = m + n\tau' = m - \frac{n}{\tau}. \tag{16}$$

Let us introduce the following "sieving" procedure from [27] in order to select within the dense ring $\mathbb{Z}[\tau]$ a Delaunay subset

$$\Sigma^P = \{x \in \mathbb{Z}[\tau] \mid x' \in P\},$$

where P is some bounded subset in $\mathbb{R}$. Then we can check from the above that

$$\mathbb{Z}_\tau^+ = \text{ positive part of } \Sigma^{(-1,\tau)}, \tag{17}$$

$$\mathbb{Z}_\tau^- = \text{ negative part of } \Sigma^{(-\tau,1)}, \tag{18}$$

and

$$\Sigma^{(-1,1)} \subset \mathbb{Z}_\tau \subset \Sigma^{(-\tau,\tau)}. \tag{19}$$

The inclusion relations (19) also mean that it is sufficient to sieve from the discrete set $\mathbb{Z}_\tau$ in order to obtain $\Sigma^{(-1,1)}$

$$\Sigma^{(-1,1)} = \{x \in \mathbb{Z}_\tau \;;\; x' \in (-1,1)\ \}. \tag{20}$$

Scaling (19) with arbitrary powers of τ leads to the interesting chain [12, 14] of embeddings (see [4] and Lemma 4.1 below for an algebraic proof)

$$\Sigma^{(-\tau^j,\tau^j)} \subset \mathbb{Z}_\tau/\tau^j \subset \Sigma^{(-\tau^{j+1},\tau^{j+1})},\ j \in \mathbb{Z}. \tag{21}$$

Let us now turn to the Meyer property of the set $\mathbb{Z}_\tau$. In this specific case, a first estimate for the finite set F in (11) is given by

$$F = \{0, \pm\frac{1}{\tau}, \pm\frac{1}{\tau^2}\}.$$

However if we do not think $\mathbb{Z}_\tau + \mathbb{Z}_\tau \subset \mathbb{Z}_\tau + F$ in terms of unique τ-expansion, more precise inclusions exist (see Section 4).

In the present content let us now prove the following:

Theorem 3.1.

$$\mathbb{Z}_\tau + \mathbb{Z}_\tau \subset \frac{\mathbb{Z}_\tau}{\tau^2}. \tag{22}$$

This result restricted to the positive part of $\mathbb{Z}_\tau$, can be found (with a totally different proof) in [10].

Proof. We will define for any $N \in \mathbb{N}$, $B_N = \{X \in {\mathbb{Z}_\tau}^+ \mid X < \tau^N\}$. It is easy to see that $B_N = \tau B_{N-1} \cup (1 + \tau^2 B_{N-2})$ and the assertion results from the two following lemmas.

Lemma 3.1. For any $N \in \mathbb{N}$ there exists N' such that

$$B_N + B_N \subset \tau^{-2} B_{N'}. \tag{23}$$

Proof. For $N = 0$ it is trivial and for $N = 1$ we use only $1 + 1 = \tau + \frac{1}{\tau^2}$. We suppose that (23) is valid till N and for $N + 1$ we have

$$\begin{aligned}
B_{N+1} + B_{N+1} &= (\tau B_N \cup (1+\tau^2 B_{N-1})) \; + \; (\tau B_N \cup (1+\tau^2 B_{N-1})) \\
&= (\tau B_N + \tau B_N) \cup (1 + \tau B_N + \tau^2 B_{N-1}) \cup (2 + \tau^2 B_{N-1} + \tau^2 B_{N-1}) \\
&\subset \tau(B_N + B_N) \cup (1 + \tau(B_N + B_N)) \cup (2 + \tau^2(B_{N-1} + B_{N-1})) \\
&\subset \tau^{-1} B_{N'} \cup (1 + \tau^{-1} B_{N'}) \cup (2 + B_{(N-1)'}).
\end{aligned}$$

In order to finish the proof we need to prove that the following holds true

$$1 + \tau^{-1} B_N \subset \; \tau^{-2} B_{N+2}, \tag{24}$$

$$2 + B_N \subset \; \tau^{-2} B_{N+3}, \tag{25}$$

hence for $(N+1)'$ we can take $\max\{N'+2, (N-1)'+3\}$.

By direct calculation we show that (24) is true for $N = 0, 1, 2$. Furthermore by induction and by using $B_{N+1} = B_N \cup (\tau^N + B_{N-1})$ we have

$$\begin{aligned}
1 + \tau^{-1} B_{N+1} &= (1 + \tau^{-1} B_N) \cup (\tau^{N-1} + 1 + \tau^{-1} B_{N-1}) \\
&\subset \tau^{-2} B_{N+2} \cup (\tau^{N-1} + \tau^{-2} B_{N+1}) \subset \tau^{-2} B_{N+3}.
\end{aligned}$$

The proof of (25) is similar. $\square$

Lemma 3.2. For any $N \in \mathbb{N}$ there exists N' such that

$$B_N \; - \; B_N \; \subset \; \pm\tau^{-2} B_{N'}. \tag{26}$$

Proof. For $N = 0$ and $N = 1$ it is trivial and for $N = 2$, we have only $\tau - 1 = \frac{1}{\tau}$. We suppose that (26) is valid till N, and for $N + 1$ we have

$$\begin{aligned}
B_{N+1} - B_{N+1} &= (\tau B_N \cup (1+\tau^2 B_{N-1})) \; - \; (\tau B_N \cup (1+\tau^2 B_{N-1})) \\
&= \pm((\tau B_N - \tau B_N) \cup (1 + \tau^2 B_{N-1} - \tau B_N) \cup \tau^2 (B_{N-1} - B_{N-1})) \\
&\subset \pm(\tau^{-1} B_{N'} \cup (1 + \tau^2 (B_{N-1} - B_{N-1})) \cup (1 - \tau + \tau^2(B_{N-1} - B_{N-1})) \cup B_{(N-1)'}) \\
&\subset \pm(\tau^{-1} B_{N'} \cup B_{(N-1)'} \cup (1 \pm B_{(N-1)'}) \cup (-\tau^{-1} \pm B_{(N-1)'}))
\end{aligned}$$

In order to finish the proof, we have to show that

$$B_N - 1 \subset \; \tau^{-1} B_{N+1} \cup \{-1\}, \tag{27}$$

and

$$B_N - \tau^{-1} \subset \; \tau^{-2} B_{N+2} \cup \{-\tau^{-1}\}. \tag{28}$$

For $x = 0$ and 1 it is obvious and for $x > 1, \; x \in B_N$, we shall take its τ-expansion

$$x = \sum_{i=k+2}^{N-1} c_i \tau^i + \tau^k$$

and (27) then follows from the two identities

$$\tau^k - 1 = \tau^{k-1} + \tau^{k-3} + \cdots + \tau^2 + \tau^{-1}\ , \quad \text{for } k > 0 \text{ even}$$

$$\tau^k - 1 = \tau^{k-1} + \tau^{k-3} + \cdots + \tau\ , \quad \text{for } k > 0 \text{ odd.}$$

The inclusion (28) is a simple consequence of (27) by using $(B_N - \tau^{-1}) = \tau^{-1}(\tau B_N - 1)$.
$\square$

Remark 3.1. We can tell more about (22) or (23). It can be shown that for all $n, m \in \mathbb{Z}$ there exists $f \in \{0, \pm(1/\tau), \pm(1/\tau^2)\}$ such that

$$x_{m+n} = x_m + x_n + f,$$

where $n \to x_n \in \mathbb{Z}_\tau$ is the bijection (13).

Remark 3.2. The content of Theorem 3.1 means that translational invariance does not hold for $\mathbb{Z}_\tau$. We cannot consider each point of the latter as the origin of another $\mathbb{Z}_\tau$ supporting and supported by the first one. However (22) also means that there is "almost" coincidence. Both sets are equal up to $\pm\frac{1}{\tau}$ as it can be guessed from Figure 3. Of course the same can be asserted about all Bravais quasilattices of the type (6).

The second result we want to give concerns the multiplicative properties of $\mathbb{Z}_\tau$. Indeed we have the following:

Theorem 3.2.

$$\mathbb{Z}_\tau\, \mathbb{Z}_\tau \subset \frac{\mathbb{Z}_\tau}{\tau^2}.$$

Proof. We shall show again that, for any $N \in \mathbb{N}$, there is a $N^\star$ such that

$$B_N \ \times\ B_N \ \subset\ \tau^{-2} B_{N^\star}. \tag{29}$$

For $N = 0, 1$ it is trivial. We suppose that (29) is valid till N, and for $N + 1$ we have

$$\begin{aligned}
B_{N+1} \times B_{N+1} &= (\tau B_N \cup (1 + \tau^2 B_{N-1}))\ \times\ (\tau B_N \cup (1 + \tau^2 B_{N-1})) \\
&= \tau^2 B_N \times B_N \cup (1 + \tau^2 B_{N-1}) \times \tau B_N \cup (1 + \tau^2 B_{N-1}) \times (1 + \tau^2 B_{N-1}) \\
&\subset B_{N^\star} \cup \tau(B_N + \tau^2 B_{N-1} \times B_N) \cup (1 + \tau^2(B_{N-1} + B_{N-1}) + \tau^4 B_{N-1} \times B_{N-1}) \\
&\subset \tau^{-2} B_{N^\star+2} \cup \tau(B_N + B_{N^\star}) \cup (1 + B_{(N-1)'} + \tau^2 B_{(N-1)^\star}) \\
&\subset \tau^{-2} B_{N^\star+2} \cup \tau^{-2} B_{(N^\star)'+1} \cup (1 + B_{(N-1)'} + \tau^2 B_{(N-1)^\star}) \\
&\subset \tau^{-2}(B_{N^\star+2} \cup B_{(N^\star)'+1} \cup B_{M'})
\end{aligned}$$

where M', N' are from Lemma 3.1., $M = \max\{(N-1)', (N-1)^\star\}$ and for $(N+1)^\star$ we can take $(N+1)^\star = \max\{(N^\star)' + 1, N^\star + 2, M'\}$. $\square$

4. Algebraic properties of $\mathbb{Z}_\beta$ for β a quadratic unit Pisot.

We now address the question of characterizing finite sets appearing in (7-8) when β is a generic quadratic unit Pisot, i.e. is solution to (9) or (10). Note that the two quasicrystallographic Pisot γ and δ from (3-4) belong to these classes. We could of course attempt to extend in this more general case the inductive methods we have employed in the previous section. However we become very soon puzzled how to manage difficulties increasing with the values of a.

4.1. Case where β is the root > 1 of the polynomial $x^2 - ax - 1, a \geq 1$.

In that case, the canonical alphabet is $A = \{0, \cdots, a\}$, the β-expansion of 1 is finite, $d(1,\beta) = 0.a1$, and every positive number of $\mathbb{Z}[\beta]$ has a finite β-expansion [11].

Let $'$ be the Galois automorphism $' : \beta \longrightarrow -\frac{1}{\beta}$. Recall that if P is some bounded subset of $\mathbb{R}$ with non empty interior, Σ^P denotes the set $\{z \in \mathbb{Z}[\beta]; z' \in P\}$. We recall that this algebraic filtering of the dense $\mathbb{Z}[\beta]$ in order to abtain the Delone set Σ^P is equivalent to the *Cut and Project method* [4, 30]. We first have

Lemma 4.1 Let β be the root > 1 of $x^2 - ax - 1$, $a \geq 1$. Then

(i) Let $z \in \mathbb{Z}[\beta]$, $z > 0$ and let $(z_i)_{-m\leq i\leq n}$ be the $\beta-$expansion of z with $z_{-m} \neq 0$. If $m > 0$ then $z \notin \Sigma^{(-1,\beta)}$, moreover if $z < \dfrac{1}{\beta}$ then $z \notin \Sigma^{(-\beta^2,\beta)}$.

$$(ii) \quad \mathbb{Z}_\beta^+ = \Sigma^{(-1,\beta)} \cap \mathbb{R}^+ \quad \mathbb{Z}_\beta^- = \Sigma^{(-\beta,1)} \cap \mathbb{R}^-, \tag{30}$$

$$(iii) \quad \Sigma^{(-1,1)} \subset \mathbb{Z}_\beta \subset \Sigma^{(-\beta,\beta)} \subset \frac{\mathbb{Z}_\beta}{\beta}, \tag{31}$$

$$(iv) \quad \mathbb{Z}_\beta + \{0, \pm\frac{1}{\beta}, \ldots, \pm\frac{a}{\beta}\} \subset \frac{\mathbb{Z}_\beta}{\beta^2}$$

Proof.
(i) Recall that $z' = \Sigma_{i=-m}^{n}(-1)^i z_i \beta^{-i}$. Since $z_m \neq 0$ then by admissibility $z_{-m+1} \leq a-1$ and the other z_i's are $\leq a$. If m is even, $z' > -a(\ldots+\beta^{m-5}+\beta^{m-3})-(a-1)\beta^{m-1}+\beta^m = -a\frac{\beta^{m-1}}{\beta^2-1} + \beta^{m-1} + \beta^{m-2} = \beta^{m-1}$, hence $z' \notin (-1,\beta)$. If m is odd, $z' < a(\ldots + \beta^{m-5} + \beta^{m-3}) + (a-1)\beta^{m-1} - \beta^m = -\beta^{m-1}$, hence $z' \notin (-1,\beta)$. Note that if $0 < z < \frac{1}{\beta}$ then $m \geq 2$.
(ii) Let $z \in \mathbb{Z}_\beta^+$ and let $(z_i)_{0\leq i\leq n}$ be its β-expansion, $z = \sum_0^n z_i\beta^i$ and $z' = \sum_0^n(-1)^i z_i\beta^{-i}$. From $\ldots - z_3\beta^{-3} - z_1\beta^{-1} \leq z' \leq z_0 + z_2\beta^{-2} + \ldots$ we know that $-1 < z' < \frac{a\beta^2}{\beta^2-1} = \beta$. Hence $\mathbb{Z}_\beta^+ \subset \sum^{(-1,\beta)} \cap \mathbb{R}^+$.
Conversely, let $z \geq 0$, $z \in \mathbb{Z}[\beta]$ such that $z' \in (-1,\beta)$ then it follows from *(i)* that the β-expansion of z has the form $(z_i)_{-m\leq i\leq n}$, where m has to be ≥ 0, hence $z \in \mathbb{Z}_\beta^+$.
(iii) The proof is a direct consequence of *(ii)*.

(iv) It is easy to see that $\mathbb{Z}_\beta^+ + \{0, \frac{1}{\beta}, \ldots, \frac{a}{\beta}\} \subset \Sigma^{(-\beta^2,\beta)} \subset \dfrac{\mathbb{Z}_\beta^+}{\beta^2}$. On the other hand, taking any nonzero $z \in \mathbb{Z}_\beta^+$, we know that $z \geq 1$. By using the inequality $\beta + a\beta < \beta^3$ we can assert that $z - \{0, \frac{1}{\beta}, \ldots, \frac{a}{\beta}\} \subset \Sigma^{(-1,\beta^3)} \cap \mathbb{R}^+ \subset \dfrac{\mathbb{Z}_\beta^+}{\beta^2}$. $\square$

Proposition 4.1. Let β be the root > 1 of $x^2 - ax - 1$, $a \geq 1$. Then

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \{0, \pm\frac{1}{\beta}\} \subset \frac{\mathbb{Z}_\beta}{\beta^2}.$$

Proof. Pick any $x, y \in \mathbb{Z}_\beta^+$, then $z = x + y \in \mathbb{Z}[\beta]$, with $z \geq 0$, hence $z' \in (-2, 2\beta)$. If $z' \in (-1, \beta)$, then Lemma 4.1(ii) gives us that $z \in \mathbb{Z}_\beta^+$. If $z' \in [\beta, 2\beta)$, then we define $g = z + \frac{1}{\beta}$ which fulfils $g > 0$, $g' = z' - \beta \in [0, \beta)$. Again, by using the previous Lemma we have that $g \in \mathbb{Z}_\beta^+$ and so $z = g - \frac{1}{\beta} \in \mathbb{Z}_\beta^+ - \frac{1}{\beta}$. Similarly, if $z' \in (-2, -1]$ then $z > 1$ and we define a positive number $g = z - \frac{1}{\beta}$ with $g' = z' + \beta \in (-1, \beta)$ which is in $\mathbb{Z}_\beta^+$. Hence we see that $z = g + \frac{1}{\beta} \in \mathbb{Z}_\beta^+ + \frac{1}{\beta}$.

Pick any $x, y \in \mathbb{Z}_\beta^+$, such that $z = x - y \geq 0$, hence $z' \in (-1 - \beta, 1 + \beta)$. If $z' \in (-1, \beta)$ then $z \in \mathbb{Z}_\beta^+$. If $z' \in [\beta, 1 + \beta)$, then $g = z + \frac{1}{\beta}$ fulfils $g > 0$, $g' = z' - \beta \in (0, 1)$, hence $z = g - \frac{1}{\beta} \in \mathbb{Z}_\beta^+ - \frac{1}{\beta}$. If $z' \in (-1 - \beta, -1] \subset (-\beta^2, \beta)$ then we apply Lemma 4.1(i) and we get that $z \geq \frac{1}{\beta}$. We can define a nonnegative number $g = z - \frac{1}{\beta}$, $g' = z' + \beta \in (-1, \beta)$ which is in $\mathbb{Z}_\beta^+$. Hence $z = g + \frac{1}{\beta} \in \mathbb{Z}_\beta^+ + \frac{1}{\beta}$. Other cases are symmetrical. $\square$

Proposition 4.2. Let β be the root > 1 of $x^2 - ax - 1$, $a \geq 1$. Then

$$\mathbb{Z}_\beta \times \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \{0, \pm\frac{1}{\beta}, \ldots, \pm\frac{a}{\beta}\} \subset \frac{\mathbb{Z}_\beta}{\beta^2}.$$

Proof. Pick any $x, y \in \mathbb{Z}_\beta$ such that $z = xy \geq 0$. From Lemma 4.1(ii) we get that $z' \in (-\beta, \beta^2)$. If $z' \in (-1, \beta)$, then $z \in \mathbb{Z}_\beta^+$. If $z' \in (-\beta, -1]$, we apply Lemma 4.1(i) and we get that $z \geq \frac{1}{\beta}$. We define a nonnegative $g = z - \frac{1}{\beta}$ with $g' = z' + \beta \in (0, \beta)$ which is in $\mathbb{Z}_\beta^+$, hence $z = g + \frac{1}{\beta} \in \mathbb{Z}_\beta^+ + \frac{1}{\beta}$. If $z' \in [(k-1)\beta, k\beta)$, $k \in \{2, \ldots, a\}$, then $g = z + \frac{k-1}{\beta}$ fulfils $g > 0$, $g' \in [0, \beta)$ and so $g \in \mathbb{Z}_\beta^+$. So we can rewrite z as $z = g - \frac{k-1}{\beta} \in \mathbb{Z}_\beta - \frac{k-1}{\beta}$. If $z' \in [a\beta, \beta^2 = a\beta + 1)$, we put $g = z + \frac{a}{\beta}$. Then $g > 0$, $g' \in (0, 1)$ and $z \in \mathbb{Z}_\beta^+ - \frac{a}{\beta}$.
Finally, for $x, y \in \mathbb{Z}_\beta$ such that $z = xy < 0$, we know from the above that $-z \in \mathbb{Z}_\beta^+ + \{0, \pm\frac{1}{\beta}, \ldots, \pm\frac{a}{\beta}\}$. $\square$

4.2. Case where β is the root > 1 of the polynomial $x^2 - ax + 1$, $a \geq 3$.

There the canonical alphabet is $A = \{0, \cdots, a-1\}$, the β-expansions of 1 is eventually periodic, $d(1, \beta) = 0.(a-1)(a-2)^\omega$, and every number of $\mathbb{Z}^+[\beta]$ (the cone of first degree polynomials in β with non-negative integer coefficients) has a finite β-expansion [11].

The interesting fact is that, if $z \in \mathbb{Z}_\beta^+$ and its β-expansion is $z = \sum_{i=0}^k z_i\beta^i$, then the β-expansion of $(z)'$ ($'$ is the Galois automorphism $\beta \longrightarrow \frac{1}{\beta}$) is $(z)' = \sum_{i=-k}^0 z_i\beta^i$ where $z_i = z_{-i}$.

Lemma 4.2. Let β be the root > 1 of $x^2 - ax + 1$, $a \geq 3$ and $z \in \mathbb{Z}[\beta] \cap \mathbb{R}^+$. Then

$$(i) \quad z \text{ has a finite } \beta\text{−expansion if and only if } z' \in (0, \infty),$$

$$(ii) \quad \mathbb{Z}_\beta^+ = \Sigma^{[0,\beta)} \cap \mathbb{R}^+ \;\; and \;\; \mathbb{Z}_\beta^- = \Sigma^{(-\beta,0]} \cap \mathbb{R}^-, \tag{32}$$

$$(iii) \quad \mathbb{Z}_\beta \subset \Sigma^{(-\beta,\beta)} \subset \mathbb{Z}_\beta + \{0, \pm\frac{1}{\beta}\}. \tag{33}$$

Proof. *(i)* If $0 < z = \sum_{-m}^n z_i\beta^i$, where $m, n \in \mathbb{N}$, $z_i \geq 0$, then $z' = \sum_{-m}^n z_i\beta^{-i} > 0$.
If $z = c\beta + d > 0$, $c, d \in \mathbb{Z}$, $z' \in (0, \infty)$, then $c/\beta + d > 0$ and so $c + d\beta > 0$.
∘ If c, $d \geq 0$, then z is in $\mathbb{Z}^+[\beta]$, and due to [11] we know that z has a finite β-expansion.
∘ Let $c < 0, d > 0$. Since there exists $l \in \mathbb{N}$ such that $\beta^{l-1} \leq -c < \beta^l$, there exists $k \in \mathbb{N}$, $k \leq l$ such that $\beta^k z \in \mathbb{Z}^+[\beta]$. To find such a k, we go through the following *procedure:* From $z = c\beta + d > 0$ and $\beta = a - 1/\beta$, it follows that $ca + d > \frac{c}{\beta}$. If $(ca + d) \geq 0$, then $\beta z = (ca + d)\beta - c \in \mathbb{Z}^+[\beta]$. Otherwise $\beta z = c_1\beta + d_1 > 0$, $c_1 < 0$, $d_1 > 0$ and $c_1 a + d_1 > \frac{ca+d}{\beta} > \frac{c}{\beta^2}$. Since $\frac{c}{\beta^l} > -1$, it follows that there exists a $k \leq l$ such that $\beta^k z \in \mathbb{Z}^+[\beta]$ and this implies that z has a finite β-expansion.
∘ Let $c > 0, d < 0$. The previous case implies that $\beta z' = d\beta + c > 0$ has a finite β-expansion and so z has a finite β-expansion.
(ii) Let $z \in \mathbb{Z}_\beta^+$ and let $\{z_i\}_0^n$ be its β-expansion. We have $z' = z_0 + z_1\beta^{-1} + \cdots + z_n\beta^{-n}$. Since $(z_i)_{0\leq i\leq n}$ is a β-expansion, we have $z_n \cdots z_0 \leq_{lex} (a-1)(a-2)\cdots(a-2)$, where $\leq_{lex}$ denotes the lexicographic ordering. In fact, it is easy to see that the forbidden blocks are of the form $(a-1)(a-2)\cdots(a-2)(a-1)$, thus if $z_n \cdots z_0$ is a β-expansion then $z_0 \cdots z_n$ is a β-expansion as well. From this it follows that $0 \leq z' = z_0 + z_1\beta^{-1} + \cdots + z_n\beta^{-n} < \beta$. Conversely, let $z \geq 0$, $z \in \mathbb{Z}[\beta]$ such that $z' \in [0, \beta)$. From *(i)* we know that the β-expansion of z is finite, $z = \sum_{-m}^n z_i\beta^i$, $z_{-m} \neq 0$, $m, n \in \mathbb{N}$. If $m = 0$, then $z \in \mathbb{Z}_\beta^+$. If $m \neq 0$, then $z = \sum_0^n z_i\beta^i + \sum_{-m}^{-1} z_i\beta^i$, and $z' = \sum_0^n z_i\beta^{-i} + \sum_{-m}^{-1} z_i\beta^{-i}$, where $0 \leq \sum_{i=0}^n z_i\beta^{-i} < \beta$. Since $z_{-m} \neq 0$ we have $\sum_{-m}^{-1} z_i\beta^{-1} \geq \beta^m$ and hence $z' \notin [0, \beta)$. Therefore $m = 0$. Hence $\mathbb{Z}_\beta^+ = \Sigma^{[0,\beta)} \cap \mathbb{R}^+$. The proof of $\mathbb{Z}_\beta^- = \Sigma^{(-\beta,0]} \cap \mathbb{R}^-$ is similar.
(iii) The first inclusion is a direct consequence of *(ii)*.
Let $z \in \Sigma^{(-\beta,\beta)}$. If $z \geq 0$ and $z' \in [0, \beta)$, then $z \in \mathbb{Z}_\beta^+$ by (i). If $z \geq 0$ and $z' \in (-\beta, 0)$, then $z' + \beta = (z + \frac{1}{\beta})' \in (0, \beta)$ and $z + \frac{1}{\beta} \geq 0$. Thus by *(i)*, $z + \frac{1}{\beta} \in \mathbb{Z}_\beta^+$ and $z \in \mathbb{Z}_\beta^+ + \{-\frac{1}{\beta}\}$. The case $z < 0$ is symmetrical. □

Proposition 4.3 Let β be the root > 1 of $x^2 - ax + 1,\ a \geq 3$. Then

$$(i) \quad \mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \frac{\mathbb{Z}_\beta^+}{\beta},$$

$$(ii) \quad \mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta + \{0, \pm\frac{1}{\beta}\}.$$

Proof. *(i)* Using the fact, that $\beta > 2$ and Lemma 4.2(ii), we find that

$$\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \Sigma^{(0,2\beta)} \subset \frac{\mathbb{Z}_\beta^+}{\beta}.$$

(ii) In general the difference of two positive β-integers may have an infinite eventually periodic β-expansion. For instance, the β-expansion of $\beta - 1$ is equal to $(a-2).(a-2)^\omega$. Nevertheless, $\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+ \subset \Sigma^{(-\beta,\beta)} \subset \mathbb{Z}_\beta + \{0, \pm\frac{1}{\beta}\}$ by Lemma 4.2(iii). $\square$

Proposition 4.4. Let β be the root > 1 of $x^2 - ax + 1,\ a \geq 3$. Then

$$\mathbb{Z}_\beta \times \mathbb{Z}_\beta \subset \frac{\mathbb{Z}_\beta}{\beta}.$$

Proof. Let $x, y \in \mathbb{Z}_\beta^+$ and $z = xy \geq 0$. Since x' and $y' \in [0, \beta)$, $z' = x'y' \in [0, \beta^2)$. Then $\frac{z'}{\beta} = (\beta z)' \in [0, \beta)$ and by Lemma 4.2(ii), $\beta z \in \mathbb{Z}_\beta^+$, hence $z \in \frac{\mathbb{Z}_\beta^+}{\beta}$. $\square$

Final Remark: Note again (see also 17-19) that the above results (30-31,32-33) concerning inclusion of sets are crucial for understanding the labelling role played by these β-integers, as we particularly stressed on in the introduction.

5. Quasiring structures and Modules on Quasirings.

We wish here to emphasize a mathematical aspect that emerges from discrete structures we have described in this paper. Precisely this concept of equivalence classes of Meyer quasilattices and related additive and multiplicative laws. We first consider our simplest example of a Meyer quasilattice, i.e. the set of τ-integers $\mathbb{Z}_\tau$. Let us restrict the Meyer definition of equivalence modulo finite sets to these F's that are finite subset of the ring $\mathbb{Z}[\tau]$. Clearly we still have

$$\mathbb{Z}_\tau + \mathbb{Z}_\tau = \mathbb{Z}_\tau \pmod{F}, \quad \mathbb{Z}_\tau \mathbb{Z}_\tau = \mathbb{Z}_\tau \pmod{F}$$

or, equivalently, in terms of equivalence classes

$$\dot{\mathbb{Z}}_\tau = \{\mathbb{Z}_\tau + F \ ; \ F \subset \mathbb{Z}[\tau] \text{ is finite}\};$$

$$\dot{\mathbb{Z}}_\tau + \dot{\mathbb{Z}}_\tau = \dot{\mathbb{Z}}_\tau, \quad \dot{\mathbb{Z}}_\tau \dot{\mathbb{Z}}_\tau = \dot{\mathbb{Z}}_\tau.$$

If we concentrate onto the elements of $\mathbb{Z}_\tau$ as representatives of the elements of the set $\dot{\mathbb{Z}}_\tau$, the above laws induce the following ones on the τ-integers. If $x+y = \eta + f, \quad f \in F = \{0, \pm\frac{1}{\tau}, \pm\frac{1}{\tau^2}\}$ we shall write $x \dot{+} y = \eta$, and similarly $x \dot{\times} y = \theta$, if $xy = \theta + g, \; g \in F$. The mathematical definition of this precisely relies upon the τ-numeration system and is clearly nonambiguous: if $x, y \in \mathbb{Z}_\tau$, and

$$|x+y| = \sum\nolimits_{-2}^{k} c_i \tau^i \ , \quad |xy| = \sum\nolimits_{-2}^{k} d_i \tau^i \tag{34}$$

then

$$x \dot{+} y = \mathrm{sign}(x+y) \sum\nolimits_{0}^{k} c_i \tau^i \ , \quad x \dot{\times} y = \mathrm{sign}(xy) \sum\nolimits_{0}^{k} d_i \tau^i . \tag{35}$$

Therefore $\dot{+}$ and $\dot{\times}$ map from $\mathbb{Z}_\tau \times \mathbb{Z}_\tau$ onto $\mathbb{Z}_\tau$.

Despite this evidence of strong departure from a ring structure, we shall adopt the term of quasiring structure. Similarly, any quasilattice of the type (6) in $\mathbb{R}^d$ will be said endowed with the structure of a module on the quasiring $\mathbb{Z}_\tau$. The reason lying behind this choice of terminology is our conviction that structural properties of Λ are very close to that of $\mathbb{Z}$-modules. For instance, it is well known that lattices are the orbit of themselves considered as a symmetry group acting on a finite subset. We have something similar with the quasiring $\mathbb{Z}_\tau$. Using the mapping $\dot{+}$ it is easy to build up $\mathbb{Z}_\tau$ from a "seed" set similarly to the crystal case. The starting set is $\mathbb{Z}_\tau^0 = \{0\}$. We denote $\Gamma = \{\pm 1\}$ and put $\mathbb{Z}_\tau^{k+1} = \mathbb{Z}_\tau^k \dot{+} \Gamma$. Then $\mathbb{Z}_\tau = \bigcup_{k=0}^{\infty} \mathbb{Z}_\tau^k$, i.e. any $x \in \mathbb{Z}_\tau$, can be reached from the starting point 0 through a finite number of quasiadditions involving elements in Γ. In this sense Γ can be considered as a "growth set". The interpretation of that fact in the Fibonacci numeration system is the successor function, which maps the Fibonacci representation of the natural number n onto the Fibonacci representation of $n+1$. Actually, we have a similar result for any quadratic unit Pisot numbers.

As for (34,35) we define laws $\dot{+}$, $\dot{-}$, $\dot{\times}$ by truncating the β-expansion of $x+y$, $x-y$, xy respectively, after the radix point. By the previous results, we know that fractional parts (i.e. parts after radix point) of these β-expansions belong to finite sets. More precisely, if $(c_i)_{-\infty \le i \le k}$ is the β-expansion of $|x+y|$, then

$$x \dot{+} y = \mathrm{sign}(x+y) \sum_{i=0}^{k} c_i \beta^i .$$

If $(c_i)_{-\infty \le i \le k}$ is the β-expansion of $|x-y|$ then

$$x \dot{-} y = \mathrm{sign}(x-y) \sum_{i=0}^{k} c_i \beta^i .$$

And if $(c_i)_{-\infty \le i \le k}$ is the β-expansion of $|xy|$ then

$$x \dot{\times} y = \mathrm{sign}(xy) \sum_{i=0}^{k} c_i \beta^i .$$

Proposition 5.1. Let β be a quadratic unit Pisot, and let $\Gamma = \{\pm 1\}$. Then

$$\mathbb{Z}_\beta = \cup_{k\geq 0} \mathbb{Z}_\beta^k, \text{ where } \mathbb{Z}_\beta^0 = \{0\} \text{ and } \mathbb{Z}_\beta^{k+1} = \mathbb{Z}_\beta^k \dot{+} \Gamma.$$

Proof. By construction we know that $\cup_{k\geq 0} \mathbb{Z}_\beta^k \subset \mathbb{Z}_\beta$. On the other hand assume that

$$\mathcal{S} = \{|x| \in \mathbb{Z}_\beta^+ | x \in \mathbb{Z}_\beta, x \notin \cup_{k\geq 0} \mathbb{Z}_\beta^k\} \neq \emptyset. \tag{36}$$

We pick the minimum of the set $\mathcal{S}$, $s_{\min} = \min\{x \in \mathcal{S}\}$. We note that $s_{\min} > 1$ and we distinguish two possible cases:

∘ Let β be the root > 1 of $x^2 = ax + 1, a \geq 1$.

By using Lemma 4.1(ii) we know that $s'_{\min} \in (-1, \beta)$. Hence if $s'_{\min} \in (0, \beta)$ then $(s_{\min} - 1)' \in (-1, \beta - 1)$ hence $(s_{\min} - 1) \in \mathbb{Z}_\beta^+$. Since $s_{\min}$ is the minimum of the set $\mathcal{S}$, we know that $(s_{\min} - 1) \in \cup_{k\geq 0} \mathbb{Z}_\beta^k$ and so $s_{\min} = (s_{\min} - 1)\dot{+}1 \in \cup_{k\geq 0} \mathbb{Z}_\beta^k$, which is a contradiction. On the other hand, if $s'_{\min} \in (-1, 0)$ then $(s_{\min} - \frac{1}{\beta})' \in (-1 + \beta, \beta)$ and so $(s_{\min} - \frac{1}{\beta}) \in \mathbb{Z}_\beta^+$. Since $s_{\min}$ is the minimum of the set $\mathcal{S}$, $(s_{\min} - \frac{1}{\beta}) \in \cup_{k\geq 0} \mathbb{Z}_\beta^k$. The fact that $s'_{\min} \in (-1, 0)$ entails that the β-expansion of $s_{\min}$ is of the form $s_{\min} = \sum_{i=1}^{j} z_i \beta^i$. Then we see that β-expansion of $(s_{\min} - \frac{1}{\beta} + 1)$ is equal to $\sum_{i=1}^{j} z_i \beta^i + \frac{a-1}{\beta} + \frac{1}{\beta^2}$ and so $s_{\min} = (s_{\min} - \frac{1}{\beta})\dot{+}1 \in \cup_{k\geq 0} \mathbb{Z}_\beta^k$.

So we get again a contradiction with (36).

∘ Let β be the root > 1 of $x^2 = ax - 1, a \geq 3$.

Here, similarly to the previous case, we show the contradiction $s_{\min} \in \cup_{k\geq 0} \mathbb{Z}_\beta^k$.

By using Lemma 4.2(ii) we know that $s'_{\min} \in (0, \beta)$. If $s'_{\min} \in (1, \beta)$ then $(s_{\min} - 1)' \in (0, \beta - 1)$ hence $(s_{\min} - 1) \in \cup_{k\geq 0} \mathbb{Z}_\beta^k$ and so $s_{\min} = (s_{\min} - 1)\dot{+}1 \in \cup_{k\geq 0} \mathbb{Z}_\beta^k$. Next, if $s'_{\min} \in (0, 1)$, then $(s_{\min} - 1 + \frac{1}{\beta})' \in (-1 + \beta, \beta)$ hence $s_{\min} = (s_{\min} - 1 + \frac{1}{\beta})\dot{+}1 \in \cup_{k\geq 0} \mathbb{Z}_\beta^k$. □

We have here two internal commutative laws which are not associative, and distributivity does not hold either $x\dot{+}(y\dot{+}z) \neq (x\dot{+}y)\dot{+}z$, $x\dot{\times}(y\dot{\times}z) \neq (x\dot{\times}y)\dot{\times}z$ and $x\dot{\times}(y\dot{+}z) \neq (x\dot{\times}y)\dot{+}(x\dot{\times}z)$. For instance $(\tau^2 + 1)\dot{\times}(1\dot{+}(\tau^2 + 1)) = \tau^5 + \tau^3$, whereas $(\tau^2 + 1)\dot{+}((\tau^2 + 1)\dot{\times}(\tau^2 + 1)) = \tau^5 + \tau^3 + 1$.

These non-associativity and non-distributivity are encoded by the range of values assumed by the three following maps from $\mathbb{Z}_\beta \times \mathbb{Z}_\beta \times \mathbb{Z}_\beta$ into $\mathbb{Z}_\beta$:

Additive associator

$$[x, y, z]_+ = ((x\dot{+}y)\dot{+}z)\dot{-}(x\dot{+}(y\dot{+}z)) \in F_+.$$

One conjectures that the above set F_+ is simply equal to $\{\pm 1\}$.

Multiplicative associator

$$[x, y, z]_\times = (x\dot{\times}y)\dot{\times}z\dot{-}x\dot{\times}(y\dot{\times}z) \in X_\times$$

and distributor

$$[x, y, z]_{\substack{\times\\+}} = x\dot{\times}(y\dot{+}z)\dot{-}(x\dot{\times}y\dot{-}x\dot{\times}z) \in X_{\substack{\times\\+}}.$$

The two sets X are not finite and a systematic rescaling of these associator and distributor in function of $\deg_\beta(x), \deg_\beta(y)$ and $\deg_\beta(z)$, is certainly needed here in order to get finite sets.

6. Five-fold quasilattices in the plane and in space

d-dimensional discrete sets of the form (6) can be built on the β-integers $\mathbb{Z}_\tau$:

$$\Lambda = \sum_{i=1}^{d} \mathbb{Z}_\tau e_i,$$

where $\{e_i\}$ is a set of d linearly independent vectors. Λ is a Meyer quasilattice by construction. More precisely we have

$$\Lambda = -\Lambda \quad \text{and} \quad \Lambda - \Lambda \subset \Lambda + F \subset \Lambda/\tau^2.$$

The set Λ is self similar

$$\tau\Lambda \subset \Lambda$$

and from Theorem 3.2. we have that

$$\mathbb{Z}_\tau \Lambda \subset \Lambda + F \subset \Lambda/\tau^2.$$

Generically, the vectors e_i's have distinct arbitrary directions in $\mathbb{R}^d$. Five-fold or ten-fold symmetries appear as locally possible if those directions are appropriately chosen. This is due to the "cyclotomic" nature of the golden mean

$$\tau = 2\cos\frac{\pi}{5} \ , \ \frac{1}{\tau} = \tau - 1.$$

By this way we can introduce in the complex plane "canonical" τ-quasilattices or τ-grids:

$$\Gamma_q \stackrel{\text{def}}{=} \mathbb{Z}_\tau + \mathbb{Z}_\tau \zeta^q \ , \ \text{where } \zeta = e^{\frac{i\pi}{5}} \text{ and } q = 1, 2, 3 \text{ or } 4.$$

The τ-grids $\Gamma_1, \ \Gamma_2$ are shown on Figure 4 and 6. Those τ-grids are not rotationally invariant with respect to the origin. However they are almost so. This is clear from the relation

$$\zeta^2 = \tau\zeta - 1. \tag{37}$$

For instance, we have from Theorem 3.1.

$$\zeta\Gamma_1 = \mathbb{Z}_\tau + (\mathbb{Z}_\tau + \tau\,\mathbb{Z}_\tau)\zeta \subset \Gamma_1 + \{0, \pm\frac{1}{\tau}, \pm\frac{1}{\tau^2}\}\zeta$$

and similar inclusions for Γ_q.

In order to see how these τ-grids are obtained by the Cut and Project method, we calculate their respective algebraically conjugate sets within the ring generated by τ and ζ, which is actually the cyclotomic ring $\mathbb{Z}[\zeta]$.

$$\mathbb{Z}[\tau] + \mathbb{Z}[\tau]\zeta \equiv \mathbb{Z}[\zeta]. \tag{38}$$

The standard automorphism (16) in $\mathbb{Z}[\tau]$ induces the following one in (38)

$$\wp = x + y\zeta \quad \longrightarrow \quad \wp^* = x' + y'\zeta^*$$

with

$$\zeta^* = \zeta^3. \tag{39}$$

Equation (39) is indeed consistent with

$$\zeta + \overline{\zeta} = \tau \to \tau' = -\frac{1}{\tau} = \zeta^* + \overline{\zeta}^*$$

and note that (39) is of order 4 since

$$(\zeta^*)^* = \overline{\zeta}\ . \tag{40}$$

From (40) and from (19-21) we have

$$\mathbb{Z}[\tau] \cap (-1,1) + (\mathbb{Z}[\tau] \cap (-1,1))\zeta^3 \subset \Gamma_1^* \subset \mathbb{Z}[\tau] \cap (-\tau,\tau) + (\mathbb{Z}[\tau] \cap (-\tau,\tau))\zeta^3.$$

We see in Figure 5 that Γ_1^* fills the τ-rhombus $\mathcal{R}_\tau = (-\tau,\tau) \times (-\tau,\tau)e^{\frac{3i\pi}{5}}$, but the equality

$$\mathbb{Z}[\zeta] \cap P = \Gamma_1^* \cap P$$

holds only for the smaller rhombus $P \equiv \mathcal{R}_1$.

This corresponds to different rhombic windows in the perpendicular plane along which the $\mathbb{Z}^4$-lattice points are projected onto Γ_1 (up to a $\frac{3\pi}{5}$ rotation). These windows are shifted with respect to each other in order to make Γ_1 symmetrical with respect to the origin. The grid Γ_q has subsets invariant with respect to $36°$-rotations about the origin. We have different possibilities to build up such sets (up to a scaling by τ^k). We can consider for instance unions of grids Γ_q

$$\Sigma_q \stackrel{\text{def}}{=} \bigcup_{j=0}^{4} \zeta^j \Gamma_q \quad \text{for} \quad q = 1,2,3,4$$

and better, we can "force" the 10-fold symmetry in a cyclotomic way by introducing the set

$$\mathbb{Z}_\tau[\zeta] \stackrel{\text{def}}{=} \sum_{j=0}^{4} \zeta^j\, \mathbb{Z}_\tau \tag{41}$$

with self-explanatory notations. Note that we could have chosen as well $\zeta^2 = e^{\frac{2i\pi}{5}}$ from the fact that $\zeta^6 = -\zeta$, $\zeta^8 = -\zeta^3$, and $\mathbb{Z}_\tau = -\mathbb{Z}_\tau$, and so

$$\mathbb{Z}_\tau[\zeta^2] = \mathbb{Z}_\tau[\zeta].$$

Clearly

$$\zeta\Sigma_q = \Sigma_q \ , \text{ for } \ q = 1,2,3,4 \quad \text{ and } \ \zeta\,\mathbb{Z}_\tau[\zeta] = \mathbb{Z}_\tau[\zeta].$$

Those sets also have nice algebraic properties:

Proposition 6.1. The sets $\Lambda_0 \stackrel{\text{def}}{=} \mathbb{Z}_\tau[\zeta]$ and $\Lambda_q \stackrel{\text{def}}{=} \Sigma_q$, *where* $q = 1,2,3,4$, *are Meyer quasilattices. In particular,* Λ_0 *is characterized by:*

$$\Lambda_0 + \Lambda_0 \subset \Lambda_0 + F_0 \ , \text{ where } F_0 = F + F\zeta \text{ and} \tag{42}$$

$$F = \{x \in \mathbb{R} \mid x = \sum_{j=0}^{6} \xi_j \tau^{-j}, \text{ with } \ \xi_j \in \{0,1\} \text{ and } \xi_j\xi_{j+1} = 0\}. \tag{43}$$

Proof. i) We have to prove that the Λ_q is, for $q \in \{0,1,2,3,4\}$, a Delaunay set. By using the formula (37) and Theorem 3.1., we obtain the following inclusions

$$\begin{aligned}\Lambda_0 &\subset \mathbb{Z}_\tau + \mathbb{Z}_\tau + \tau(\mathbb{Z}_\tau + \mathbb{Z}_\tau) + ((\mathbb{Z}_\tau + \mathbb{Z}_\tau) + \tau(\mathbb{Z}_\tau + \mathbb{Z}_\tau))\zeta \\ &\subset \mathbb{Z}_\tau/\tau^4 + \mathbb{Z}_\tau/\tau^4\zeta = \Gamma_1/\tau^4\end{aligned} \tag{44}$$

and

$$\Lambda_q \subset (\mathbb{Z}_\tau + \mathbb{Z}_\tau) + (\mathbb{Z}_\tau + \mathbb{Z}_\tau)\zeta \subset \mathbb{Z}_\tau/\tau^2 + (\mathbb{Z}_\tau/\tau^2)\zeta = \Gamma_1/\tau^2, \text{ for } q = 1,2,3,4 \ ,$$

i.e. each Λ_q is embedded in a thinner "τ-grid" which has of course the Delaunay property.

On the other hand each of those Λ_q contains a looser τ-grid (for example $\Gamma_1 \subset \Lambda_0$), which is also Delaunay. From this fact it follows that the Λ_q's, $q \in \{0,1,2,3,4\}$, are Delaunay sets.

ii) there exists a finite F_q such that $\Lambda_q - \Lambda_q = \Lambda_q + \Lambda_q \subset \Lambda_q + F_q$, $q \in \{0,1,2,3,4\}$. At first we prove ii) for Λ_0. From (44) we have that

$$\Lambda_0 + \Lambda_0 \subset \Gamma_1/\tau^4 + \Gamma_1/\tau^4 \subset \Gamma_1/\tau^6,$$

then by using

$$\mathbb{Z}_\tau/\tau^6 \subset \mathbb{Z}_\tau + F,$$

where F is defined in (43) we get (42). Similarly, it is obvious that Λ_q, for $q \in \{1,2,3,4\}$, is a Meyer quasilattice. $\square$

Note that we could also prove the same for the 10-fold set $\bigcup_{q=1}^4 \Sigma_q$. The "quasi-cyclotomic" ring $\mathbb{Z}_\tau[\zeta]$ that we have introduced in (41) has aesthetic nice properties which can be seen in Figure 8. Its algebraic conjugate set is displayed in Figure 9. The double decagonal nature originates from the property (19). One can assert that

$$D_1 \cap \mathbb{Z}[\zeta] = D_1 \cap (\mathbb{Z}_\tau[\zeta])^* \quad \text{whereas} \quad (\mathbb{Z}_\tau[\zeta])^* \subset D_{2\tau^2} \cap \mathbb{Z}[\zeta],$$

where D_α is the decagon with radius α and centered at the origin.

The existence of the symmetric Delaunay set $\mathbb{Z}_\tau[\zeta]$ within the dense ring $\mathbb{Z}[\zeta]$ is reminiscent of the discrete reduction of the cyclotomic ring

$$\mathbb{Z}[e^{\frac{2i\pi}{n}}] = \sum_{q=0}^{n-1} \mathbb{Z}e^{\frac{2iq\pi}{n}} = \mathbb{Z} + \mathbb{Z}e^{\frac{2i\pi}{n}},$$

when $n = 1, 2, 3, 4$ and 6, i.e when n is crystallographic. In the present case, replacing $\mathbb{Z}$ by $\mathbb{Z}_\tau$ leads to something similar:

$$\mathbb{Z}_\tau + \mathbb{Z}_\tau e^{\frac{i\pi}{5}} \subset \mathbb{Z}_\tau[e^{\frac{i\pi}{5}}] \subset (\mathbb{Z}_\tau + \mathbb{Z}_\tau e^{\frac{i\pi}{5}})/\tau^4.$$

For other noncrystallographic n it would be necessary to find an appropriate Pisot-cyclotomic number β (see [14]). $\beta = \gamma = 1 + \sqrt{2}$ and $\beta = \delta = 2 + \sqrt{3}$ from (3,4) are such examples.

We wish to end the present section by mentioning the three-dimensional version of the Γ_q's and $\mathbb{Z}_\tau[\zeta]$. These sets seem to be fundamental for quasicrystals and they are fully described in [4]. Some insights are just given here.

A canonical three-dimensional τ-grid is given by

$$\Gamma = \mathbb{Z}_\tau\vec{\alpha} + \mathbb{Z}_\tau\vec{\beta} + \mathbb{Z}_\tau\vec{\gamma},$$

where $\vec{\alpha}, \vec{\beta}$ and $\vec{\gamma}$ are position vectors of three five-fold icosahedron vertices forming an equilateral triangle . For instance,

$$\vec{\alpha} = \frac{1}{2}(1, \frac{1}{\tau}, 0), \quad \vec{\beta} = \frac{1}{2}(0, 1, \frac{1}{\tau}), \quad \vec{\gamma} = \frac{1}{2}(\frac{1}{\tau}, 0, 1).$$

Again we can force the structure in order to have a complete five-fold symmetry by considering the set

$$\sum_{i=1}^{6} \mathbb{Z}_\tau\vec{\alpha}_i, \tag{45}$$

when the sum runs on one-half of the set of all icosahedron vertices. By construction the set algebraically conjugate to (45) will densely fill a contracted icosahedron in space.

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Figure captions

Figure 1. Penrose quasilattice as a subset of the τ-grid Γ_1 (Figure 4)

Figure 2. Diffraction pattern of Penrose quasilattice of Figure 1 as a subset of the τ-grid Γ_1

Figure 3. $\mathbb{Z}_\tau$-numbers

Figure 4. τ-grid Γ_1

Figure 5. Algebraic conjugate Γ_1^* of the τ-grid Γ_1

Figure 6. τ-grid Γ_2

Figure 7. Algebraic conjugate Γ_2^* of the τ-grid Γ_2

Figure 8. Quasicyclotomic ring $\mathbb{Z}_\tau[e^{\frac{i\pi}{5}}]$

Figure 9. Galois conjugate of the quasicyclotomic ring $\mathbb{Z}_\tau[e^{\frac{i\pi}{5}}]$

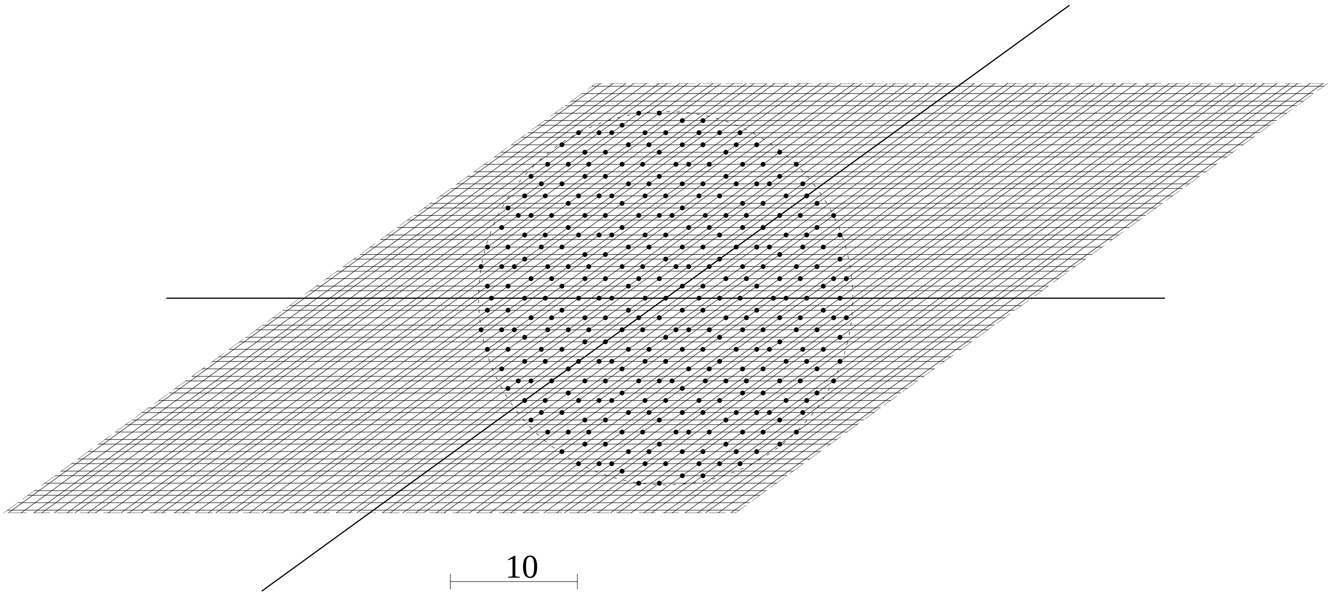

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Figure 1: Penrose quasilattice as a subset of the τ-grid Γ_1 (see Figure 4)

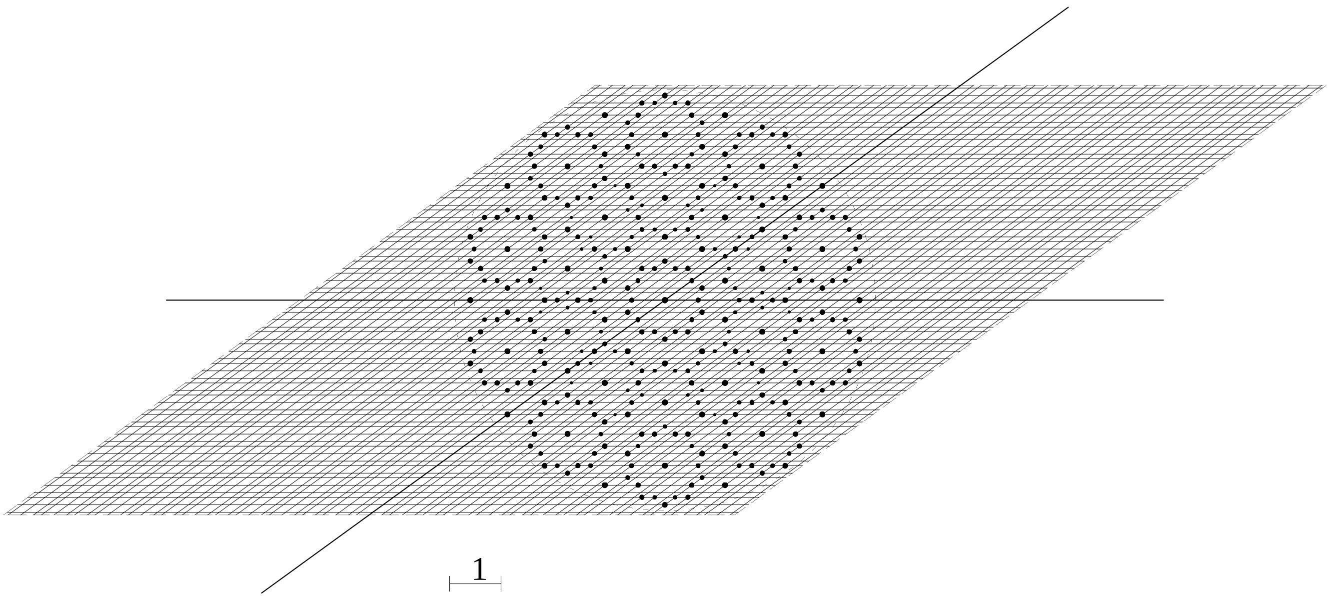

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Figure 2: Diffraction pattern of Penrose quasilattice of Figure 1
as a subset of the τ-grid Γ_1

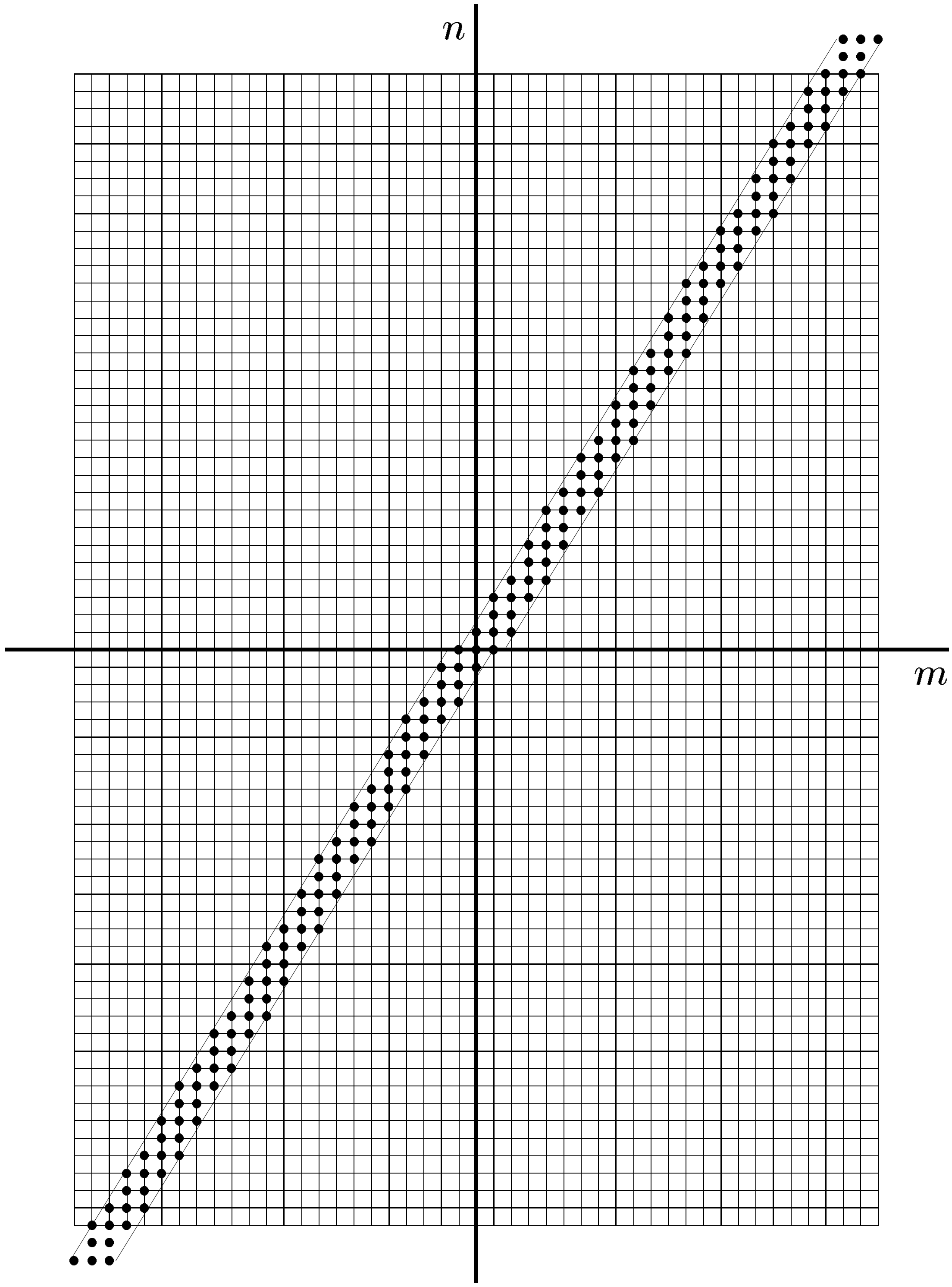

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Figure 3: $\mathbb{Z}_\tau$-numbers

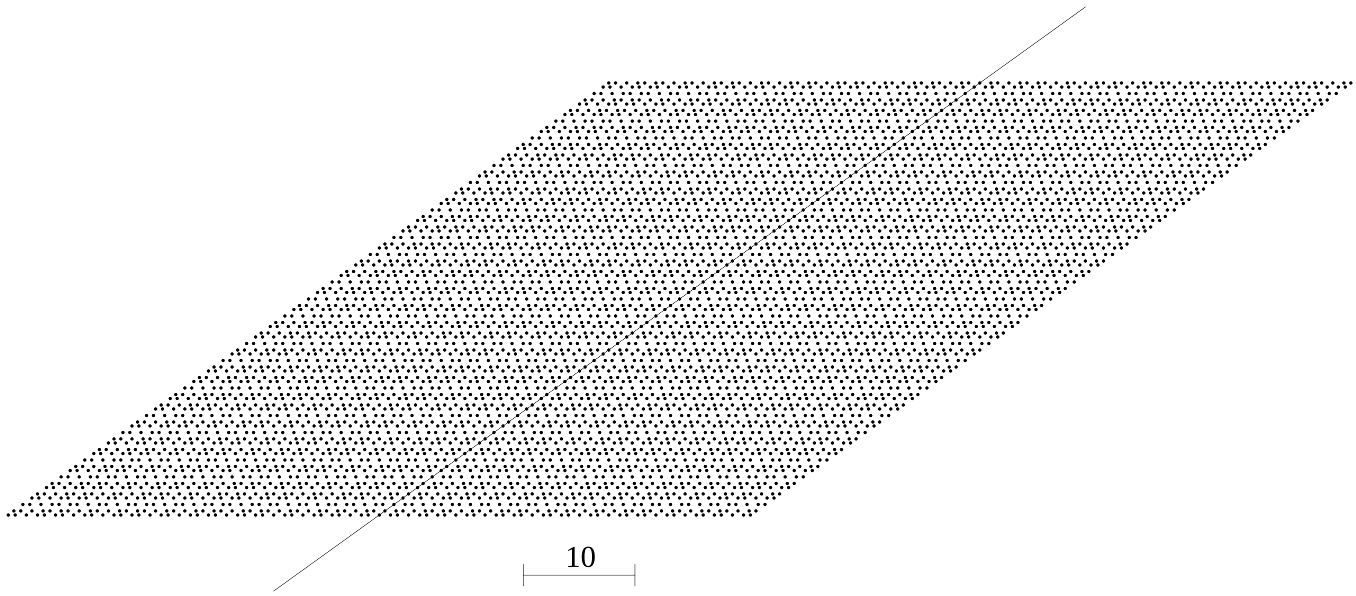

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Figure 4: τ-grid Γ_1

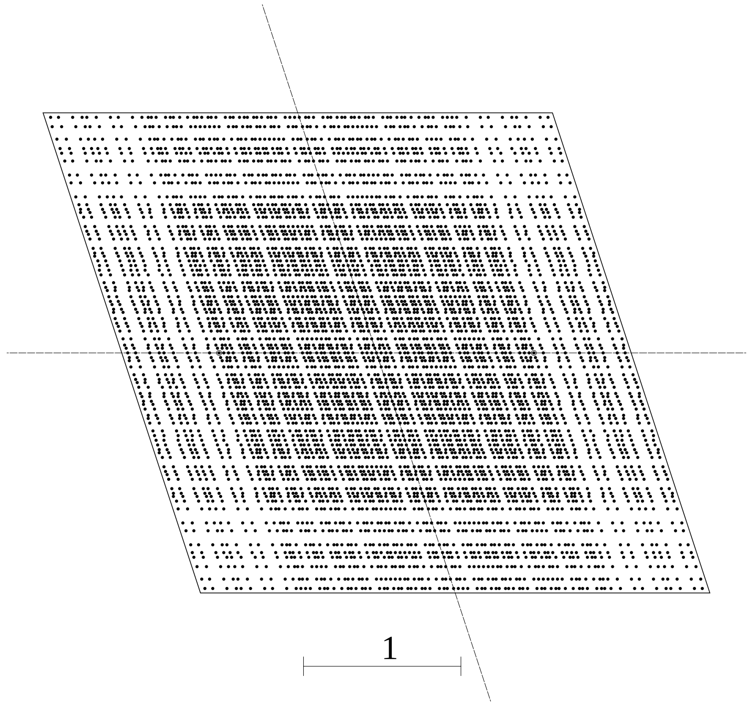

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Figure 5: Algebraic conjugate Γ_1^* of the τ-grid Γ_1

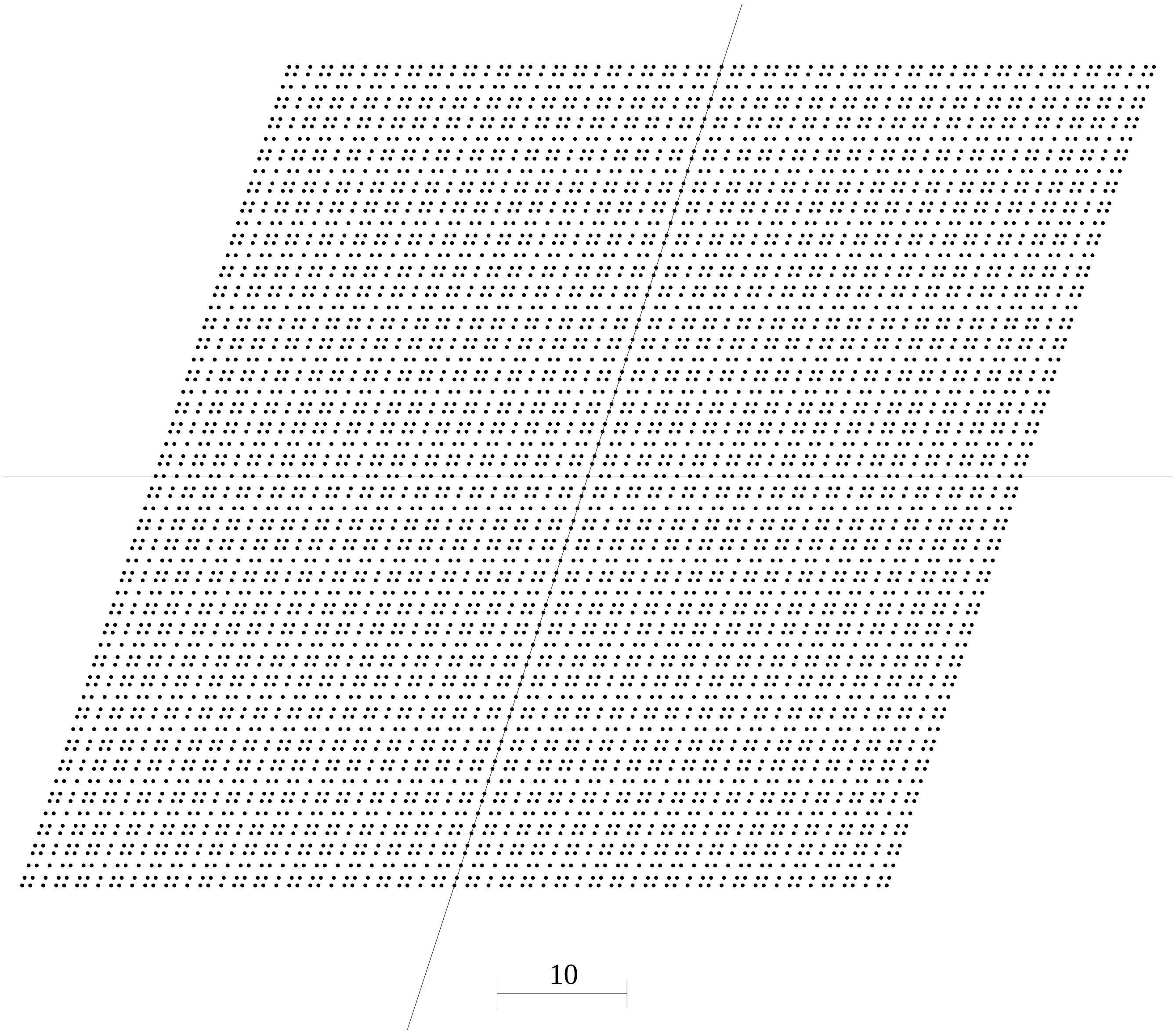

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Figure 6: $\tau-$grid Γ_2

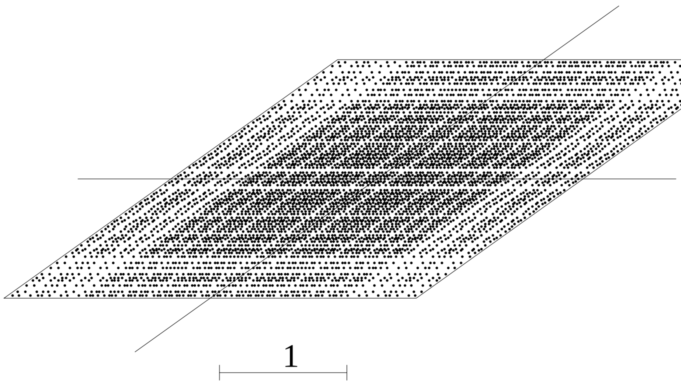

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Figure 7: Algebraic conjugate Γ_2^* of the τ-grid Γ_2

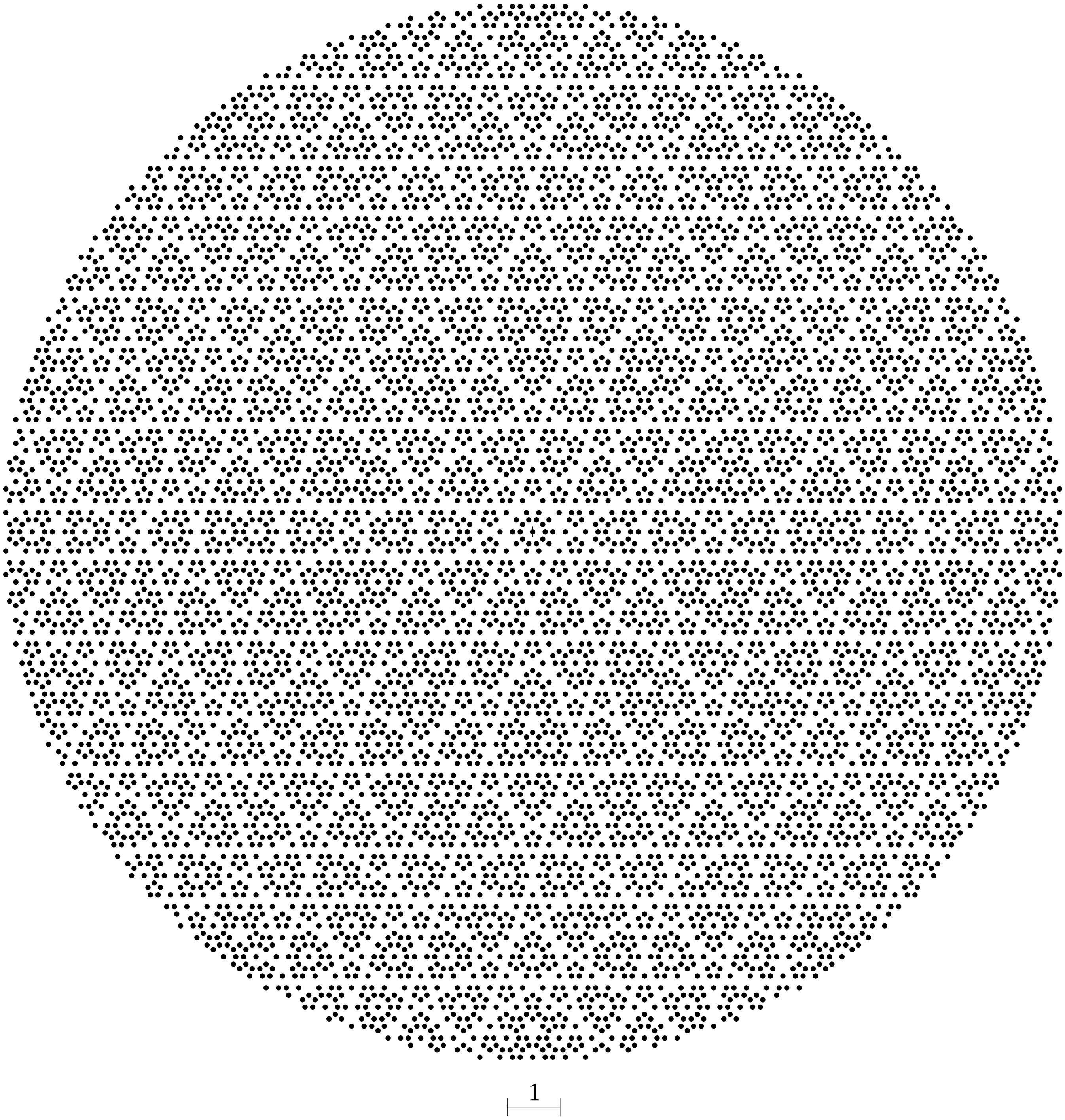

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Figure 8: Quasicyclotomic ring $\mathbb{Z}_\tau[e^{\frac{i\pi}{5}}]$

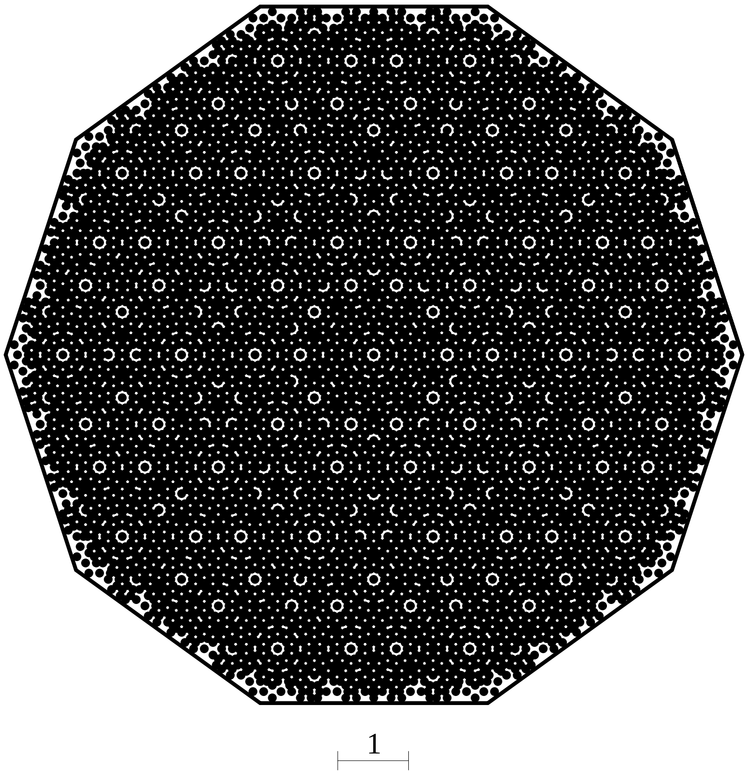

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Figure 9: Galois conjugate of the quasicyclotomic ring $\mathbb{Z}_\tau[e^{\frac{i\pi}{5}}]$