ADDITIVE AND MULTIPLICATIVE PROPERTIES OF POINT SETS BASED ON BETA-INTEGERS

by

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Abstract. — To each number $\beta > 1$ correspond abelian groups in $\mathbb{R}^d$, of the form $\Lambda_\beta = \sum_{i=1}^d \mathbb{Z}_\beta \mathbf{e}_i$, which obey $\beta\Lambda_\beta \subset \Lambda_\beta$. The set $\mathbb{Z}_\beta$ of beta-integers is a countable set of numbers : it is precisely the set of real numbers which are polynomial in β when they are written in "basis β", and $\mathbb{Z}_\beta = \mathbb{Z}$ when $\beta \in \mathbb{N}$. We prove here a list of arithmetic properties of $\mathbb{Z}_\beta$: addition, multiplication, relation with integers, when β is a quadratic Pisot-Vijayaraghavan unit (quasicrystallographic inflation factors are particular examples). We also consider the case of a cubic Pisot-Vijayaraghavan unit associated with the seven-fold cyclotomic ring. At the end, we show how the point sets Λ_β are vertices of d-dimensional tilings.

Résumé. — À chaque nombre $\beta > 1$ correspondent des groupes abéliens dans $\mathbb{R}^d$, de la forme $\Lambda_\beta = \sum_{i=1}^d \mathbb{Z}_\beta \mathbf{e}_i$, et qui satisfont $\beta\Lambda_\beta \subset \Lambda_\beta$. L'ensemble $\mathbb{Z}_\beta$ des beta-entiers est un ensemble dénombrable de nombres, qui est formé de tous les réels qui sont polynomiaux en β lorsqu'on les écrit en "base β", et qui se confond avec $\mathbb{Z}$ lorsque β est un naturel > 1. Un ensemble de propriétés arithmétiques de $\mathbb{Z}_\beta$, addition, multiplication, relation avec les entiers, sont ici présentées lorsque β est un nombre de Pisot-Vijayaraghavan quadratique unitaire quelconque. Nous rappelons que les facteurs d'inflation en quasicristallographie en sont des cas particuliers. Nous traitons aussi le cas d'un nombre de Pisot cubique unitaire associé à l'anneau cyclotomique à symétrie d'ordre 7. Enfin, nous montrons comment les ensembles de points Λ_β peuvent être vus comme les nœuds de pavages dans $\mathbb{R}^d$.

1. Introduction

The conceptual role played by lattices of the form

$$\Lambda = \sum_{i=1}^{d} \mathbb{Z}\,\mathbf{e}_i,$$

as well as by integers in Crystallography is well known. Lattice nodes are geometrical support of atoms in "perfect" or "ideal" crystals, as they are support of Bragg peaks in the diffraction pattern of the latter (Poisson formulae and related spectral measure theory). One can easily imagine that some "standard" quasilattice as well exists for "ideal" quasicrystal. Indeed, numerous experimental evidences force us to rank quasicrystal in the category of aperiodic structures with long range order. In particular, Bragg peaks beyond a certain intensity level are supported by model sets, and the cut and projection method has become like a paradigm in (almost) all geometrical models of quasicrystals. It turns out that these model sets can be always considered as a part of a more basic Delaunay set of the type

$$\Lambda = \sum_{i=1}^{d} \mathbb{Z}_\beta \, \mathbf{e}_i,$$

suitably rescaled and/or decorated if need be (see [**14, 15**] for instance). Here, β is one of the three quadratic unit Pisot-cyclotomic numbers [**1**] precisely encountered in quasicrystallography

$$\beta = \frac{1+\sqrt{5}}{2}, \text{ or } 1+\sqrt{2}, \text{ or } 2+\sqrt{3}, \tag{1}$$

and $\mathbb{Z}_\beta$ is the so-called set of "beta-integers". Loosely speaking, the set $\mathbb{Z}_\beta$ is the set of real numbers which are polynomial in β when they are written in "base β" (see Definition 3.1). Of course this set is not closed under ordinary addition and multiplication. So we could think we lose most of the nice algebraic properties of the lattices like their narrow relation with group theory and the subsequent possibilities of labelling and classifying crystals in Nature. Actually, we show in this paper that we can endow $\mathbb{Z}_\beta$ with adapted addition $\oplus$ and multiplication $\otimes$ which allow to partially restore those lattice algebraic features

$$\begin{aligned} \Lambda \oplus \Lambda &= \Lambda \\ \mathbb{Z}_\beta \otimes \Lambda &= \Lambda. \end{aligned}$$

Moreover the difference between $x \oplus y$ and $x+y$, resp. $x \otimes y$ and xy, is uniformously bounded. The group structure of $\mathbb{Z}_\beta$, when β is a quadratic Pisot unit, was shown in [**6**]. The proof given here is new and simpler. Moreover the techniques can be used for the multiplicative case.

Thus it becomes clear we can initialize some labelling and classification programs which are based on these properties. Actually we can do better. The increasing sequence of Delaunay sets $\{\Lambda/\beta^k\}_{k\in\mathbb{Z}}$ offers an ideal discretization scheme for $\mathbb{R}^d$. As a matter of fact, they prove to be reliable backgrounds for numerical simulations (e.g. Monte Carlo) of quasicrystalline ground states [**12**].

The paper is organized as follows. After the recalling of some definitions and properties, we consider the case when β is a quadratic Pisot unit. We give several combinatorial and algebraic properties, allowing to define addition and multiplication laws on the set of beta-integers. It follows that the set $\mathbb{Z}_\beta$ is a group isomorphic to the group $\mathbb{Z}$ of integers. We could also define on $\mathbb{Z}_\beta$ a simple multiplication law that would allow us to consider $\mathbb{Z}_\beta$ as a ring isomorphic to $\mathbb{Z}$, but we reject it, because it does not fit well with the ordinary multiplication of real numbers. Instead, we shall propose another multiplication law which fits much better to the ordinary multiplication and to our quasicrystallographic motivations. Unfortunately, this law is not associative nor distributive and so we lose the ring structure.

We then study a cubic Pisot unit, associated with a seven-fold symmetry. We give similar properties to the quadratic case. In particular we characterize the finite set F such that $\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$.

We explain in Section 8 in what sense beta-integers can be of some use in the describing of some Delaunay sets in $\mathbb{R}^d$. Actually, we shall restrict ourselves to the two-dimensional five-fold structures, *i.e.* to the case $\beta = \tau$, for pedagogical purposes. The standard example is the set of nodes of a Penrose tiling. Therefore it will be shown that such nodes are selected as points in the τ-lattice or τ-grid

$$\Lambda = \mathbb{Z}_\tau + \mathbb{Z}_\tau e^{\frac{i\pi}{5}}$$

by following an elementary algebraic procedure.

In the appendix, we show that, for the case in which β is a quadratic Pisot unit, the infinite word generated by the associated substitution is a Sturmian word — the generic case being the one associated with the golden mean.

2. Meyer sets and quasicrystals

We recall here several definitions and results from Meyer that can be found in [**16, 17, 18, 19, 20**].

A set $\Lambda \subset \mathbb{R}^d$ is said to be *uniformly discrete* if there exists $r > 0$ such that every ball of radius r contains at most a point of Λ. A set Λ is said to be *relatively dense* if there exists $R > 0$ such that every ball of radius R contains at least a point of Λ. If both conditions are satisfied, Λ is said to be a *Delaunay set*.

A *cut and projection scheme* is the following

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times G & \xrightarrow{\pi_2} & G \\ & & \cup & & \\ & & D & & \end{array}$$

where G is a locally compact abelian group, called the *internal space*, $\mathbb{R}^d$ is called the *physical space*, D is a lattice, *i.e.* a discrete subgroup of $\mathbb{R}^d \times G$ such that $(\mathbb{R}^d \times G)/D$ is compact. The projection $\pi_1|_D$ is 1-to-1, and $\pi_2(D)$ is dense in G.

Let $M = \pi_1(D)$ and set $* = \pi_2 \circ (\pi_1|_D)^{-1}$

$$* : M \longrightarrow G$$

The set $\Lambda \subset \mathbb{R}^d$ is a *model set* if there exist a cut and projection scheme and a relatively compact set $\Omega \subset G$ of non-empty interior such that

$$\Lambda = \{x \in M \mid x^* \in \Omega\}.$$

The set Ω is called a *window*.

***Example* 2.1.** — As an illustration, we describe one type of Fibonacci chain. Let $\tau = \frac{1+\sqrt{5}}{2}$. Consider the following cut and projection scheme

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R} \times \mathbb{R} & \xrightarrow{\pi_2} & \mathbb{R} \\ & & \cup & & \\ & & \mathbb{Z}^2 & & \end{array}$$

Here $\pi_1(\mathbb{Z}^2) \sim \mathbb{Z}[\tau] = \{a + b\tau \mid a, b \in \mathbb{Z}\}$.

The Fibonacci chain $\mathcal{F}$ is

$$\begin{aligned}\mathcal{F} &= \{x = a + b\tau \mid x' = a - \frac{b}{\tau} \in \Omega = [0,1)\} \\ &= \{\ldots, -\tau^3, -\tau, 0, \tau^2, \tau^3 + 1, \tau^4, \ldots\}\end{aligned}$$

It is the set of left endpoints of a quasiperiodic tiling of $\mathbb{R}$ with 2 tiles L et S, of respective length τ^2 and τ, generated by the substitution rules

$$\begin{aligned} L &\mapsto LLS \\ S &\mapsto LS \end{aligned}$$

Starting from SL in both directions we get a biinfinite word

$$\cdots LLSLS \mid LLSLLSLS \cdots$$

Note that

$$\tau^2 \mathcal{F} \subset \mathcal{F}$$

and that this tiling is, by construction, *stone inflation*, which means that all the tiles when scaled by the factor τ^2 can be packed face-to-face from the original ones.

Remark that the set $\mathcal{F} \cup (-\mathcal{F})$ is also a model set, since

$$\mathcal{F} \cup (-\mathcal{F}) = \{x \in \mathbb{Z}[\tau] \mid x' \in (-1, 1)\} = \mathcal{F} - \mathcal{F} = (\mathcal{F} + \{-1, 0\}) \setminus \{-1\}.$$

This set is invariant under multiplication by τ. The symmetrical tiling associated with it has 3 tiles with respective length τ^{-1}, 1 and τ. ◊

A set $\Lambda \subset \mathbb{R}^d$ is said to be a *Meyer set* if it is a Delaunay set and if there exists a finite set F such that

$$\Lambda - \Lambda \subset \Lambda + F.$$

Such sets are called *quasicrystals* by Meyer.

***Theorem* 2.2.** — *A model set is a Meyer set.*
Conversely if Λ is a Meyer set, there exist a finite set F and a model set Λ_0 such that $\Lambda \subset \Lambda_0 + F$.

We now recall some definitions on numbers. A *Pisot number* (more precisely a Pisot-Vijayaraghavan number) is an algebraic integer > 1 such that all its Galois conjugates have modulus strictly less than one. A *Salem number* is an algebraic integer such that every conjugate has modulus smaller than or equal to 1, and at least one of them has modulus 1.

The following result from Meyer makes the connection between Meyer sets and those algebraic integers.

***Theorem* 2.3.** — *If $\Lambda \subset \mathbb{R}^d$ is a Meyer set and if $\beta > 1$ is a real number such that $\beta\Lambda \subset \Lambda$ then β is a Pisot or a Salem number.*
Conversely for each d and for each Pisot or Salem number β, there exists a Meyer set $\Lambda \subset \mathbb{R}^d$ such that $\beta\Lambda \subset \Lambda$.

3. Representation of numbers

3.1. Representation of real numbers. — Let $\beta > 1$ be a real number. A *beta-representation* of a real number $x \geq 0$ is an infinite sequence $(x_i)_{i \leq k}$, such that

$$x = \sum_{i \leq k} x_i \beta^i$$

for a certain integer k.

A particular beta-representation — called the *beta-expansion* — can be computed by the "greedy algorithm" (see [**24**] and [**22**]). Denote by $\lfloor y \rfloor$ and $\{y\}$ the integer part and the fractional part of a real number y. There exists $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Let $x_k = \lfloor x/\beta^k \rfloor$ and $r_k = \{x/\beta^k\}$. Then for $i < k$, put $x_i = \lfloor \beta r_{i+1} \rfloor$, and $r_i = \{\beta r_{i+1}\}$. We get an expansion $x = x_k \beta^k + x_{k-1} \beta^{k-1} + \cdots$. If $x < 1$ then $k < 0$, and we put $x_0 = x_{-1} = \ldots = x_{k+1} = 0$. The beta-expansion of x is denoted by

$$\langle x \rangle_\beta = x_k x_{k-1} \ldots x_1 x_0 \cdot x_{-1} x_{-2} \ldots$$

The digits x_i obtained by this algorithm are integers from the set $A = \{0, \ldots, \lceil \beta \rceil - 1\}$, called the *canonical alphabet*, where $\lceil \beta \rceil$ denotes the closest integer to β from above. If an expansion ends in infinitely many zeros, it is said to be *finite*, and the ending zeros are omitted.

There is a representation which plays an important role in the theory. The *beta-expansion* of 1, denoted by $d_\beta(1)$, is computed by the following process [**24**]. Let the *beta-transform* be defined on $[0,1]$ by $T_\beta(x) = \beta x \bmod 1$. Then $d_\beta(1) = (t_i)_{i \geq 1}$, where $t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor$. Note that $d_\beta(1)$ belongs to $A^{\mathbb{N}}$. Recall that if β is a Pisot number, then $d_\beta(1)$ is eventually periodic [**3, 25**]. A number β such that $d_\beta(1)$ is eventually periodic is traditionally called a *beta-number*. Since these numbers were introduced by Parry [**22**], we propose to call them *Parry numbers*. When $d_\beta(1)$ is finite, β is said to be a *simple* Parry number.

***Definition* 3.1.** — The set of *beta-integers* is the set of real numbers such that their beta-expansion is polynomial,

$$\begin{aligned} \mathbb{Z}_\beta &= \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_k \cdots x_0\} \\ &= \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+) \end{aligned}$$

where $\mathbb{Z}_\beta^+$ is the set of non-negative beta-integers.

$\mathbb{Z}_\beta$ is self-similar and symmetrical with respect to the origin

$$\beta \mathbb{Z}_\beta \subset \mathbb{Z}_\beta \ , \ \mathbb{Z}_\beta = -\mathbb{Z}_\beta.$$

It is shown in [**5**] that if β is a Pisot number then $\mathbb{Z}_\beta$ is a Meyer set.

3.2. Representation of integers. — Let $U = (u_n)_{n \geq 0}$ be an increasing sequence of integers with $u_0 = 1$. A *representation in the system* U, or a *U-representation*, of a non-negative integer N is a finite sequence of integers $(d_i)_{0\leq i\leq k}$ such that

$$N = \sum_{i=0}^{k} d_i u_i.$$

Such a representation will be written $d_k \cdots d_0$, most significant digit first.

Among all possible U-representations $d_k \cdots d_0$ of a given positive integer N one is distinguished and called the *greedy U-representation* of N: the greatest in the lexicographical ordering. It is obtained by the following greedy algorithm (see [**10**]): Given integers m and p let us denote by $q(m,p)$ and $r(m,p)$ the quotient and the remainder of the Euclidean division of m by p.
Let $k \geq 0$ such that $u_k \leq N < u_{k+1}$ and let $d_k = q(N,u_k)$ and $r_k = r(N,u_k)$, $d_i = q(r_{i+1},u_i)$ and $r_i = r(r_{i+1},u_i)$ for $i = k-1,\cdots,0$. Then $N = d_k u_k + \cdots + d_0 u_0$.

The greedy representation is often called the *normal* representation. The greedy representation of N in the system U is denoted by $\langle N\rangle_U$.

By convention the greedy representation of 0 is the empty word ε. Under the hypothesis that the ratio u_{n+1}/u_n is bounded by a constant as n tends to infinity, the integers of the greedy U-representation of any positive integer N are bounded and contained in a *canonical* finite alphabet A_U associated with U.

4. Numeration system associated with a Parry number

We recall here results from [**4**]. These results are in fact valid for any real number $\beta > 1$, but we present them in the cases we shall consider in this paper.

Let β be a Parry number. There are two cases to consider :
Case 1. The beta-expansion of 1 is finite, $d_\beta(1) = t_1 \cdots t_m$.
Then

$$P(X) = X^m - t_1 X^{m-1} - \cdots - t_m$$

is the *characteristic polynomial* of β. To P is associated a linear recurrent sequence of integers U_β, defined by

$$u_{n+m} = t_1 u_{n+m-1} + \cdots + t_m u_n$$

$$u_0 = 1,\ u_i = t_1 u_{i-1} + \cdots + t_i u_0 + 1,\ 1 \leq i \leq m-1.$$

It will be useful for our purpose to extend the recurrence to elements with negative indices such as u_{-n}.
Case 2. The beta-expansion of 1 is infinite, $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+p})^\omega$.
Then

$$P(X) = X^{m+p} - t_1 X^{m+p-1} - \cdots - t_{m+p} - X^m + t_1 X^{m-1} + \cdots + t_m$$

is the *characteristic polynomial* of β. To P is associated a linear recurrent sequence of integers U_β, defined by

$$u_{n+m+p} = t_1 u_{n+m+p-1} + \cdots + t_{m+p} u_n + u_{n+m} - t_1 u_{n+m-1} - \cdots - t_m u_n$$

$$u_0 = 1,\ u_i = t_1 u_{i-1} + \cdots + t_i u_0 + 1,\ 1 \leq i \leq m+p-1.$$

We also extend the recurrence to elements with negative indices such as u_{-n}.

Note that the characteristic polynomial is a multiple in $\mathbb{Z}[X]$ of the minimal polynomial of β.

The increasing sequence U_β defines the *numeration system associated with* β.

Let us consider the set $\mathbb{Z}[X, X^{-1}]$ of formal polynomials in X and X^{-1} with integer coefficients. We define two applications. The first one is

$$\begin{array}{rcl} \pi_\beta : \mathbb{Z}[X, X^{-1}] & \to & \mathbb{Z}[\beta, \beta^{-1}] \subset \mathbb{R} \\ F = f_k X^k + \cdots + f_{-j} X^{-j} & \mapsto & f_k \beta^k + \cdots + f_{-j} \beta^{-j}. \end{array}$$

The second application is

$$\begin{array}{rcl} \pi_{U_\beta} : \mathbb{Z}[X, X^{-1}] & \to & \mathbb{Z} \\ F = f_k X^k + \cdots + f_{-j} X^{-j} & \mapsto & f_k u_k + \cdots + f_{-j} u_{-j}. \end{array}$$

Now let

$$\begin{array}{rcl} \Phi : \mathbb{Z}[\beta, \beta^{-1}] & \to & \mathbb{Z} \\ f_k \beta^k + \cdots + f_{-j} \beta^{-j} & \mapsto & f_k u_k + \cdots + f_{-j} u_{-j}. \end{array}$$

Remark that Φ is a $\mathbb{Z}$-linear map. We then have the following result [**4**].

***Proposition* 4.1**. — *Let $x \in \mathbb{Z}_\beta^+$ such that its beta-expansion is $\langle x \rangle_\beta = x_k \cdots x_0$, and let $N = \Phi(x)$. Then the greedy U_β-representation of N is exactly $\langle N \rangle_{U_\beta} = x_k \cdots x_0$.*

The set $\mathbb{Z}_\beta$ is ordered: for $n \geq 0$ denote by b_n the n-th beta-integer, and set $b_{-n} = -b_n$. As a consequence of Proposition 4.1, we have that

$$\Phi(b_n) = n$$

thus the function

$$\begin{array}{rcl} \mathbb{Z} & \to & \mathbb{Z}_\beta \\ n & \mapsto & b_n \end{array}$$

is a bijection.

***Example* 4.2**. — The numeration system associated with $\tau = (1 + \sqrt{5})/2$ is the Fibonacci numeration system with $u_0 = 1$ and $u_1 = 2$.

$\mathbb{Z}$	Fibonacci	$\mathbb{Z}_\tau$
1	1	1
2	10	τ
3	100	τ^2
4	101	$\tau^2 + 1$
5	1000	τ^3
6	1001	$\tau^3 + 1$

$\Diamond$

5. One-dimensional tiling and substitution associated with a Parry number

Tiling. — Let β be a Parry number. Positive beta-integers can be considered as vertices of a self-similar tiling of the positive real line with a finite number of tiles [**26**] : the lengths of the tiles are exactly the $T^i_\beta(1)$, for $i \geq 0$.

More precisely, if $d_\beta(1) = t_1 \cdots t_m$, there are m tiles, of respective lengths $1, T^1_\beta(1) = \beta - t_1$, $T^2_\beta(1) = \beta^2 - t_1\beta - t_2$, ..., $T^{m-1}_\beta(1) = \beta^{m-1} - t_1\beta^{m-2} - \cdots - t_{m-1}$.

If $d_\beta(1) = t_1 \cdots t_m(t_{m+1} \cdots t_{m+p})^\omega$, there are $m+p$ tiles, of respective lengths 1, $T^1_\beta(1) = \beta - t_1$, $T^2_\beta(1) = \beta^2 - t_1\beta - t_2$, ..., $T^{m+p-1}_\beta(1) = \beta^{m+p-1} - t_1\beta^{m+p-2} - \cdots - t_{m+p-1}$.

This tiling is equivalently described by a substitution.

Substitution. — Let β be a Parry number. To β one associates in a canonical way a beta-substitution σ_β [**9**]. We denote by a^k the word obtained by concatenating k letters a.

When $d_\beta(1) = t_1 \cdots t_m$, σ_β is defined on the alphabet $\mathbb{A} = \{a_0, \cdots, a_{m-1}\}$, by

$$\sigma_\beta : \begin{cases} a_0 & \mapsto & a_0^{t_1} a_1 \\ a_1 & \mapsto & a_0^{t_2} a_2 \\ & \cdots & \\ a_{m-2} & \mapsto & a_0^{t_{m-1}} a_{m-1} \\ a_{m-1} & \mapsto & a_0^{t_m}. \end{cases}$$

If $d_\beta(1) = t_1 \cdots t_m(t_{m+1} \cdots t_{m+p})^\omega$, σ_β is defined on the alphabet $\mathbb{A} = \{a_0, \cdots, a_{m+p-1}\}$ by

$$\sigma_\beta : \begin{cases} a_0 & \mapsto & a_0^{t_1} a_1 \\ a_1 & \mapsto & a_0^{t_2} a_2 \\ & \cdots & \\ a_{m+p-2} & \mapsto & a_0^{t_{m+p-1}} a_{m+p-1} \\ a_{m+p-1} & \mapsto & a_0^{t_{m+p}} a_{m+1}. \end{cases}$$

The substitution is a morphism from the monoid $\mathbb{A}^*$ into itself. One can see the letters a_i of $\mathbb{A}$ as the tiles defined above, with length of a_i equal to $\ell(a_i) = T^i_\beta(1)$. Then the substitution acts on the tiles as multiplication by β, and the interval $[0, \beta^n[$ is tiled by the word $\sigma^n_\beta(a_0)$.

We point out that a_0 is always a proper prefix of $\sigma_\beta(a_0)$. Thus for any integer $j \in \mathbb{N}$, $\sigma^j_\beta(a_0)$ is a prefix of $\sigma^{j+1}_\beta(a_0)$ and we can define an infinite word ω which is fixed under the action of σ_β. Let ω be the infinite word $\sigma^\infty_\beta(a_0)$ generated by σ_β, and let ω_n be the prefix of length n of ω.

Our purpose will be now to show the unique relation between the beta-integers and beta-substitutions. Denote by $|w|$ the length of a word w and by $|w|_a$ the number of letters a in w. We have

Proposition 5.1. — [**8, 7, 9**] *The characteristic polynomial of the matrix of the substitution σ_β is equal to the characteristic polynomial of β.*
The n-th term u_n of the sequence U_β is equal to $|\sigma^n_\beta(a_0)|$.

Let n be an integer ≥ 1. There exist an unique integer $\kappa = \kappa(n)$ and an unique "admissible" sequence $(w_i, c_i)_{i=0,\ldots,\kappa}$ in $\mathbb{A}^* \times \mathbb{A}$ such that w_κ is not empty and
(i) $w_\kappa c_\kappa$ is a prefix of $\sigma_\beta(a)$, where $a \in \mathbb{A}$
(ii) if $1 \leq i \leq \kappa$ then $w_{i-1}c_{i-1}$ is a prefix of $\sigma_\beta(c_i)$
(iii) $\omega_n = \sigma_\beta^\kappa(w_\kappa)\sigma_\beta^{\kappa-1}(w_{\kappa-1})\cdots(w_0)$
(iv) the sequence $|w_\kappa|\,|w_{\kappa-1}|\cdots|w_0|$ is the U_β-representation of n.

This passage from words to real numbers gives us the required relation between beta-integers and prefixes of the infinite word ω. Recall that b_n denotes the n-th beta-integer. From the previous results, we have, for $n \geq 1$

$$b_n = \sum_{a_i \in \mathbb{A}} |\omega_n|_{a_i} T_\beta^i(1).$$

We introduce a function which will play a central role in the proofs of the additive and multiplicative structure of $\mathbb{Z}_\beta$. Let a be in $\mathbb{A}$. The *counting beta-function* $\rho_a(n)$ for $n \in \mathbb{Z}$ is defined as

$$\rho_a(n) = \operatorname{sgn}(n)|\omega_{|n|}|_a.$$

In the following sections our aim will be to show that the set of beta-integers can be equipped with a suitable addition law $\oplus$ which makes it a group isomorphic to the group of integers, in the case when β is a quadratic Pisot unit, and in a simple cubic case, namely for the root > 1 of the polynomial $X^3 - 2X^2 - X + 1$. This law will be *compatible* with the ordinary addition, which means that if $b_m + b_n \in \mathbb{Z}_\beta$, then $b_m + b_n = b_m \oplus b_n$.

We will also equip $\mathbb{Z}_\beta$ with a multiplication law $\otimes$, which, despite its non-associativity and its non-distributivity, is compatible with the ordinary multiplication, and which means that if $b_m b_n \in \mathbb{Z}_\beta$, then $b_m b_n = b_m \otimes b_n$.

6. Quadratic case

Here we consider only the case in which β is a quadratic Pisot unit. There are two cases.

Case 1: β is the root > 1 of the polynomial $X^2 - aX - 1$, with $a \geq 1$. — The canonical alphabet is equal to $A = \{0, \ldots, a\}$, the beta-expansion of 1 is finite, equal to $d_\beta(1) = a1$, and every positive number of $\mathbb{Z}[\beta]$ has a finite beta-expansion [**11**].

Denote $\mathbb{A} = \{L, S\}$. The substitution σ_β is defined by

$$\sigma_\beta : \begin{cases} L & \mapsto & L^a S \\ S & \mapsto & L. \end{cases}$$

So here we have: $\ell(L) = 1$, $\ell(S) = T_\beta(1) = \beta - a = 1/\beta$.

From Proposition 5.1 (i) and (ii) we deduce that words w_i, $0 \leq i \leq \kappa$ are empty or $w_i \in \{L, \ldots, L^a\}$.

We thus have the following result.

Proposition 6.1. — *Let β be the root > 1 of the polynomial $X^2 - aX - 1$, with $a \geq 1$. Then the following statements hold for any $n, m \in \mathbb{Z}$.*
(i) $n - b_n = (1 - \ell(S))\rho_S(n)$.
(ii) Let $x = c_1\beta + c_2$ be in $\mathbb{Z}[\beta]$. Then x is equal to b_n for some n iff $\rho_S(\Phi(x)) = c_1$.
(iii) $\rho_S(n+m) - \rho_S(n) - \rho_S(m) \in \{0, \pm 1\}$.
(iv) $\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \{0, \pm(1 - \frac{1}{\beta})\}$.

Proof. — (i) We have $n = \rho_L(n) + \rho_S(n)$, and $b_n = \rho_L(n) + \rho_S(n)T_\beta(1) = n - \rho_S(n) + \rho_S(n)(1/\beta)$.
(ii) Since u_1 is equal to $a+1$, we get $\Phi(x) = c_1(a+1) + c_2$. We have $x = c_1(a + 1/\beta) + c_2 = c_1 T_\beta^1(1) + ac_1 + c_2$. Therefore the equalities $x = b_n$, $\Phi(x) = n$ and $\rho_S(\Phi(x)) = c_1$ are equivalent.
(iii) Let us take any $n, m \in \mathbb{Z}$. Suppose that there is a number $p \in \mathbb{Z}$ such that $\rho_S(n+m) - \rho_S(n) - \rho_S(m) = p$. Then from (i) follows that $b_{n+m} - b_n - b_m = p(1/\beta - 1)$. Let $'$ be the Galois automorphism $' : \beta \mapsto -1/\beta$. In [**5**] we proved that for any $n \in \mathbb{N}$, $b'_n \in (-1, \beta)$ and so $(b_{n+m} - b_n - b_m)' \in (-1 - 2\beta, 2\beta + 1)$. We deduce that $-1 - 2\beta < p(-\beta - 1) < 2\beta + 1$ and so $p \in \{0, \pm 1\}$.
(iv) It is a consequence of (i) and (iii). It follows that for any $m, n \in \mathbb{Z}$, the difference between $b_n + b_m$ and b_{m+n} lies in the set $\{0, \pm(1 - 1/\beta)\}$. This set is minimal, *i.e.* we cannot remove any element to hold the inclusion. □

In Proposition 6.1 (iv) we show that any number of the type $x + y$, where $x, y \in \mathbb{Z}_\beta$ can be written as $x + y = z + \eta(1 - \frac{1}{\beta})$, where $z \in \mathbb{Z}_\beta, \eta \in \{0, \pm 1\}$. This writing is unique. Indeed, let us suppose that there exist $x, y \in \mathbb{Z}_\beta$, $z, z' \in \mathbb{Z}_\beta$ and $\eta, \eta' \in \{0, \pm 1\}$ such that

$$z + \eta(1 - \frac{1}{\beta}) = x + y = z' + \eta'(1 - \frac{1}{\beta}).$$

Then $\Phi(\eta - \eta' + (\eta' - \eta)/\beta) = (\eta - \eta')u_0 + (\eta' - \eta)u_{-1} = 0$ since $u_0 = u_{-1} = 1$, therefore $\Phi(z') = \Phi(z)$ and $z = z'$.

Using this fact, we can define on $\mathbb{Z}_\beta$ an internal operation of *addition* $\oplus : \mathbb{Z}_\beta \times \mathbb{Z}_\beta \to \mathbb{Z}_\beta$,

$$x \oplus y = z.$$

Then clearly

$$b_m \oplus b_n = b_{m+n}.$$

Note that $\oplus$ is compatible with the ordinary addition. Indeed, let b_m and b_n be in $\mathbb{Z}_\beta$. From Proposition 6.1 we know that $b_m = c_1\beta + c_2$, $b_n = c_3\beta + c_4$, where $c_1 = \rho_S(m)$, $c_2 = m - (a+1)\rho_S(m)$ and $c_3 = \rho_S(n)$, $c_4 = n - (a+1)\rho_S(n)$. Thus, $b_m + b_n$ is in $\mathbb{Z}_\beta$, *i.e.* $b_m + b_n = b_k$ for some k, if and only if $k = m + n$.

The previous results imply that

Theorem 6.2. — *Let β be the root > 1 of the polynomial $X^2 - aX - 1$, with $a \geq 1$. The set $\mathbb{Z}_\beta$ equipped with addition $\oplus$ is a group isomorphic to $\mathbb{Z}$.*

Of course, it seems now straightforward to define an operation of *multiplication* $\otimes : \mathbb{Z}_\beta \times \mathbb{Z}_\beta \to \mathbb{Z}_\beta$ on the group $\mathbb{Z}_\beta$ in order to obtain a ring isomorphism between $\mathbb{Z}$ and $\mathbb{Z}_\beta$. Unfortunately, such operation would give us $\beta \otimes \beta \neq \beta^2$. However at the price of losing the ring structure, let us try to find such operation which is compatible with the ordinary multiplication, that is, which satisfies the condition:

$$xy \in \mathbb{Z}_\beta \Rightarrow x \otimes y = xy.$$

The existence of such an operation is proved by the following result.

***Proposition* 6.3**. — *Let β be the root > 1 of the polynomial $X^2 - aX - 1$, with $a \geq 1$, and let $\otimes : \mathbb{Z}_\beta \times \mathbb{Z}_\beta \to \mathbb{Z}_\beta$ be defined by*

$$b_m \otimes b_n = b_{(mn - a\rho_S(m)\rho_S(n))}. \tag{2}$$

Then
(i) if $b_m b_n$ is in $\mathbb{Z}_\beta$ then $b_m \otimes b_n = b_m b_n$,
(ii) in general, $b_m b_n - b_m \otimes b_n \in \{0, \pm 1, \ldots, \pm a\}(1 - \frac{1}{\beta})$.

Proof. — (i) Suppose that $n, m \in \mathbb{Z}$ and $b_m b_n \in \mathbb{Z}_\beta$. Then from Proposition 6.1 we know that $b_m = c_1\beta + c_2, b_n = c_3\beta + c_4$, where $c_1 = \rho_S(m)$, $c_2 = m - (a+1)\rho_S(m)$ and $c_3 = \rho_S(n)$, $c_4 = n - (a+1)\rho_S(n)$. Now we can rewrite $b_m b_n$ as $b_m b_n = \beta\,(ac_1c_3 + c_1c_4 + c_2c_3) + c_1c_3 + c_2c_4$. Using the assumption that $b_m b_n \in \mathbb{Z}_\beta$ we obtain that

$$b_m b_n = b_k, \text{ where } k = a^2 c_1 c_3 + (a+1)\,(c_1c_3 + c_1c_4 + c_2c_3) + c_2c_4.$$

Now we express $b_m \otimes b_n$. Because $mn = ((a+1)c_1 + c_2)((a+1)c_3 + c_4) = (a^2 + a)c_1c_3 + (a+1)\,(c_1c_3 + c_1c_4 + c_2c_3) + c_2c_4$, we see that $b_m \otimes b_n = b_k$.
(ii) Suppose that $b_m \otimes b_n = b_k$ with $k = (mn - a\rho_S(m)\rho_S(n))$, and that $b_m = c_1\beta + c_2$, $b_n = c_3\beta + c_4$. In general, $b_m b_n \in \mathbb{Z}[\beta]$ and so there exist coefficients $c, d \in \mathbb{Z}$ such that $b_m b_n = c + \frac{d}{\beta}$. We can see that $c + d = (a+1)(ac_1c_3 + c_1c_4 + c_2c_3) + c_1c_3 + c_2c_4 = k$. Due to this fact we deduce that $b_m b_n - b_m \otimes b_n \in p(\frac{1}{\beta} - 1)$, where $p \in \mathbb{Z}$. Similarly to the proof of Proposition 6.1 we obtain that $p \in \{0, \pm 1, \ldots, \pm a\}$. More precisely, for any $n \in \mathbb{Z}$, $b'_n \in (-\beta, \beta)$ and so $(b_m b_n - b_m \otimes b_n)' \in (-\beta(\beta+1), \beta(\beta+1))$. From the relations $-\beta(\beta+1) < p(-\beta-1) < \beta(\beta+1)$ we get the result. □

The operation $\otimes$ defined by (2) is commutative but is not associative, as shown on the following example.

***Example* 6.4**. — Let $\beta = \tau$. Then $\omega = LSLLSLSL\ldots$. Take $b_2 = \tau$ and $b_4 = \tau^2 + 1$. Then $b_2 \otimes b_4 = \tau^3 + \tau = b_7$. Now, $(b_2 \otimes b_4) \otimes b_4 = b_7 \otimes b_4 = b_{(28-3)} = b_{25} = \tau^6 + \tau^3 + 1$. On the other hand $b_4 \otimes b_4 = b_{(16-1)} = b_{15} = \tau^5 + \tau$, and $b_2 \otimes (b_4 \otimes b_4) = \tau^6 + \tau^2 = b_{24}$, thus the operation $\otimes$ is not associative. ◊

Moreover, the three sets consisting of $\{((x \otimes y) \otimes z) \oplus (-x \otimes (y \otimes z)) \mid x, y, z \in \mathbb{Z}_\beta\}$ and the sets obtained by circular permutation are not finite. The same holds true for the set $x \otimes (y \oplus z) \oplus (-(x \otimes y) \oplus (x \otimes z))$. However, it would be interesting to give precise estimates on such sets in order to understand the deviation from a pure ring structure.

In the case of the golden mean τ, $x \oplus y$ (respectively $x \otimes y$) is actually the τ-integer closest to $x + y$ (respectively xy).

Case 2: β is the root > 1 of the polynomial $X^2 - aX + 1$, with $a \geq 3$. — The canonical alphabet is equal to $A = \{0, \ldots, a-1\}$, the beta-expansion of 1 is eventually periodic, equal to $d_\beta(1) = (a-1)(a-2)^\omega$, and every positive number of $\mathbb{Z}[\beta]$ has an eventually periodic beta-expansion, which is finite for numbers from $\mathbb{N}[\beta]$, [**11**].

The substitution σ_β is defined on $\mathbb{A} = \{L, S\}$ by

$$\sigma_\beta : \begin{cases} L & \mapsto & L^{a-1}S \\ S & \mapsto & L^{a-2}S. \end{cases}$$

We have that $\ell(L) = 1$, $\ell(S) = T_\beta(1) = \beta - (a-1) = 1 - 1/\beta$.

From Proposition 5.1 (i) and (ii) we deduce that words w_i, $0 \leq i \leq \kappa$ are empty or $w_i \in \{L, \ldots, L^{a-1}\}$. We thus have.

***Proposition* 6.5**. — *Let β be the root > 1 of the polynomial $X^2 - aX + 1$, with $a \geq 3$. Then for any $n, m \in \mathbb{Z}$ the following properties hold true:*
(i) $n - b_n = \rho_S(n)/\beta$.
(ii) Let $x = c_1\beta + c_2$ be in $\mathbb{Z}[\beta]$. Then x is equal to b_n for some n iff $\rho_S(\Phi(x)) = c_1$.
(iii) $\rho_S(n+m) - \rho_S(n) - \rho_S(m) \in \{0, \pm 1\}$.
(iv) $\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \{0, \pm\frac{1}{\beta}\}$

Proof. — The proof is similar to that of Proposition 6.1.
(ii) Use just the fact that u_1 is equal to a here.
(iii) Let $'$ be the Galois automorphism $' : \beta \mapsto 1/\beta$. For any $n \in \mathbb{N}$ we have $b'_n \in [0, \beta)$ (see [**5**]). □

Similarly to the previous case, we can define an addition law on $\mathbb{Z}_\beta$. From Proposition 6.5 (iii), we see that any number of the form $x + y$, where $x, y \in \mathbb{Z}_\beta$, can be uniquely written as $x + y = z + \frac{\eta}{\beta}$, where $z \in \mathbb{Z}_\beta, \eta \in \{0, \pm 1\}$. Thus we define on $\mathbb{Z}_\beta$ an internal operation of *addition* $\oplus : \mathbb{Z}_\beta \times \mathbb{Z}_\beta \to \mathbb{Z}_\beta$

$$x \oplus y = z.$$

Thus

$$b_m \oplus b_n = b_{m+n}$$

and $\oplus$ is compatible with the ordinary addition. As above, we have

***Theorem* 6.6**. — *Let β be the root > 1 of the polynomial $X^2 - aX + 1$, with $a \geq 3$. The set $\mathbb{Z}_\beta$ equipped with addition $\oplus$ is a group isomorphic to $\mathbb{Z}$.*

The operation of multiplication is defined by using the same criterion as in the previous case:

***Proposition* 6.7**. — *Let β be the root > 1 of the polynomial $X^2 - aX + 1$, with $a \geq 3$, and let $\otimes : \mathbb{Z}_\beta \times \mathbb{Z}_\beta \to \mathbb{Z}_\beta$ be defined as*

$$b_m \otimes b_n = b_{(mn - \rho_S(m)\rho_S(n))}.$$

Then
(i) if $b_m b_n$ is in $\mathbb{Z}_\beta$ then $b_m \otimes b_n = b_m b_n$,
(ii) in general $b_m b_n - b_m \otimes b_n \in \{0, 1, \ldots, (a-1)\}\frac{\mathrm{sgn}(b_m b_n)}{\beta}$.

Proof. — The proof is similar to that of Proposition 6.3.
(i) Suppose $n, m \in \mathbb{Z}$ and $b_m b_n \in \mathbb{Z}_\beta$. Then from Proposition 6.5. we know that $b_m = c_1\beta + c_2, b_n = c_3\beta + c_4$, where $c_1 = \rho_S(m), c_2 = m - a\rho_S(m)$ and $c_3 = \rho_S(n), c_4 = n - a\rho_S(n)$. Now we can rewrite $b_m b_n$ as $b_m b_n = \beta\,(ac_1c_3 + c_1c_4 + c_2c_3) - c_1c_3 + c_2c_4$. Using the assumption that $b_m b_n \in \mathbb{Z}_\beta$ we obtain that

$$b_m b_n = b_k, \text{ where } k = (a^2 - 1)c_1c_3 + a\,(c_1c_4 + c_2c_3) + c_2c_4.$$

Now we express $b_n \otimes b_m$. Because $mn = (ac_1 + c_2)(ac_3 + c_4) = a^2c_1c_3 + a\,(c_1c_4 + c_2c_3) + c_2c_4$, we see that $b_m \otimes b_n = b_k$.
(ii) Suppose that $n, m \in \mathbb{Z}$ and $b_m \otimes b_n = b_k$, where $k = (mn - \rho_S(m)\rho_S(n))$. Since $b_m b_n \in \mathbb{Z}[\beta]$, there exist coefficients $c, d \in \mathbb{Z}$ such that $b_m b_n = c + d(1 - \frac{1}{\beta})$. Similarly to the proof of Proposition 6.3 we have $c + d = k$. Thus we can assert that $b_m b_n - b_m \otimes b_n = \frac{p}{\beta}$, where $p \in \mathbb{Z}$. Now we follow the same argumentation scheme as in the proof of Proposition 6.3, having for any $n \in \mathbb{N}, b'_n \in [0, \beta)$ and $\operatorname{sgn}(b_m b_n) = \operatorname{sgn}(b_m \otimes b_n)$. The latter follows from the fact that for any $n \in \mathbb{N},\ \rho_S(n) \leq n$. Thus, if $b_m b_n \geq 0$ then $(b_m b_n - b_m \otimes b_n)' \in [0, \beta^2)$ which gives us the relations $0 \leq p\beta < \beta^2$. Symmetrically, if $b_m b_n < 0$ then $(b_m b_n - b_m \otimes b_n)' \in (-\beta^2, 0]$ and $-\beta^2 < p\beta \leq 0$. □

7. Cubic case

We now consider the case in which $\beta = 1 + 2\cos(2\pi/7)$ is the root > 1 of the polynomial $X^3 - 2X^2 - X + 1$. The associated cyclotomic ring presents a seven-fold symmetry. In that case, the canonical alphabet is $A = \{0, 1, 2\}$ and $d_\beta(1) = 2(01)^\omega$. Note that, contrarily to the quadratic case, in general the integers do not have a finite beta-expansion: for instance, the beta-expansion of 3 is equal to $10 \cdot 11(10)^\omega$.

Take $\mathbb{A} = \{L, S, M\}$. The substitution σ_β is defined by

$$\sigma_\beta : \begin{cases} L & \mapsto & LLS \\ S & \mapsto & M \\ M & \mapsto & LS. \end{cases}$$

We have $\ell(L) = 1,\ \ell(S) = T_\beta(1) = \beta - 2$ and $\ell(M) = T_\beta^2(1) = \beta^2 - 2\beta$. From Proposition 5.1 (i) and (ii) we deduce that w_i is empty or $w_i \in \{L, LL\}$ for $0 \leq i \leq \kappa$.

In the ring $\mathbb{Z}[\beta]$ we define the Galois ring automorphism (conjugation) of order two

$$\prime : \beta \mapsto \beta' = \beta^2 - 2\beta.$$

We see that $\beta' = \beta_1, (\beta')' = \beta_2 = -\beta^2 + \beta + 2$ are the two other (conjugate) roots of the equation $X^3 = 2X^2 + X - 1$, *cf.* [**1**]. This automorphism is the trace on $\mathbb{Q}(\beta)$ of the Galois automorphism of the field $\mathbb{Q}(\zeta)$, with $\zeta = e^{2i\pi/7}$, defined by the map $\zeta \mapsto \zeta^2$. It is of order 3, and $\mathbb{Q}(\beta)$ is invariant under its action.

In [**5**] we proved that for the first quadratic case there exists a window $(-1, \beta)$ such that

$$\mathbb{Z}_\beta^+ = \{x \in \mathbb{Z}[\beta],\ x \geq 0 \mid x' \in (-1, \beta)\} \tag{3}$$

and for the second quadratic case there exists a window $[0, \beta)$ such that

$$\mathbb{Z}_\beta^+ = \{x \in \mathbb{Z}[\beta],\ x \geq 0 \mid x' \in [0, \beta)\}. \tag{4}$$

The situation in the cubic case is similar but slightly more difficult. Because the Galois automorphism is of order two, we have to consider at least two windows. Let (c_1, c_2) and (d_1, d_2) be two intervals of $\mathbb{R}$, and denote $\Sigma_1(c_1, c_2) = \{x \in \mathbb{Z}[\beta],\ x \geq 0\ |x' \in (c_1, c_2)\}$ and $\Sigma_2(d_1, d_2) = \{x \in \mathbb{Z}[\beta],\ x \geq 0\ |x'' \in (d_1, d_2)\}$. In fact, for our purpose it is sufficient to find two minimal intervals $(c_1, c_2) \subset \mathbb{R},\ (d_1, d_2) \subset \mathbb{R}$ such that

$$\mathbb{Z}_\beta^+ \subset \Sigma_1(c_1, c_2) \cap \Sigma_2(d_1, d_2). \tag{5}$$

Of course, we know that, from the point of view of the cut and projection method, the inclusion with only two intersecting sets is not exhaustive but a closer look at the structure of $\mathbb{Z}_\beta$ reveals deep differences with the quadratic case. The analysis of the structure of $\mathbb{Z}_\beta$ shows that it is not possible anymore to find a finite number of windows in order to obtain an equation similar to (3) or (4). Thus, in the cubic case, cut-and-project sets and beta-integers are not so much compatible as they were in the quadratic case.

Now, in order to find minimal intervals in (5), we use a recurrence formula for $\mathbb{Z}_\beta$. Let us denote by B_N the set $B_N = \{x \in \mathbb{Z}_\beta^+ \mid x < \beta^N\}$. It is obvious that $\mathbb{Z}_\beta^+ = \cup_{N \geq 1} B_N$. The following proposition determines how to compute B_N through a recurrence method.

Proposition 7.1. — *Let $N \geq 1$ be an integer. Then B_N admits a partition of the form*

$$B_N = X_N \cup Y_N \cup \beta Y_{N-1}$$

where

$$X_0 = \{0, 1\},\ Y_0 = \{2\},\ X_1 = \{0, 1, \beta, \beta + 1\},\ Y_1 = \{2, \beta + 2\}$$

and for $N \geq 1$,

$$\begin{aligned} X_{N+1} &= \beta X_N \cup \beta^2 Y_{N-1} \cup (\beta X_N + 1) \\ Y_{N+1} &= (\beta X_N + 2) \cup (\beta^2 Y_{N-1} + 1). \end{aligned} \tag{6}$$

Proof. — Splitting B_N into the three non-intersecting sets $X_N, Y_N, \beta Y_{N-1}$ helps us to build all the beta-expansions with non-negative powers with the aid of the recurrence method. From (6) it follows that the set X_N contains all x in B_N whose beta-expansion $\langle x \rangle_\beta$ does not end with letter 2 or words 20, $2(01)^k$, $2(01)^k0$. On the contrary the set Y_N contains all x in B_N whose beta-expansion $\langle x \rangle_\beta$ ends with words 2, $2(01)^k$. ☐

Using the induction formula (6) we obtain for any $N \in \mathbb{N}$ the following

$$\begin{aligned} X_N &\subset \Sigma_1(0, \beta) \cap \Sigma_2(\frac{\beta_2}{1 - \beta_2^2}, \frac{1}{1 - \beta_2^2}), \\ Y_N &\subset \Sigma_1(\frac{1}{1 - \beta_1^2}, \beta + 1) \cap \Sigma_2(\frac{\beta_2}{1 - \beta_2^2} + \beta_2, \frac{1}{1 - \beta_2^2} + 1). \end{aligned}$$

From this we immediately obtain the required conjugate windows

$$\mathbb{Z}_\beta^+ \subset \Sigma_1(0, \beta + 1) \cap \Sigma_2(\beta_2 - \beta, \frac{-\beta}{\beta_2} + 1). \tag{7}$$

Proposition 7.2. — *Let β be the root >1 of the polynomial X^3-2X^2-X+1 and let F be the finite set*

$$F=\pm\{0,-\beta^{-1},-2\beta^{-1},-3\beta^{-1},\beta-3,2(\beta-3),\beta^2-\beta-4,\\ \beta^2-3\beta+2,2\beta^2-5\beta+1,3\beta^2-7\beta,2\beta^2-3\beta-5,\beta^2-4\beta+5,2\beta^2-6\beta+4,-3\beta^2+8\beta-3\}. \tag{8}$$

Then for any $m,n\in\mathbb{Z}$ the following holds true:
(i) $n-b_n=\rho_M(n)(1-\beta^2+2\beta)+\rho_S(n)(3-\beta)$,
(ii) $x=c_1\beta^2+c_2\beta+c_3\in\mathbb{Z}[\beta]$ is a beta-integer if and only if

$$\rho_M(\Phi(x))=c_1 \text{ and } \rho_S(\Phi(x))=2c_1+c_2,$$

(iii) $b_m+b_n-b_{m+n}\in F$,
(iv) $\rho_S(m)+\rho_S(n)-\rho_S(m+n)\in\pm\{0,1,2\}$, $\rho_M(m)+\rho_M(n)-\rho_M(m+n)\in\pm\{0,1,2,3\}$ and $\rho_L(m)+\rho_L(n)-\rho_L(m+n)\in\pm\{0,1,2,3\}$,
(v) $\mathbb{Z}_\beta+\mathbb{Z}_\beta\subset\mathbb{Z}_\beta+F$.

Proof. — (i) and (ii) are similar to the proof of Proposition 6.5.
(iii) Suppose that $b_m+b_n=a\,\ell(S)+b\,\ell(M)+c\,\ell(L)$ then $a+b+c=m+n$. It follows that $b_m+b_n-b_{m+n}=p(\ell(S)-\ell(L))+s(\ell(M)-\ell(L))$, where $p,s\in\mathbb{Z}$. Suppose that $m,n\geq 0$, then from (7) we obtain $\mathbb{Z}_\beta^++\mathbb{Z}_\beta^+\subset\Sigma_1(0,2\beta+2)\cap\Sigma_2(2\beta_2-2\beta,\frac{-2\beta}{\beta_2}+2)$ and so we get two inequalities

$$\begin{aligned} -\beta-1< &\quad (b_m+b_n-b_{m+n})' \quad <2\beta+2\\ \frac{\beta}{\beta_2}-1-2\beta+2\beta_2< &\quad (b_m+b_n-b_{m+n})'' \quad <\beta-\beta_2-\frac{2\beta}{\beta_2}+2. \end{aligned} \tag{9}$$

Now, replacing $b_m+b_n-b_{m+n}$ by $p(\beta-3)+s(\beta_1-1)$ a technical exhaustive analysis leads to the following list of possible combinations of $(p,s)\in\mathbb{Z}^2$

$$(0,0),(0,1),(0,-1),(0,-2),(0,-3),(1,0),(1,1),(1,-1),(1,-2), \tag{10}$$

$$(1,-3),(0,-1),(-1,-1),(-1,2),(-1,-2),(-1,3),(-2,0),(-2,1),\\ (-2,3),(2,-2),(2,-1),(2,-3),(-2,2),(-1,0),(-1,1). \tag{11}$$

To end the proof we assume that $m\geq 0,n\leq 0$ and $m+n\geq 0$. Using the inclusion $\mathbb{Z}_\beta^+-\mathbb{Z}_\beta^+\subset\Sigma_1(-\beta-1,\beta+1)\cap\Sigma_2(\frac{\beta}{\beta_2}-1-\beta+\beta_2,1-\frac{\beta}{\beta_2}+\beta-\beta_2)$ we get the inequalities

$$-2\beta-2<(p(\beta-3)+s(\beta_1-1))'<\beta+1$$

$$\frac{2\beta}{\beta_2}-2-\beta+\beta_2<(p(\beta-3)+s(\beta_1-1))''<2\beta-2\beta_2+1-\frac{\beta}{\beta_2}.$$

We note that they are similar to (9). The possible combinations of $(p,s)\in\mathbb{Z}^2$ are symmetrical to (10). From the above and from the symmetry of $\mathbb{Z}_\beta=\mathbb{Z}_\beta^+\cup\mathbb{Z}_\beta^-$, it follows that for any $m,n\in\mathbb{Z}$, $b_m+b_n-b_{m+n}\in\pm\{0,\beta_1-1,2(\beta_1-1),3(\beta_1-1),\beta-3,2(\beta-3),\beta^2-\beta-4,\beta^2-3\beta+2,2\beta^2-5\beta+1,3\beta^2-7\beta,2\beta^2-3\beta-5,\beta^2-4\beta+5,2\beta^2-6\beta+4,-3\beta^2+8\beta-3\}$.
(iv) It is a consequence of the proof of (iii)-(10). ☐

As a consequence of the previous result we obtain that any number of type $x+y$, where $x, y \in \mathbb{Z}_\beta$, can be uniquely written as $x+y=z+f$, where $z \in \mathbb{Z}_\beta$, $f \in F$, and F is defined in (8). It means that an operation of addition can be defined in a similar way as it was in the quadratic case, *i.e.* $\oplus : \mathbb{Z}_\beta \times \mathbb{Z}_\beta \to \mathbb{Z}_\beta$,

$$x \oplus y = z.$$

We have that

$$b_m \oplus b_n = b_{m+n}$$

and $\oplus$ is compatible with the addition. Thus the following result holds true.

***Theorem* 7.3.** — *Let β be the root >1 of the polynomial X^3-2X^2-X+1. Then $\mathbb{Z}_\beta$ equipped with the addition $\oplus$ is a group isomorphic to $\mathbb{Z}$.*

To be complete, let us remark that one can define a multiplication law, which satisfies (2), but the formula seems rather complicate:

$$b_m \otimes b_n = b_{mn-\Delta},$$

where

$$\Delta = [26\rho_M(m)\rho_M(n) + 9\rho_M(m)(\rho_S(n) - 2\rho_M(n)) + \\ 9\rho_M(n)(\rho_S(m) - 2\rho_M(m)) + 2(\rho_S(m) - 2\rho_m(m))(\rho_S(n) - 2\rho_S(n))].$$

8. Tau-integer labelling of two-dimensional structures

It is well known that the condition $2\cos\frac{2\pi}{n} \in \mathbb{Z}$ characterizes n-fold Bravais lattices in $\mathbb{R}^2$ (and in $\mathbb{R}^3$). Let us put $\zeta = e^{\frac{2\pi i}{n}}$, $\zeta^n = 1$. If we consider the $\mathbb{Z}$-module in the plane :

$$\mathbb{Z}[\zeta] = \mathbb{Z} + \mathbb{Z}\zeta + \mathbb{Z}\zeta^2 + \cdots + \mathbb{Z}\zeta^{n-1},$$

we get the cyclotomic ring of order n. This n-fold structure is generically dense in $\mathbb{C}$, except precisely for the crystallographic cases, that is, for $n=1$, $n=2$, $n=3$, $n=4$ and $n=6$. We indeed check that $\mathbb{Z}[\zeta] = \mathbb{Z}$ for $n=1$ or 2, $\mathbb{Z}[\zeta] = \mathbb{Z} + \mathbb{Z}i$ for $n=4$ (square lattice), and $\mathbb{Z}[\zeta] = \mathbb{Z} + \mathbb{Z}e^{\frac{i\pi}{3}}$ for the hexagonal cases $n=3$ and $n=6$.

If n is not crystallographic, $2\cos\frac{2\pi}{n}$ is an algebraic integer of degree $m \le \lfloor (n-1)/2 \rfloor$. A *cyclotomic Pisot number* with symmetry of order n is a Pisot number β such that

$$\mathbb{Z}[2\cos\frac{2\pi}{n}] = \mathbb{Z}[\beta].$$

Then $m = \varphi(n)/2$, where φ is the Euler function, and

$$\mathbb{Z}[\beta] + \mathbb{Z}[\beta]\zeta \equiv \mathbb{Z}[\zeta]$$

is a ring invariant under rotation of order n (see [**1**]). The quasicrystalline numbers of Eq. (1) are all cyclotomic Pisot units. As a consequence of the results presented above, if β is a quadratic Pisot unit, or the root of X^3-2X^2-X+1, then if $(\mathbf{e}_i)$ is a base of $\mathbb{R}^d$

$$\Lambda = \sum_{i=1}^{d} \mathbb{Z}_\beta \mathbf{e}_i$$

is a Meyer set and a lattice for the law $\oplus$. Moreover $\mathbb{Z}_\beta \otimes \Lambda \subset \Lambda$.

We shall adopt the generic name of *beta-lattice* for such a Λ. An example of a beta-lattice in the plane is a point set of the form

$$\Gamma_q = \mathbb{Z}_\beta + \mathbb{Z}_\beta \zeta^q$$

for $1 \leq q \leq n-1$.

Let

$$\Lambda_q = \bigcup_{j=0}^{n-1} \Gamma_q \zeta^j$$

and

$$\mathbb{Z}_\beta[\zeta] = \sum_{j=0}^{n-1} \mathbb{Z}_\beta \zeta^j$$

Then $\mathbb{Z}_\beta[\zeta]$ and Λ_q for $1 \leq q \leq n-1$ are Meyer sets.

Let us now focus on the simplest case, namely $n = 5$ or 10. It is more convenient to introduce the root of unity $\xi = e^{\frac{i\pi}{5}}$, since $\tau = 2\cos\pi/5 = \xi + \xi^c$, where ξ^c is the complex conjugate of ξ. We obtain the set

$$\mathbb{Z}_\tau[\xi] \;\equiv\; \mathbb{Z}_\tau + \mathbb{Z}_\tau \xi + \mathbb{Z}_\tau \xi^2 + \mathbb{Z}_\tau \xi^3 + \mathbb{Z}_\tau \xi^4, \tag{12}$$

which we can call the *five-fold cyclotomic quasiring*. This discrete set $\mathbb{Z}_\tau[\xi]$ is displayed in Fig. 1.

If we now consider the following τ-lattices in the plane,

$$\Gamma_q = \mathbb{Z}_\tau + \mathbb{Z}_\tau \xi^q, \quad q = 1,\ 2,\ 3, \text{or } 4, \tag{13}$$

we can easily guess that none of them is identical to (12). This fact is obvious by simple inspection of Fig. 2 in which the set Γ_1 is displayed.

On the other hand, we can prove ([**5**]) the following inclusions

$$\Gamma_q \subset \mathbb{Z}_\tau[\xi] \subset \frac{\Gamma_q}{\tau^4}. \tag{14}$$

One can understand from this illustrative example that the τ-lattices Γ_q are "universal" labelling frames for a large class of planar pentagonal or decagonal quasilattices Λ of interest in physics. Universal means that it is always possible to embed Λ into a suitably deflated or inflated version of such "τ-square" papers

$$\Lambda \subset \Gamma_q/\tau^j,\ j \in \mathbb{Z}.$$

This embedding is understood through cut and projection method and labelling procedures. We just have to define the algebraic dual operation extending the Galois conjugation. Due to the following algebraic relations based on the cyclotomic nature of $\tau = \xi + \xi^c = 2\cos\frac{\pi}{5}$, we have

$$\xi^2 = -1 + \tau\xi,\ \xi^3 = -\tau + \tau\xi,\ \xi^4 = -\tau + \xi.$$

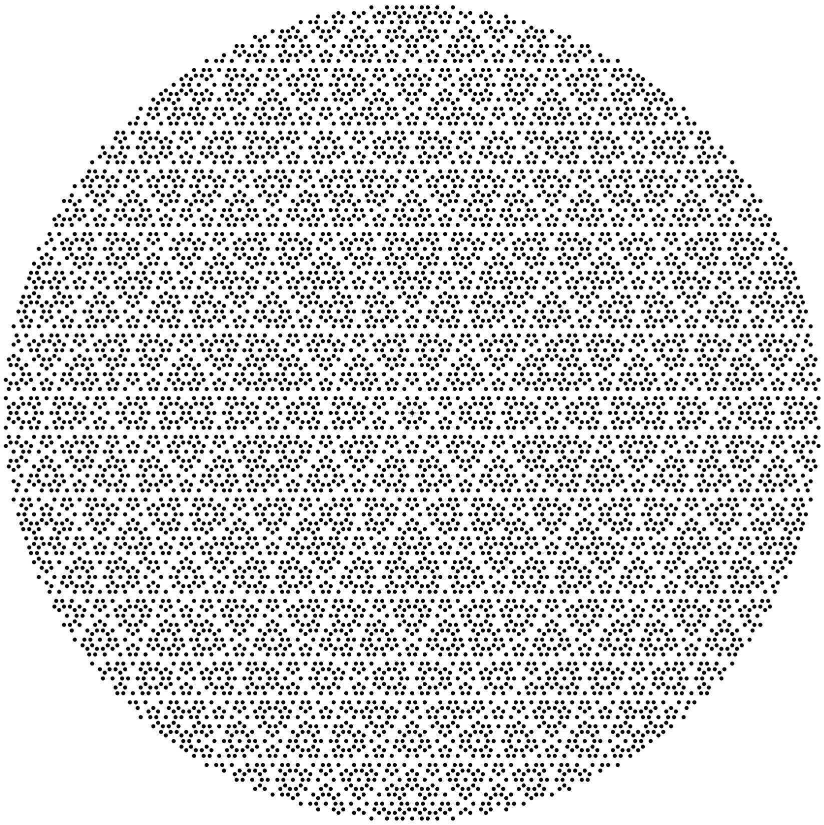

FIGURE 1. Five-fold cyclotomic quasiring

It follows that $\mathbb{Z}[\xi] = \mathbb{Z}[\tau] + \mathbb{Z}[\tau]\xi$. Consistently to the fact that $\tau' = 2\cos\frac{3\pi}{5} = \xi^3 + (\xi^c)^3$, we introduce the automorphism in $\mathbb{Z}[\xi]$,

$$z = m + n\tau + (p + q\tau)\xi \;\longrightarrow\; z^\star = m - n\frac{1}{\tau} + (p - q\frac{1}{\tau})\xi^3,$$

and the filtering procedure (possibly involving a "phason" shift $\Psi \in \mathbb{C}$),

$$\Sigma_\Psi^P = \{z \in \mathbb{Z}[\xi] \mid z^\star - \Psi \in P\} \;=\; (\mathbb{Z}[\xi] \cap (P + \Psi))^\star,$$

where P is bounded in the plane. Then it is just a matter of choice of appropriate scale and origin in order to get $P + \Psi \subset \mathbb{Z}'_\tau + \mathbb{Z}'_\tau\xi^{3q} = \Gamma_q^\star$, with the notations of (14), and for a certain $q = 1, 2, 3, 4$. It follows the embedding into a τ-lattice Γ_q and hence the labelling of the quasilattice points with those of Γ_q:

$$\Sigma_\Psi^P = \{z \in \mathbb{Z}_\tau + \mathbb{Z}_\tau\xi^q = \Gamma_q \text{ such that } z^\star - \Psi \in P\}.$$

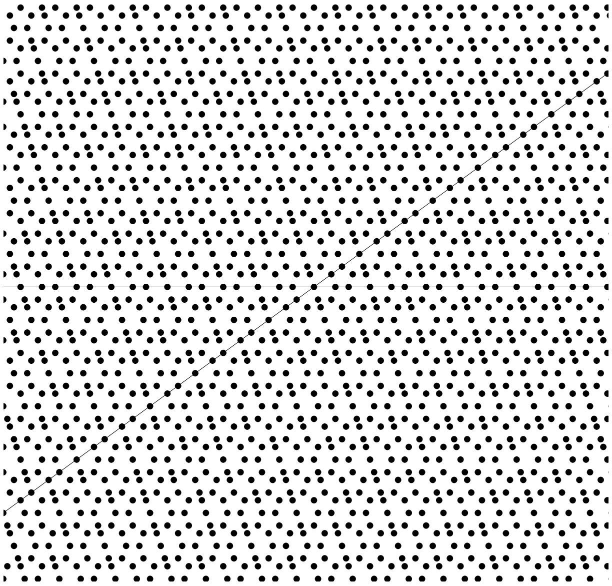

FIGURE 2. τ-lattice Γ_1

A nice example of such an embedding/labelling is provided by Penrose tilings [**23**] and their diffraction patterns. We show in Fig. 3 how the set of Penrose tiling vertices is actually a subset of the lattice Γ_1.

The Bragg peaks, beyond a given intensity, of the corresponding diffraction pattern are shown in Fig. 4.

For understanding Fig. 3, we start from a τ-lattice of the type $\Gamma_q = \mathbb{Z}_\tau + \mathbb{Z}_\tau \xi^q$, and follow a precise algebraic filtering in order to get a Penrose set as a subset. The procedure is based on both algebraic colouring and conjugation [**21**]. The introduction of colour stems from the fact that the integral coordinates $(n_0, n_1, n_2, n_3, n_4)$ of a cyclotomic number $z = n_0 + n_1\xi + n_2\xi^2 + n_3\xi^3 + n_4\xi^4$ in $\mathbb{Z}[\xi]$ are not unique. The quantity $n_0 - n_1 + n_2 - n_3 + n_4$ is defined modulo 5, due to the identity $1 - \xi + \xi^2 - \xi^3 + \xi^4 = 0$. Therefore, there are five (algebraic) colours, each one corresponding to an equivalence class modulo 5. Hence there exists a colouring ring homomorphism $\chi : \mathbb{Z}[\xi] \to \mathbb{Z}/5\mathbb{Z}$ given by

$$z = n_0 + n_1\xi + n_2\xi^2 + n_3\xi^3 + n_4\xi^4 \mapsto \chi(z) = \overline{n_0 - n_1 + n_2 - n_3 + n_4},$$

where the overbar designates the equivalence class modulo 5. Note that the colour of τ is $\bar{3}$, and the latter is the unique root of the equation $x^2 = x + 1$ in $\mathbb{Z}/5\mathbb{Z}$. Also note that colour is left unchanged under algebraic conjugation, since $\chi(-\frac{1}{\tau}) = \chi(1-\tau) = \bar{3}$, and $\chi(\xi^\star) = (\chi(\xi))^3 = \overline{-1}$. Let $\Psi = \sum_{i=0}^4 \gamma_i \xi^{2i} \in \mathbb{Z}[\xi]$ such that $\sum_{i=0}^4 \gamma_i = 0$. The set Π_Ψ of vertices of the Penrose tiling with associated phason Ψ is the union of four "coloured" discrete subsets $\Sigma(k) \subset \mathbb{Z}[\xi]$, one per colour $\bar{k} \in \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \}$, ($\bar{0}$

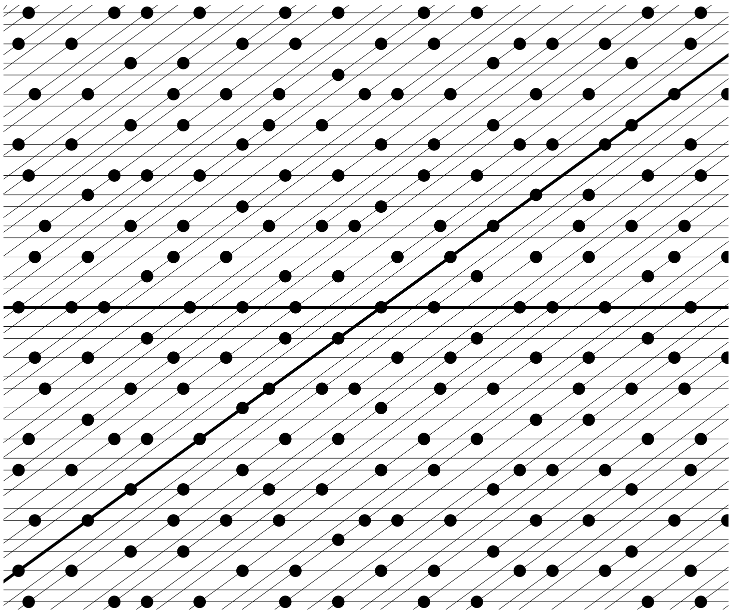

FIGURE 3. Penrose quasilattice as a subset of the τ-lattice Γ_1

is excluded), and selected by imposing pentagonal windows in the Galois conjugate space. More precisely,

$$\Pi_\Psi = \bigcup_{k=1}^{4} \Sigma(k), \text{ where } \Sigma(k) = \{z \in \mathbb{Z}[\xi] \mid \chi(z) = \bar{k},\ z^\star - \Psi \in P_{\bar{k}}\}, \tag{15}$$

where $P_{\bar{1}}$ is the pentagonal convex hull of the five points $1, \xi^2, \xi^4, \xi^6, \xi^8$ (all with colour $\bar{1}$), $P_{\bar{4}} = -P_{\bar{1}}$, $P_{\bar{3}} = \tau P_{\bar{1}}$, and $P_{\bar{2}} = -\tau P_{\bar{1}}$. Note that the other pentagons have vertices with colour $\bar{4}$, $\bar{3}$, and $\bar{2}$ respectively. This is consistent with (15). Let us now denote by R_q the rhomboid convex hull of $\{1+\xi^q, -1+\xi^q, -1-\xi^q, 1-\xi^q\}$, for $q = 1, 2, 3, 4$. Those lozenges can also be written as $R_q = [-1, 1] + [-1, 1]\xi^q$. It is shown in [**5**] that

$$\mathbb{Z}'_\tau \subset (-\tau, \tau) \cap \mathbb{Z}[\tau] \text{ and } \mathbb{Z}'_\tau \cap [-1, 1] = \mathbb{Z}[\tau] \cap [-1, 1].$$

From that and (13) follows that

$$\mathbb{Z}[\xi] \cap R_q = (\mathbb{Z}'_\tau + \mathbb{Z}'_\tau \xi^q) \cap R_q = \Gamma^\star_{q-3} \cap R_q.$$

Next we notice that $P_{\bar{1}} \cup P_{\bar{4}} \cup \frac{1}{\tau} P_{\bar{2}} \cup \frac{1}{\tau} P_{\bar{3}} \subset R_q$ for $q = 2$ or 3, whereas

$$\frac{1}{\tau} P_{\bar{1}} \cup \frac{1}{\tau} P_{\bar{4}} \cup \frac{1}{\tau^2} P_{\bar{2}} \cup \frac{1}{\tau^2} P_{\bar{3}} \subset R_q$$

for $q = 1$ or 4. So a suitable scaling allows one to view Penrose vertices as elements of the labelling lattice Γ_q. As a matter of fact, for $q = 1$ or 4, and with $\Psi = 0$, we

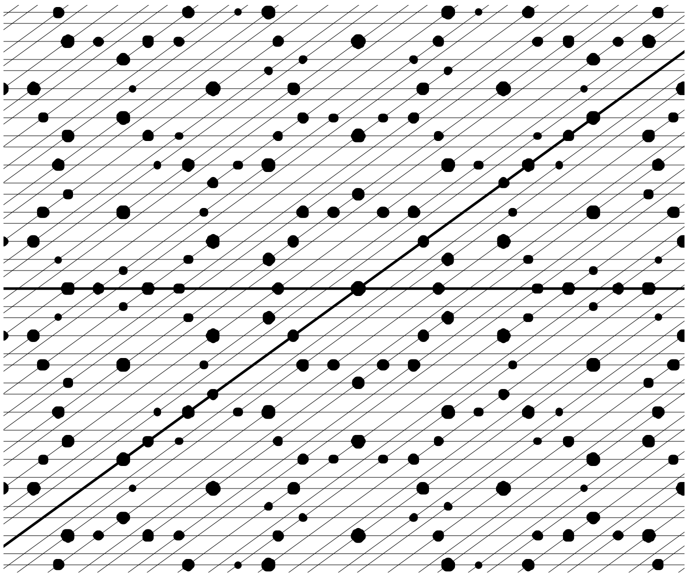

FIGURE 4. Diffraction pattern of Penrose quasilattice as a subset of the τ-lattice Γ_1

have

$$\tau\Pi_0 = \bigcup_{k=1}^{4} \{z \in \Gamma_q \mid \chi(z) = \overline{3k},\ z^\star \in -\frac{1}{\tau} P_{\bar{k}}\},$$

whereas for $q = 2$ or 3 we have

$$\tau^2\Pi_0 = \bigcup_{k=1}^{4} \{z \in \Gamma_q \mid \chi(z) = \overline{-k},\ z^\star \in \frac{1}{\tau^2} P_{\bar{k}}\}.$$

Now, an arbitrary Penrose tiling Π of the plane with two types of rhombic tiles, like it appears in familiar pictures, is as "democratic" as its one-dimensional counterpart, namely the Fibonacci chain. This means that it is in some sense optimal with respect of the inclusion property

$$\Pi - \Pi \subset \Gamma_q,$$

for an appropriate choice of scale. For instance, if we put the length of the rhomb edges equal to τ^4, choose one of the vertices as the origin O, and one of the edges issued from 0 as the complex τ^4, then $\Pi - \Pi \subset \Gamma_1$.

9. Conclusion

We would like to end this paper with some comments on the relationships between model sets and the set of beta-integers. In order to make things as simple as possible, let us come back to the example of the Fibonacci chain developed in Example 2.1.

Recall that the Fibonacci chain $\mathcal{F}$ is

$$\mathcal{F} = \{x \in \mathbb{Z}[\tau] \mid x' \in [0,1)\}.$$

It is a model set. Actually its is the one-dimensional toy prototype in quasicrystalline studies, see [**13**]. The tiling associated with it has 2 tiles, and is not symmetrical. No point is favoured. In this sense we can say that the Fibonacci chain is an homogeneous set.

The set $\mathcal{F} \cup (-\mathcal{F})$ is also a model set,

$$\mathcal{F} \cup (-\mathcal{F}) = \{x \in \mathbb{Z}[\tau] \mid x' \in (-1,1)\}.$$

In constrast, this set has a specific point, the origin! However, this symmetrical tiling is more complex, in the sense that it has three tiles.

On the other hand, the set $\mathbb{Z}_\tau$ of τ-integers is symmetrical. The tiling associated with it has 2 tiles. This set is not a model set, but it is a Meyer set. Let us now show that the τ-integers are above all perfect labelling frames for the model sets of the above type.

In [**5**] we have shown that

$$\mathcal{F} \cup (-\mathcal{F}) = \{x \in \mathbb{Z}_\tau \mid x' \in (-1,1)\}$$

which means that the points of $\mathbb{Z}_\tau$ enumerate the model set $\mathcal{F} \cup (-\mathcal{F}) = \mathcal{F} - \mathcal{F}$. From that follows that, for each x_0 in $\mathcal{F}$, $\mathcal{F} \subset \mathbb{Z}_\tau + \{x_0\}$. Thus, if we choose a point in $\mathcal{F}$ for origin point of $\mathbb{Z}_\tau$, then all the points of $\mathcal{F}$ are in $\mathbb{Z}_\tau$, and therefore the τ-integers form a labelling of the Fibonacci chain $\mathcal{F}$.

More generally, let Ω be a bounded window in $\mathbb{R}$ and let

$$X(\Omega) = \{x \in \mathbb{Z}[\tau] \mid x' \in \Omega\}$$

be a more generic one-dimensional model. It is always possible to find a λ in $\mathbb{Z}[\tau]$ and an integer j in $\mathbb{Z}$ such that $\Omega \subset (-\tau^j, \tau^j) + \lambda$. Let $\Delta = \tau^{-j}(\Omega - \lambda) \subset (-1,1)$, and set

$$Z(\Delta) = \{z \in \mathbb{Z}[\tau] \mid z' \in \Delta\}.$$

Then from [**5**]

$$Z(\Delta) = \{z \in \mathbb{Z}_\tau \mid z' \in \Delta\}$$

and so

$$X(\Omega) = (-1)^j \tau^{-j} Z(\Delta) + \lambda'.$$

Therefore $\mathbb{Z}_\tau$ enumerates the model set $X(\Omega)$. This property can be extended to 2 and 3-dimensional sets relevant to quasicrystalline studies.

10. Appendix

In the case in which β is a quadratic Pisot unit, the statements given in Proposition 6.1 (iii) and in Proposition 6.5 (iii) mean that in the infinite word generated by the substitution σ_β the difference between the number of S's in two different factors of same length is bounded by 1. In fact, this result can be obtained using the theory of Sturmian words.

A *Sturmian word* is an infinite word such that the number of factors of length n of the infinite word is equal to $n+1$. They are defined on 2-letter alphabets, for instance $\mathbb{A} = \{L, S\}$. The paradigm is the Fibonacci word, defined by the substitution associated with the golden ratio. An equivalent definition is the following one. An infinite word $w = (w_i)_{i \geq 0} \in \mathbb{A}^{\mathbb{N}}$ is said to be *balanced* if for each $n \geq 1$, and for each $i, j \geq 0$, the difference between the number of S's (or equivalently of L's) in $w_i \cdots w_{i+n-1}$ and in $w_j \cdots w_{j+n-1}$ is bounded by 1. Then an infinite aperiodic word is Sturmian if and only if it is balanced (see [**2**]). A substitution σ is *Sturmian* if $\sigma(s)$ is a Sturmian word for every Sturmian word s.

For the golden ratio τ we have

$$\sigma_\tau : \begin{cases} L & \mapsto & LS \\ S & \mapsto & L. \end{cases}$$

Let us denote by $\widetilde{\sigma_\tau}$ the mirror of the substitution σ_τ, that is,

$$\widetilde{\sigma_\tau} : \begin{cases} L & \mapsto & SL \\ S & \mapsto & L \end{cases}$$

and by E the exchange

$$E : \begin{cases} L & \mapsto & S \\ S & \mapsto & L. \end{cases}$$

The following result is straightforward.

***Proposition* 10.1**. — *If β is the root > 1 of the polynomial $X^2 - aX - 1$, with $a \geq 1$, then $\sigma_\beta = \sigma_\tau (E\sigma_\tau)^{a-1}$.*
If β is the root > 1 of the polynomial $X^2 - aX + 1$, with $a \geq 3$, then $\sigma_\beta = \widetilde{\sigma_\tau}\sigma_\tau (E\sigma_\tau)^{a-3}$.

From these results it follows

***Theorem* 10.2**. — *When β is a quadratic Pisot unit, the infinite word generated by the substitution σ_β is a Sturmian word.*

Proof. — It is a consequence of the result saying that any substitution composed of the substitutions σ_τ, $\widetilde{\sigma_\tau}$ and E is Sturmian, see [**2**]. $\square$

Remark that for the cubic case, Proposition 7.2 (iv) gives a property of "quasibalance" for the infinite word generated by the substitution, which seems to be interesting to study.

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