An introduction to map enumeration
Notes

Guillaume Chapuy

September 5, 2014

These lecture notes are here to keep a trace in order to help people who have attended the lectures. This a draft, and is meant to remain in this form.

Contents

0 Introductory remarks

0.1 What this course is (not) about

These lecture notes are an introduction to *map enumeration.* They contain:

- an introduction to counting maps with generating functions (both in the plane and other surfaces);
- an introduction to the bijective counting of these objects.

The following subject is an important motivation for this content and will only be very briefly mentionned:

- random planar maps/ random maps on surfaces.

If you're interested in one of the following subjects you will find some information in the exercises:

- the enumeration of one-face maps;
- the link between maps and algebraic combinatorics.

The following subjects will not be mentioned at all:

- the link between map enumeration and algebraic geometry (moduli spaces of curves, Hurwitz numbers);
- the link between map enumeration and matrix integrals.

0.2 References

Look at the webpage (from my webpage) for both the main references from which the material of these notes is taken, and for complements. Some other references are indicated in the text.

1 Chapter 1: The oral tradition

The purpose of this chapter is to define maps, and more importantly, practice the different definitions. This material is not so easy to learn from books or papers and mostly belongs to the oral tradition (so these written notes may not be very useful either).

1.1 Topological disclaimer

This course is *not* about topology. Actually, we will not do any topology at all: all the underlying objects will be combinatorial. There will be some topological definitions at the beginning but they will quickly be forgotten: we will admit their equivalence with some combinatorial definitions. However, we will rely very much on the the topological intuition.

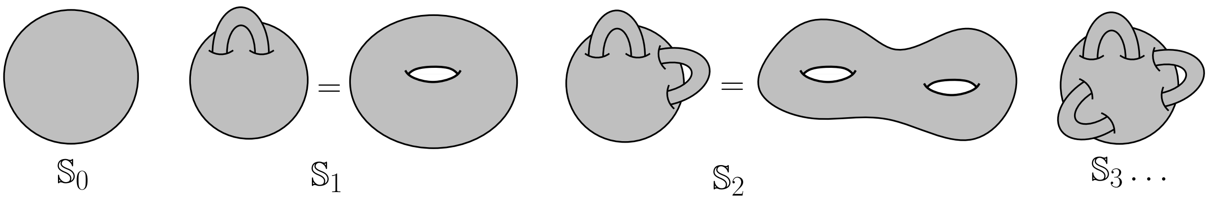
1.2 Surfaces and graphs

For us, a *surface* will be a connected, closed, oriented, 2-dimensional manifold. Luckily, surfaces are completely classified, and we admit the following (nontrivial but classical) result in topology:

Theorem 1 (Classification of surfaces, see e.g. Massey's textbook)**.** *Every surface is homeomorphic the sphere with g handles attached, for some nonnegative integer g, called the* genus*.*

Since we will be interested in surfaces only up to homeomorphism, we will talk about *the* surface of genus g without ambiguity, and we will denote it by $\mathbb{S}_g$. The first low genus surfaces are:

- the 2-sphere $\mathbb{S}_0$.
- the *torus* $\mathbb{S}_1$, that can be viewed as a sphere with one handle attached.
- the 2-*torus* $\mathbb{S}_2$, or *double-torus*, obtained by attaching a handle to a torus, etc.



Note that one could also consider *non-orientable* surfaces (and maps): the classification is not much more complicated, and a lot of the material of this course could be generalized to that setting (but not all). However we choose to avoid these complications, and we will not mention non-orientable surfaces anymore in these notes.

We now turn to *graphs*. All our graphs will be finite, and may have *loops* and/or multiple edges. In other words, what we call graphs is called multigraphs by others.

For us it will often be convenient to think of multigraphs as made of *half-edges* instead of edges. Each edge is made of two half-edges matched together, and each vertex represents an incidence between a set of half-edges (note that this representation makes natural the possibility of loops and multiple edges).

Unless otherwise mentioned, all graphs will be *connected*. The *degree* of a vertex is the number or edges incident to it, loops counting twice. Equivalently, the degree of a vertex is the number of half-edges incident to it.

1.3 Maps: three (four?) definitions

1.3.1 As embedded graphs.

A *map* $\mathfrak{m}$ is a proper embedding[1] of a graph G into a surface $\mathbb{S}$, in such a way that the connected components of $\mathbb{S} \setminus G$, called the *faces*, are homeomorphic to disks.

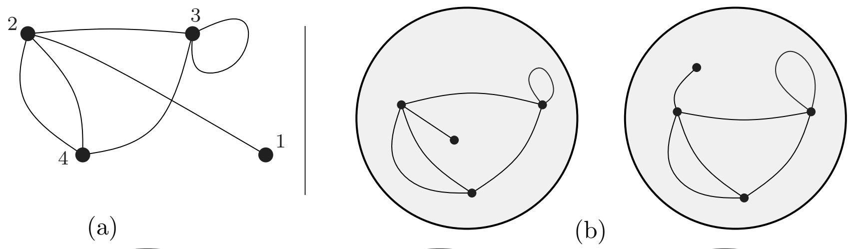
G and $\mathbb{S}$ are called respectively the *underlying graph* and *underlying surface* of $\mathfrak{m}$, and we sometimes use, with a slight abuse of notation, $\mathfrak{m} = (G, \mathbb{S})$ to denote the map (the embedding being implicit in the notation).

Each edge in a map is made of two *half-edges*, obtained by removing its middlepoint. A *corner* is an angular sector defined by two consecutive half-edges in the neighbourhood of a vertex. A map with n edges has $2n$ corners. There is a canonical correspondence between corners and half-edges, by associating a corner with the half-edge bordering it clockwise. A *rooted map* is a map with a distinguished corner (equivalently, a distinguished half-edge; equivalently, a distinguished and oriented edge) called the *root corner* (*root half-edge*, *root edge*, respectively). The vertex and face incident to the root corner are called the *root vertex* and *root face*, respectively.

The *degree* of a vertex in its degree in the underlying graph. The *degree* of a face is the numbet of corners incident to it. Hence the sum of the vertex degrees equals the sum of the face degrees, equals the number of half-edges, and equals twice the number of edges.

Maps are considered up to orientation-preserving homeomorphisms. Rooted maps are considered up to orientation-preserving homeomorphisms preserving the root corner.

See Figures 1– 2 for illustrations of the definition of a map.



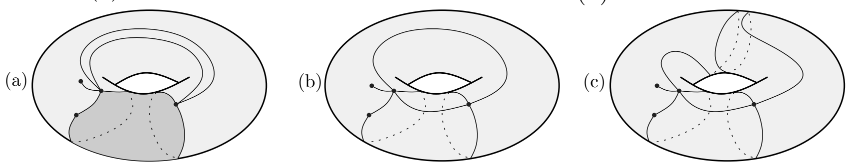


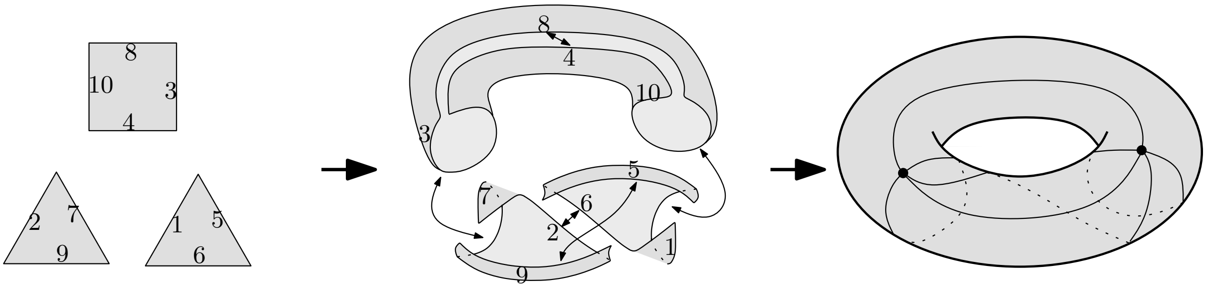
Figure 2: (a) An embedding of a graph in the torus that violates the definition of a map, since the dark face is not simply connected. (b) An embedding of the same graph on the torus that is a valid map. (c) Another embedding that defines the same map, since the two are homeomorphic (even though they are not homotopic).

Remark 1 (On the difference between a graph and a map)**.** Note that if $\mathfrak{m}$ is a map of underlying graph G and surface $\mathbb{S}$, the half-edges incident to any vertex v of G are **cyclically ordered** by the clockwise orientation of the neighbourhood of v in $\mathbb{S}$ (recall that $\mathbb{S}$ is endowed with an orientation). This cyclic ordering is preserved by orientation-preserving homeomorphisms, so it is a well-defined on the map. We will see in a few paragraphs that this cyclic ordering is all the information needed to describe the map from the graph.

[1]we do not define properly what a “proper embedding” is here. Intuitively, this is a continuous drawing of the graph in the surface, where edges do not cross.

1.3.2 As polygons gluings

A convenient way to construct a map is to proceed as follows. Start with a finite collection of polygons, with an even total number of sides, say $2n$. For convenience, assume that the sides of these polygons are labelled from 1 to $2n$.



Given a fixed-point free involution on $[1..2n]$ (a matching), we can form a (non necessarily connected) surface by gluing these polygons together along their sides according to that matching, as on the figure above. On each connected component of that surface, the sides of the polygon define the embedding of a graph. By construction, the faces of this embedding are the original polygons, so the embedding is a map as defined above.

Intuitively, it is clear that any map $\mathfrak{m} = (G, \mathbb{S})$ can be obtained from this procedure. Indeed, by cutting the surface $\mathbb{S}$ along the edges of G, one obtains a collection of components which are homoemorphic to polygons, and one can reconstruct $\mathbb{S}$ by gluing their sides back together. However, proving this fact would require too much topology and is out of our scope, so let's admit:

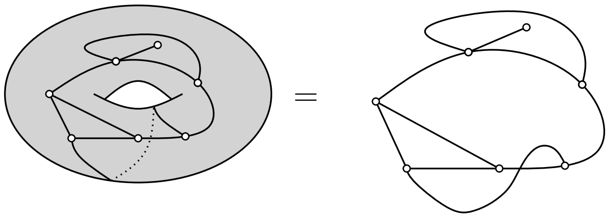
Proposition 1. *Any map can be constructed as a polygon gluing.*

1.3.3 As graphs with rotation systems at vertices

Let G be a graph. A *rotation system* is the data of a cyclic ordering of half-edges incident to each vertex of G. As mentioned in the previous remark, the clockwise ordering associates a rotation system to the underlying graph of any map. The following statement says that this is all the information we need:

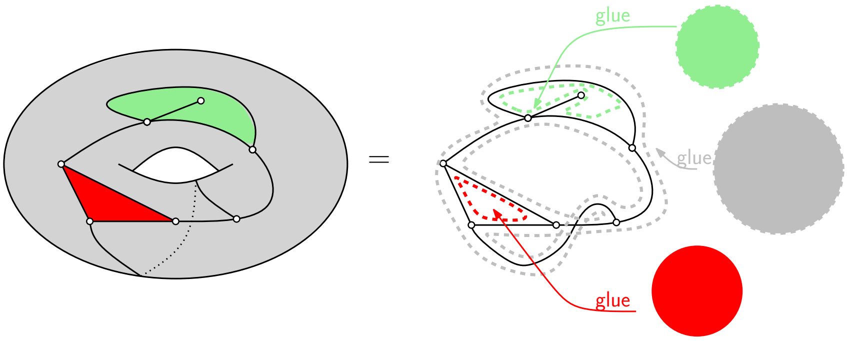
Theorem 2 (Classical topology result, see e.g. Mohar&Thomassen's book)**.** *There is a bijection between maps and (connected) graphs equipped with a rotation system. The bijection is given by considering the clockwise ordering of half-edges around each vertex.*

This theorem gives an easy way to represent a map (of arbitrary genus) on a sheet of paper: we just draw the graph in such a way that the clockwise orientation (on paper) of half-edges around vertices correspond to cyclic ordering given by the rotation system. For example, these two drawings represent the same map.



Note that this representation may contains edge-crossings, but we do not care: the only meaningful information is the graph structure and the local neighbourhood of each vertex, that gives the clockwise ordering of half-edges around vertices.

While it is clear how to go from a map to its rotation system, let us say a few words about the opposite direction. We rely on the last and next pictures. The map on the left has 3 faces, that for simplicity we can color with three different colors. Notice that the boundary of these faces can be recovered from the picture on the right, by "walking" along the edge-sides of the graph (using the rotation system to decide where to continue each time we reach a new corner). We can then reconstruct the surface back by "gluing a disk" (a face) along each of these boundaries, as tentatively depicted by the following figure:

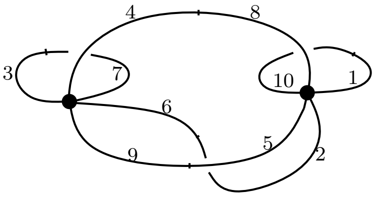


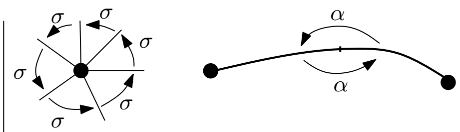
Remark 2. In order to be able to work with maps easily, understanding the last picture is very important.

1.3.4 As systems of permutations

The pure combinatorialists may not be satisfied with the previous paragraphs, so let us give a purely combinatorial definition of maps in terms of permutations[2].

Start with a map with n edges, viewed as a graph equipped with a rotation system, and consider an arbitrary labelling of its $2n$ half-edges by the numbers in $[1..2n]$:

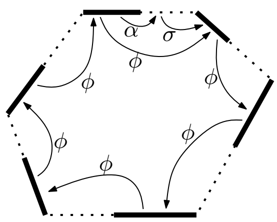




$\sigma = (1, 8, 10, 5, 2)(3, 9, 6, 7, 4)$

$\alpha = (1, 10)(2, 6)(3, 7)(4, 8)(5, 9)$

$\phi = \alpha\sigma = (1, 5, 6)(2, 7, 9)(3, 4)$



Given such an object (called a *labelled map*), we can form two permutations σ and α of $[1..2n]$ as follows:

- α is a fixed-point free involution whose cycles are given by the edges of G.
- σ is the permutation that maps a half-edge to the half-edge incident to it in clockwise direction, around the vertex it belongs to.

[2]We will only brifely mention it, but this representation makes a fundamental link between map enumeration and the symmetric group and symmetric functions. The deep algebraic structures of these objects can be fruitfully used in map enumeration – see the exercise sheet for more on this.

The most remarkable feature of this encoding is that the cycles of the permutation $\phi = \sigma\alpha$ are in bijection with the faces of the map (see the figure above).

This leads to the following definition:

Definition 1 (Combinatorial definition of a labelled map)**.** Let $n \geq 1$ be an integer. A *combinatorial labelled map* of size n is a triple of permutations $\mathfrak{m} = (\sigma, \alpha, \phi)$ in $(\mathbb{S}_{2n})^3$ such that

- α is a fixed point free involution
- $\sigma\alpha = \phi$
- the subgroup generated by σ, α, and ϕ acts transitively on $[1..2n]$.

The elements of $[1..2n]$ are abusively called *half-edges* and the cycles of σ and ϕ are called respectively the *vertices* and *faces* of $\mathfrak{m}$.

Note that from any triple (σ, α, ϕ) satisfying the first two properties we could construct a graph with a rotation system, but it will not necessarily be connected. The third property (transitivity) is here to ensure connectivity. It is clear that labelled maps are in bijection with graphs equipped with a rotation system and labelled on their half-edges, hence, from what precedes, they are in bijection with topological maps labelled on their half-edges.

Definition 2. A *combinatorial rooted map* is an equivalence class of combinatorial labelled maps under renumberings of $[1..2n]$ preserving 1. The half-edge labelled 1 is called the *root half-edge.*

Theorem 3. *Combinatorial rooted maps are in bijection with (topological) rooted maps.*

Remark 3. We leave the reader check the following "rigidity" statement: each rooted map of size n corresponds to exactly $(2n-1)!$ different labelled maps. This amounts to prove that an automorphism of a labelled map fixing the root half-edge is necessarily the identity (Hint: use the transitivity assumption!). Therefore rooted maps and labelled maps are really similar objects: there is only a "multiplicative factor" between them.

Remark 4. *Unrooted* maps can be defined as equivalence classes of labelled maps up to relabelling of $[1..2n]$ (without asking that 1 is preserved). The counting of these objects is not simply related to the one of rooted/labelled maps since a labelled map can have nontrivial automorphisms. We will not mention unrooted maps in these notes (see the exercise sheet for a few words on them).

1.4 Duality

Note that in the permutation representation of a map, the permutations σ and ϕ. More precisely the mapping $(\sigma, \alpha, \phi) \longrightarrow (\phi^{-1}, \alpha, \sigma^{-1})$ is an involution on labelled maps, called *duality.*

Graphically, the dual map $\tilde{\mathfrak{m}}$ of a map $\mathfrak{m}$ is constructed by adding a new vertex inside each face of $\mathfrak{m}$, and adding a new edge for each edge of $\mathfrak{m}$, that links the two (new) vertices corresponding to the two faces bordering the edge (Figure 3).

In summary, duality sends vertices to faces, faces to vertices, and edges to edges. It is easy to see topologically that the dual map is indeed a map on the same surface (each face of the dual map is a neighbourhood of one of the vertices of the original map and is clearly simply connected). By choosing an appropriate convention we can also view duality as sending half-edges to half-edges (for example associate a half-edge to the dual half-edge that belongs to its right). This enables to consider duality on rooted maps, by choosing the dual root half-edge... as the dual of the root half-edge.

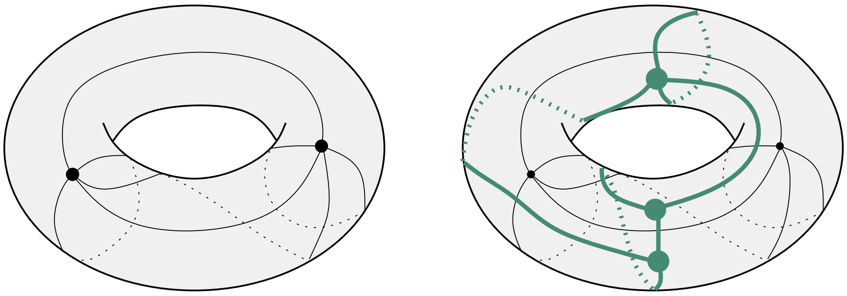


Figure 3: A map of genus 1 and its dual map (green edges).

1.5 Euler's formula

Theorem 4 (Euler's formula)**.** *If* $\mathfrak{m}$ *is a map of genus* g *with* n *edges,* f *faces, and* v *vertices, then one has:*

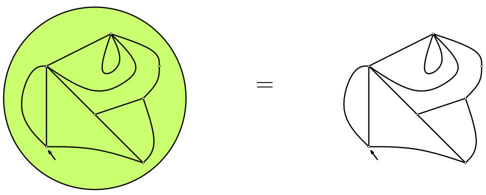
$$v + f = e + 2 - 2g.$$

Note that this formula enables one to recover the genus easily from the representation of a map as a rotation system (or triples of permutations). We just need to count faces, vertices, and edges.

1.6 Examples: one-face maps, planar maps, Catalan trees

1.6.1 Planar maps.

Rooted maps of genus 0 are usually called *rooted planar maps*. The reason for this terminology is that they can be represented in the plane by sending the root face to infinity:



Planar maps have several properties that general maps do not have. For example, the existence of an algebraic dual, or the fact that each spanning tree of a planar map induces a *dual spanning tree*, a spanning tree of the dual map (Figure 4).

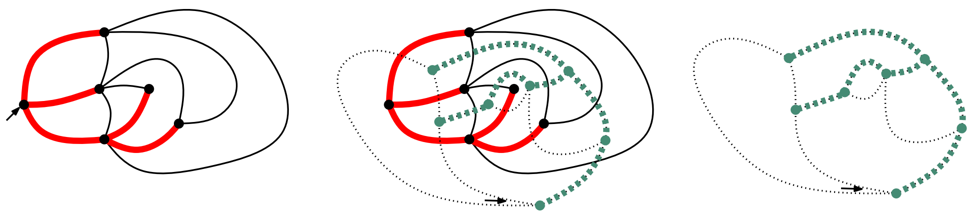


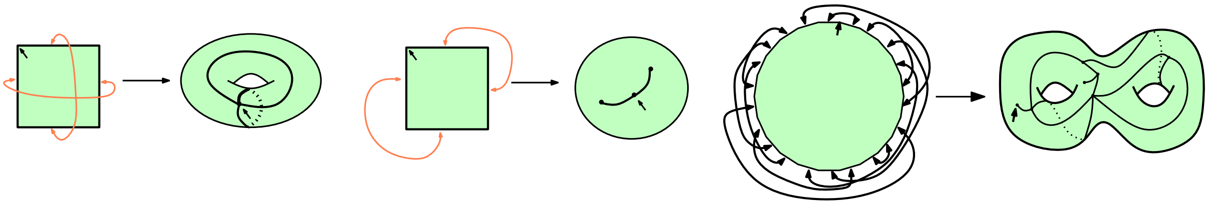
Figure 4: Left: a rooted planar map with equipped with a spanning tree (in red); Center: superimposition of the map and its dual (dotted edges) – the dual edges of edges not in the spanning tree form a spanning tree of the dual map (in green); Right: the dual map and the dual spanning tree. Root corners are indicated with arrows.

Note that these two trees have respectively v and f vertices, hence $v-1$ and $f-1$ edges, and indeed in the planar case Euler's formula writes:

$$(v-1)+(f-1)=e.$$

1.6.2 One-face maps

One-face maps are, with planar maps, one of the most interesting subfamilies of maps (from various viewpoints, including enumeration). A one-face map with n edges can be thought as a polygon with $2n$ sides in which the edges have been glued together in pairs:



In the permutation represention, it is convenient to see a rooted one-face map as a pair (σ,α) such that $\sigma\alpha=(1,2,\ldots,2n)$. Note that in this case the transitivity assumption is not necessary, therefore any fixed-point free involution α gives rise to a rooted one face map by letting $\sigma:=(1,2,\ldots,2n)\alpha$. Equivalently, any matching on $[1..2n]$ gives rise to rooted a one-face map by starting with a polygon with sides labelled $1,2,\ldots,2n$ clockwise, and gluging them according to the matching. We thus have:

Proposition 2. *The number of rooted one-face maps with n edges in* $(2n)!!:=\frac{(2n)!}{2^n n!}$.

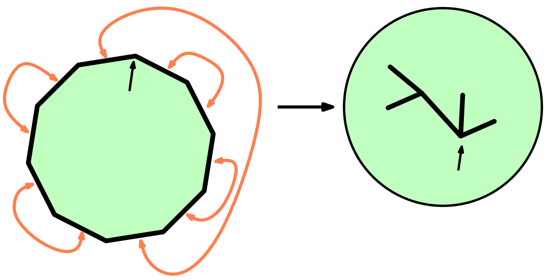
One-face maps are interesting for (at least) two reasons. (1) They are the basic building blocks of the combinatorics of maps on surfaces, as we will see; (2) They are interesting on their own, and many things can be said about them.

Note that the last proposition does not say anything on the enumeration of one-face maps of a given genus. This is clearly a more complicated problem (see the exercise sheet).

1.6.3 Plane trees

We start with an unusual definition: a *rooted plane tree* is a rooted one-face map of genus 0.

Note that by Euler's formula such a map with n edges has $n+1$ vertices, hence it is a tree. Moreover, if we view it as a graph with rotation system, such a map is a tree with a distinguished root corner, and a cyclic ordering of half-edges (i.e. an ordering of chlidren) around each vertex. So we recover rooted plane (=ordered) trees as we usually define them in combinatorics.



Plane trees (that we sometimes call Catalan trees) play a very important role in the combinatorics of maps (they are the basic building blocks, as we will see). We recall (and we invite the reader to check he/she recalls at least two proofs of this) that the number of plane trees with n edges is given by the Catalan number:

$$\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}.$$

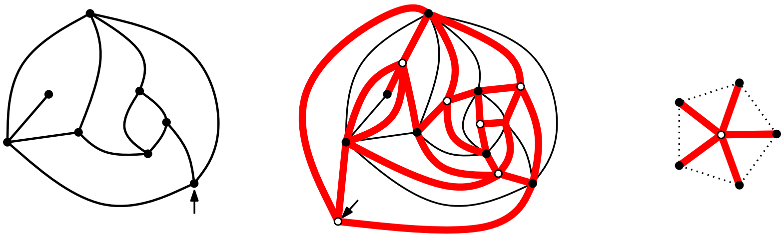
1.7 Tutte's quadrangulation bijection

We conclude this introductory chapter with a nice and useful bijection due to Tutte.

A *quadrangulation* is a map in which all faces have degree 4. In the planar case, a classical result asserts that the underlying graph of any quadrangulation is bipartite[3]. Such a statement is not true in positive genus: not all quadrangulations are bipartite. Bipartite quadrangulations will be of special interest to us:

Proposition 3 (Tutte)**.** *For each g and n there is an explicit bijection between rooted maps of genus g with n edges and rooted bipartite quadrangulations of genus g with n faces.*

Tutte's bijection is better explained with a figure:



(a) A map $\mathfrak{m}$

(b) its associated bipartite quadrangulation $\mathfrak{q}$ (thick edges)

(c) the local rules of the construction around a face of $\mathfrak{m}$

Figure 5: **Tutte's bijection.** Given a (not necessarily bipartite) map $\mathfrak{m}$ of genus g with n edges, add a new (white) vertex inside each face of $\mathfrak{m}$, and link it by a new edge to each of the corners incident to the face. The bipartite quadrangulation $\mathfrak{q}$ is obtained by erasing all the original edges of $\mathfrak{m}$, *i.e.* by keeping only the new (white) vertices, the old (black) vertices, and the newly created edges. The root edge of $\mathfrak{q}$ is the one created from the root corner of $\mathfrak{m}$ (which is enough to root $\mathfrak{q}$ if we demand that its root vertex is white). (a) and (b) display an example of the construction for a map of genus 0 (embedded on the sphere). Root corners are indicated by arrows.

The main feature that makes Tutte's bijection very useful is that it transforms arbitrary maps into maps with bounded face degrees. And it turns out that for the bijective approach to map enumeration, working with bounded face degrees makes things much easier.

For bijections, maps with bounded face degrees will be easier to deal with than general maps. That's why Tutte's bijection will be very useful to us.

[3]More generally, any planar map having all face degrees even is bipartite. Equivalently, all the cycles of its underlying graph have even length: this is easy to see, since in the plane a cycle necessarily encloses a bounded region, and since this region has to be the union of some faces (so by summing the face degrees inside the region, and by noticing that each inner edge is counted twice, it follows that the length of the boundary, i.e. of the cycle we started from, is even).

2 Chapter 2: Counting planar maps – solving the Tutte equation

The purpose of this chapter is to write the Tutte equation (which is the best way to be sure we understand what a planar map is), to say a word on equations with one catalytic variable, and to show Tutte's result on the number of planar maps.

Theorem 5 (Tutte 1954)**.** *The number of rooted planar maps with n edges is equal to*

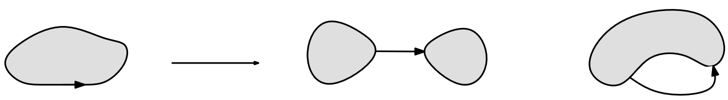
$$m_0(n) = \frac{2 \cdot 3^n}{n+2}\mathrm{Cat}(n).$$

2.1 The Tutte equation

We let $m_0(n)$ be the number of rooted maps of genus 0 with n edges, and we let $f_0(t) := \sum_n m_0(n)t^n$ be their generating function. By convention we consider the "single isolated vertex" as a planar map (the *trivial* map, with 0 edges and 1 face), so we have $m_0(0) = 1$.

NOTATION: *In this chapter, all maps will be planar and rooted. We will also omit the "genus 0" subscript and write $f(t)$ for $f_0(t)$.*

Say we want to count planar maps with generating functions, in the same way as we count rooted plane trees: delete the root edge, and see what we are left with. When we delete the root edge in a (nontrivial) map, two things can happen:



A. Maybe we disconnect the map. In this case we are left with two planar maps. Conversely starting from two rooted maps we can add an edge between them, so the generating function for case A. is $tf(t)^2$.

B. Maybe we do not disconnect the map. In this case we are left with one (smaller) planar map. Conversely, starting with one (small) planar map, the number of ways we can add an edge inside its root face to obtain a (big) map in case B. depends on the degree of the root face.

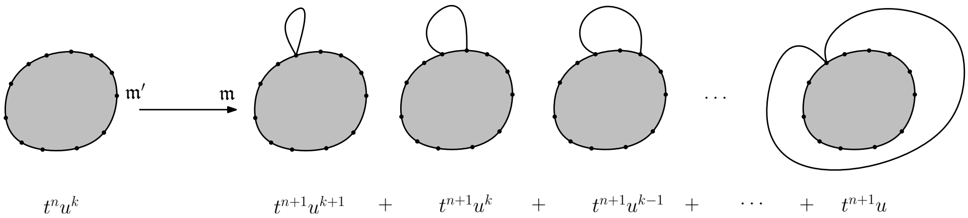
Remark 5 (Key remark)**.** By root-edge deletion we **can't** write an equation for the generating function $f(t)$ of rooted planar maps. However, we **can** do it if we track the degree of the root face, i.e. if instead we consider the generating function $F(t, u)$, where t (the main variable) marks the number of edges and u (the "catalytic variable") marks the degree of the root face.

NOTATION: *The main variable, counting edges, will always be denoted by t – and no substitution will be made on it. We will often forget the "main variable" in the notation, i.e. we will write $F(u)$ for $F(t; u)$ or f for $f(t)$.*

Theorem 6 (The Tutte equation)**.** *The generating function of planar maps satisfies the equation:*

$$F(u) = 1 + tu^2 F(u)^2 + tu\frac{uF(u) - F(1)}{u-1}.$$

Proof. The first term corresponds to the trivial map, the second one to case A. above. The fact that case B. is counted by the third term deserves a more careful inspection. Each map $\mathfrak{m}$ in case B. can be obtained in a unique way from a map $\mathfrak{m}'$ with one less edge by adding an edge as on the following figure:



If $\mathfrak{m}'$ has n edges and root-face-degree k, then $\mathfrak{m}$ has $n+1$ edges and root-face degree $1, 2, \ldots, k+1$. In terms of monomials, going from $\mathfrak{m}'$ to $\mathfrak{m}$ corresponds to the operation:

$$\begin{aligned} t^n u^k \longrightarrow & t^{n+1}\left(u + u^2 + u^3 + \cdots + u^{k+1}\right) \\ &= tu\frac{t^n u^{k+1} - t^n}{u-1}. \end{aligned}$$

By summing over $\mathfrak{m}'$ in the set of all rooted planar maps, we obtain that the generating function for case B. is $tu\frac{uF(u)-F(1)}{u-1}$ as wanted. ☐

2.2 Polynomial equations with one catalytic variable

We just wrote the Tutte equation. What kind of equation is this? This is a polynomial, **non linear**, equation with one catalytic variable. It is not obvious to solve it. Note in particular that we can't simply get rid of the catalytic variable u: if we set $u = 1$, then $\frac{uF(u)-F(1)}{u-1}$ involves the derivative $F'_u(1)$, and we are left with one equation but two unknowns ($F(1)$ and $F'_u(1)$). Such equations are characteristic of the enumeration of planar maps. For example one of the exercises deals with planar *triangulations* (an interesting subfamily of planar maps), for which instead of the operator

$$\Delta : A(u) \longmapsto \frac{A(u) - A(1)}{u-1},$$

we have the operator

$$\Delta_0 : A(u) \longmapsto \frac{A(u) - A(0)}{u}.$$

Equations involving Δ or Δ_0 often appear in combinatorics, and we will just say a word about them in this course.

2.2.1 Digression: linear equations and the kernel method

Linear equations with one catalytic variable are classical in the enumeration of **paths**. Say for example that you want to write an equation for walks of on $\mathbb{N}_{\geq 0}$, taking steps in $\{\pm 1\}$, starting from 0, simply by looking at what happens step by step. Then the generating function $D(u) \equiv D(t; u)$ of such walks, where t marks the number of steps and u marks the final position satisfies:

$$D(u) = 1 + tuD(u) + t\frac{D(u) - D(0)}{u}.$$

The classical way to solve this equation to use the *kernel method*, i.e. write:

$$K(u)D(u) = 1 - \frac{t}{u}D(0) \tag{1}$$

where $K(u) = 1 - tu - \frac{t}{u}$ is the *kernel*. Now let $U(t)$ be the only series such that $K(U(t)$ =0, i.e.:

$$U = t + tU^2 = \frac{1 - \sqrt{1-4t^2}}{2t}$$

Then substituting $u = U(t)$ in (1) give $1 - \frac{t}{U(t)}D(0) = 0$, so we can determine $D(0)$:

$$D(0) = \frac{U(t)}{t} = \frac{1 - \sqrt{1-4t^2}}{2t}.$$

(Of course we recover the generating function of Dyck-paths, i.e. Catalan numbers) We then find $D(u)$ by plugging $D(0)$ in the original equation. This wonderful trick of cancelling the kernel works for all linear equations with one catalytic variable and is known as the *kernel method*.

2.2.2 How to solve the Tutte equation?

The Tutte equation involves one unknown function $F(t,u)$, and its substition $F(t,1)$. However we can think of these as two independant unknowns. More precisely we let

$$E(F,f,u) := -F + 1 + tu^2F^2 + tu\frac{uF - f}{u-1},$$

so that the Tutte equation reads $E(F(u), F(1), u) = 0$. Then we have:

Proposition 4. *The equation* $E(F,f,u) = 0$ *has a* unique *solution* (F,f) *in* $\mathbb{C}[u][[t]] \times \mathbb{C}[[t]]$.

Proof. Compute the series expansion (in t) of $F(t,u)$ recursively using the equation. By doing so, observe that in order to have $F(t,u) \in C[u][[t]]$, we need to have $f = F(1)$, order by order in t. □

Since the solution to the Tutte equation is unique, one way of solving it is to guess the solution: this what Tutte did in the first place. Then Tutte and Brown designed a constructive way of solving this equation: a variant of the kernel method called the *quadratic method*. Much later, Bousquet-Mélou and Jehanne desinged a general way of solving *all* (up to reasonable technical asumptions) polynomial equations with one catalytic variable, in a way that generalizes both the kernel and quadratic method.

Theorem 7 (Bousquet-Mélou & Jehanne 2005)**.** *Let* $Q(y_0, y_1, \ldots, y_k; u, t)$ *and* $R(u)$ *be polynomials with coefficients in* $\mathbb{Q}$. *Then the unique f.p.s. solution of the equation*

$$F(u) = R(u) + tQ(F(u), \Delta F(u), \Delta^2 F(u), \ldots, \Delta^k F(u); u, t)$$

is algebraic over $\mathbb{Q}[t,u]$. *Moreover there is an explicit algorithm to determine an equation satisfied by* $F(u)$.

2.3 The case of the Tutte equation

The Tutte equation is an instance of the BMJ theorem, with $k = 1$, and a polynomial of degree 2 in F. We will not prove the BMJ theorem, nor describe the algorithm for general k. But we will explain the algorithm for $k = 1$ and apply it to our case. We start with the equation:

$$E(F(u), f, u) = 0$$

as above, where we look for a solution $(F(u), f) \in \mathbb{Q}[u][[t]] \times \mathbb{Q}[[t]]$. As in the kernel method, we want to find a good series $U(t)$ whose substitution in the equation leads to some simplification. First note that if $U(t)$ is *any* series in $\mathbb{Q}[[t]]$ with $U(0) = 0$, the substitution is well defined and we have:

$$E(F(U(t)), f, U(t)) = 0$$

in $\mathbb{Q}[[t]]$. Now differentiate the Tutte equation with respect to u, we have:

$$F'(u)E_1'(F(u), f, u) + E_3'(F(u), f, u) = 0.$$

Therefore, if we find $U(t)$ such that $E_1'(F(u), f, u) = 0$, we will *also* have $E_3'(F(u), f, u) = 0$. We will thus be left with the BMJ system:

$$\begin{cases} E(F(U(t)), f, U(t)) & = & 0 \\ E_1'(F(U(t)), f, U(t)) & = & 0 \\ E_3'(F(U(t)), f, U(t)) & = & 0 \end{cases}$$

Recall that E is an explicit polynomial, so so are E_1' and E_3'. We thus have a system with **three** equations and **three** unknowns! $(F(U(t)), U(t)$, and $f)$. Moreover, the Tutte equation begin contractant in the space of formal power series, the system is not singular and we can perform polynomial elimination.

Remark 6.

The BMJ strategy works in a similar way in the case $k \geq 1$, but in general the BMJ system involves k auxiliary series $U_1(t), \ldots U_k(t)$, leading to a system of $3k$ equations with $3k$ unknowns. The main point of the BMJ theory is to prove that the system of equations thus obtained is always non singular (which is not clear for $k > 1$. The BMJ theory also provides refinements of the strategy, enabling to reduce the number of equations from $3k$ to $2k$ (we refer to the original paper for these developments).

In the case of the Tutte equation the BMJ system takes the following form:

$$\begin{cases} F & = & 1 + tu^2F^2 + tu\frac{uF-f}{u-1} \\ 1 & = & 2tu^2F + \frac{tu^2}{u-1} \\ 0 & = & 2tuF^2 + \frac{t(Fu^2-2Fu+f)}{(u-1)^2} \end{cases}$$

The elimination can be performed by hand (or better: with a computer) in several ways. It turns out (we'll see why later) that we obtain a nice parametrization by letting $T : \frac{U(t)}{3(U(t)-1}$. This leads to our wanted result:

Theorem 8 (Tutte)**.** *The generating function $f(t) = \sum_n m_0(n)t^n$ is given by:*

$$f(t) = T - tT^3$$

where $T = 1 + 3tT^2$. Equivalently, $f(t)$ is the unique fps solution of the equation:

$$1 - 16t + 18tf - 27t^2f^2 = 0.$$

By applying Lagrange inversion we obtain:

Corollary 1. *The number of rooted planar maps with n edges is given by:*

$$m_0(n) = \frac{2 \cdot 3^n \cdot (2n)!}{n!(n+2)!} = \frac{2 \cdot 3^n}{n+2}\mathrm{Cat}(n).$$

Remark 7.

- It is surprising to find such a simple formula! Is there a bijective explanation? Yes! but it is not obvious at all (wait until Chapter 3).
- Catalan numbers satisfy the asymptotics $\mathrm{Cat}(n) \sim (constant) n^{-3/2} 4^n$. Hence we have:

  $$m_0(n) \sim c \cdot n^{\mathbf{-5/2}} 12^n.$$

  The counting exponent $-\frac{5}{2}$ is a characteristic of the **universality class of planar maps**. This means that, all reasonable families of rooted planar maps have the same exponent (for example triangulations, or 3-connected maps). The notion of *universality class* is not well defined, it is rather an accumulation of facts. Others characteristics of the universality class of planar maps are: non-linear polynomial equations with one catalytic variable, distances of order $n^{1/4}$, scaling limit being the Brownian map). This contrasts with the universality class of rooted trees (counting exponent $n^{-3/2}$, linear polynomial equations with one catalytic variable, Drmota-Lalley-Woods type equations, distances of order $n^{1/2}$, scaling limit the continuum random tree).

3 Chapter 3: Counting genus g maps à la Bender and Canfield

The purpose of this chapter is to write the Tutte equations for arbitrary genus (which is a good way to be sure we understand what a map is), to explain how they can be solved by recursion on the genus, and to give Bender and Canfield's result on the rationality and main simgularity of the generating function of genus g maps.

Theorem 9 (Bender & Canfield 1986)**.** *Let $g \geq 1$. The generating function $f_g(t) := \sum_n m_g(n)t^n$ of rooted maps of genus g by the number of edges is an element of $\mathbb{Q}(\rho)$ where $\rho = \sqrt{1-12t}$.*
When n tends to infinity, the number $m_g(n)$ of rooted maps of genus g with n edges is equivalent to:

$$m_g(n) \sim t_g n^{\frac{5(g-1)}{2}} 12^n,$$

for some $t_g > 0$.

3.1 The "Tutte equation" (a.k.a. loop equation, Schwinger-Dyson equation)

We fix an integer g, and we are interested in finding the number $m_g(n)$ of maps of genus g with n edges, or the generating function $F_g(t) = \sum_n m_g(n)t^n$.

We start as in the planar case: we look at what happens when we remove the root edge in a map of genus g. This is less obvious than in the planar case, so we have to be more careful not to forget any case. Let $\mathfrak{m}$ be a genus g map and let $\mathfrak{e}$ be its root edge. There are two cases:

A. in $\mathfrak{m}$, the edge $\mathfrak{e}$ is bordered twice by the same face F.

B. in $\mathfrak{m}$, the edge $\mathfrak{e}$ is bordered by two different faces F_1 and F_2.

In case B., the situation is similar to the planar case: removing the edge $\mathfrak{e}$ merges the two faces F_1 and F_2, thus giving rise to a map with one less edge and one less face. By Euler's formula, this map has the same genus as $\mathfrak{m}$, genus g. This can be dealt with the catalytic operator Δ as in the planar case.

In case A, there are two different possibilities. If the removal of $\mathfrak{e}$ disconnects the map, the situation is similar to the planar case: we obtain two different connected components $\mathfrak{m}_1$ and $\mathfrak{m}_2$, and by Euler's formula their genus g_1 and g_2 satisfy $g_1 + g_2 = g$. If the removal of $\mathfrak{e}$ does not disconnect the map, the face F is split into two different faces. We thus obtain a map $\mathfrak{m}'$ with one less edge and one more face: by Euler's formula, this map has genus $g-1$.

Therefore we have:

$$F_g(u) = tu\Delta F_g(u) + tu^2 \sum_{g_1+g_2=g} F_{g_1}(u)F_{g_2}(u) + tu^2 \left.\frac{v\partial}{\partial v}\right|_{v=u} F^{(2)}_{g-1}(u,v)$$

where $F^{(2)}_{g-1}(u,v)$ is the generating function of rooted maps of genus $g-1$ carrying an additional marked face, where u marks the size of the root-face and v marks the face of the other marked face.

Remark 8 (Comments on the equation)**.**

- We remark that, if $g \geq 1$ and if we consider the series $F_g(u)$ as unknown and all the series concerning a lower genus as known, what we have is a *linear* equation with one catalytic variable. It is tempting to solve it using the kernel method, and using induction on the genus!

- On the other hand, in order to be able to write the equation, we needed to introduce the series $F_{g-1}^{(2)}(u,v)$ of maps carrying an extra root face. So we need to write an equation for this series as well. In order to do that, and to be able to apply induction, the price to pay is to consider the series $F_h^{(k)}$ of maps of genus h carrying k extra faces, for all h and k.

  NOTATION: *We will need to manipulate series with an arbitrary number of parameters. If I is some set of integers, $I = \{i_1, i_2, \ldots, i_k\}$, we will note $v_I = (v_{i_1}, v_{i_2}, \ldots, v_{i_k})$ and write for example $F(v_I)$ if F is a function of k parameters.*

Definition 3. For $k, h \geq 0$ we let $F_h^{(k)}(t; u; v_1, \ldots, v_k) \equiv F_h^{(k)}(u; v_{[1..k]})$ be the generating function of all rooted maps of genus h carrying k marked faces, numbered from 1 to k, and different from the root face. The variables t, u, and u_i record the number of edges, the degree of the root-face, and the degree of the i-th marked face, respectively (for $i \in [1..k]$).

Theorem 10 (The Tutte/loop/Schwinger-Dyson equations for arbitrary genus)**.** *Let $g, k \geq 0$ be integers. Then we have:*

$$F_g^{(k)}(u; v_{[1..k]}) = tu^2 \sum_{\substack{g'+g''=g \\ I \uplus J = [1..k]}} F_{g'}^{(|I|)}(u; v_I) F_{g''}^{(|J|)}(u; v_J) + tu \frac{uF_g^{(k)}(u; v_{[1..k]}) - F_g^{(k)}(1; v_{[1..k]})}{u-1}$$

$$+ tu^2 \left. \frac{v\partial}{\partial v} \right|_{v=u} F_{g-1}^{(k+1)}(u; v, v_{[1..k]}) + \sum_{i=1}^{k} tuv_i \frac{uF_g^{(k-1)}(u; v_{[1..k]\setminus\{i\}}) - v_i F_g^{(k-1)}(v_i; v_{[1..k]\setminus\{i\}})}{u - v_i}. \quad (2)$$

Proof. We proceed as in the planar case. Take a map $\mathfrak{m}$ of genus g with root edge $\mathfrak{e}$, with k extra marked faces. The following cases can happen when we remove the root-edge (it is good to look at the figure while reading the text):

1. **A. In $\mathfrak{m}$, $\mathfrak{e}$ is bordered twice by the same face**

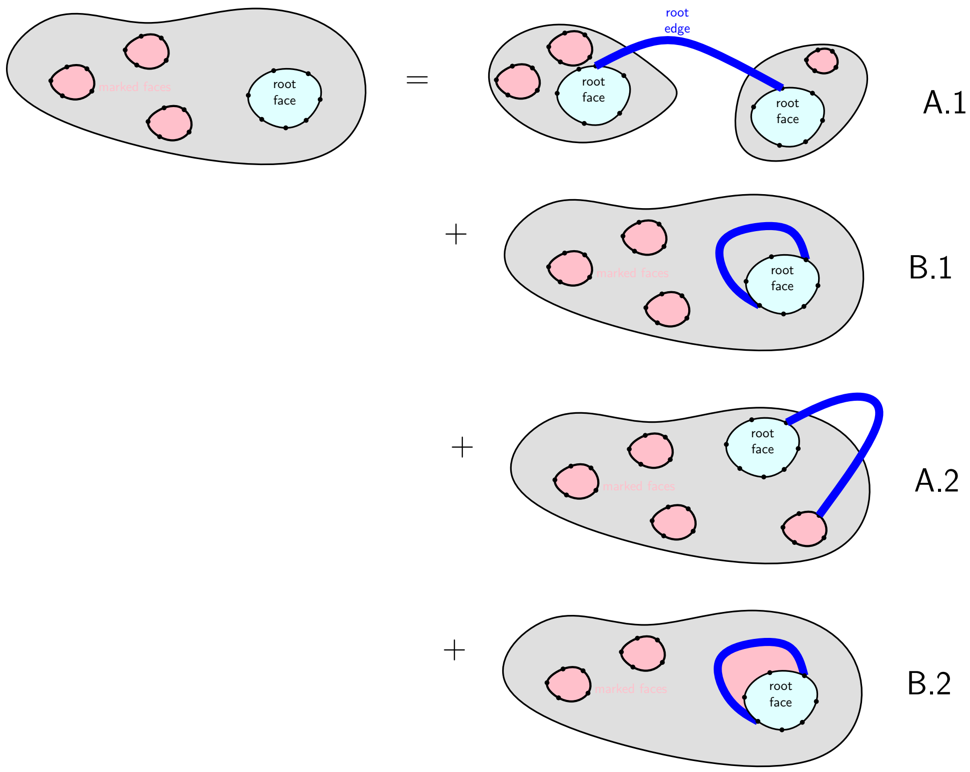
   A.1 Removing $\mathfrak{e}$ disconnects the map. We thus obtain two maps whose genus sum to g. Moreover the k marked faces are distributed between the two maps. This case corresponds to the term:

   $$tu^2 \sum_{\substack{g'+g''=g \\ I \uplus J = [1..k]}} F_{g'}^{(|I|)}(u; v_I) F_{g''}^{(|J|)}(u; v_J)$$

   A.2 Removing $\mathfrak{e}$ does not disconnect the map. As discussed above, we obtain a map of genus $g-1$, with an additional marked face (so $k+1$ marked faces). This extra face will be merged with the root face when we add $\mathfrak{e}$ back. Therefore this is counted by the term:

   $$tu^2 \left. \frac{v\partial}{\partial v} \right|_{v=u} F_{g-1}^{(k+1)}(u; u, v_{[1..k]}).$$

   Notice the derivative, that accounts for the fact that we have to choose where to attach the root edge in the marked face.



2. **B. In $\mathfrak{m}$, $\mathfrak{e}$ is bordered by two different faces, the root-face F and another face F'.** In this case, removing $\mathfrak{e}$ creates a map $\mathfrak{m}'$ in which the faces are merged together. Two things can happen:

   B.1 F' is not one of the marked faces. This case is similar to the planar case. Recall that adding the edge $\mathfrak{e}$ back to $\mathfrak{m}'$ is taken into account in the world of generating functions by the operator:

$$t^n u^k \longrightarrow t^{n+1}\Big(u + u^2 + u^3 + \cdots + u^{k+1}\Big)$$
$$= tu\frac{t^n u^{k+1} - t^n}{u-1}.$$

   Therefore the term corresponding to this case is

$$tu\frac{uF_g^{(k)}(u; v_{[1..k]}) - F_g^{(k)}(1; v_{[1..k]})}{u-1}.$$

   B.2 F' is one of the marked faces (say the i-th one). This case is very similar to $B.1$, but we have not to forget the power of v_i contributed by the i-th face. Combinatorially, adding the edge $\mathfrak{e}$ to $\mathfrak{m}'$ splits the root face into two faces and corresponds to the generating function operation:

$$t^n u^k \longrightarrow t^{n+1}\Big(uv_i^{k+1} + u^2 v_i^k + u^3 v_i^{k-1} + \cdots + u^{k+1} v_i\Big)$$
$$= tuv_i\frac{t^n u^{k+1} - v_i^{k+1} t^n}{u-1}.$$

   Therefore the term corresponding to this case is

$$\sum_{i=1}^{k} tuv_i \frac{uF_g^{(k-1)}(u; v_{[1..k]\setminus\{i\}}) - v_i F_g^{(k-1)}(v_i; v_{[1..k]\setminus\{i\}})}{u - v_i}.$$

□

3.2 Bender and Canfield's recursive kernel method

Observe that, if $(k,g) \neq (0,0)$, the above equation can be put in the form:

$$K(u)F_g^{(k)}(u, v_{[1..k]}) = \frac{tu}{u-1} F_g^{(k)}(1, v_{[1..k]}) + \text{STUFF} \tag{3}$$

where

- $K(u) = 1 - 2tu^2 F_0^{(0)}(u) - \frac{tu}{u-1}$ is the **kernel**. This quantity is **independent of** g **and** k. It is **explicit** since we have solved for $F_0^{(0)}$ in the previous chapter.
- STUFF involves only series $F_h^{(\ell)}$ with $(h,\ell) < (g,k)$ for the lexicographical order. Therefore it is is tempting to use **induction on** $g+k$ and to claim that STUFF is an explicit quantity, in which case we have nothing more than a **linear** equation with one catalytic variable, amenable to the kernel method.

Proposition 5 (Bender and Canfield recursive algorithm)**.** *The generating series* $F_g^{(k)}$ *can be computed inductively on* (g,k) *(for the lexicographical order) as follows:*

- *if* $(k,g) = (0,0)$ *the answer is known from the previous chapter.*
- *Else, let* $u_0(t)$ *be the only power series such that* $K(u_0(t)) = 0$*. Since* K *is explicit, so is* u_0 *and indeed one can check that* $u_0(t) = \frac{5-\sqrt{1-12t}}{4+2t}$*. Substitute* $u = u_0(t)$ *in equation* (3) *to obtain the expression of* $F_g^{(k)}(1, v_{[1..k]})$ *(assuming that STUFF is known, by induction). Inject this value back in* (2) *to obtain the expression of* $F_g^{(k)}(u, v_{[1..k]})$*.*

3.3 Proof of the first part of Theorem 9 (rationality)

We are now ready to prove the wanted structure result. To this end we are going (again) to rely on induction, and it is convenient to consider (univariate) power series:

$$H_g^{k,\alpha} = \left(\frac{\partial^{\alpha_0}}{\partial u^{\alpha_0}} \prod_i \frac{\partial^{\alpha_i}}{\partial v_i^{\alpha_i}} \right) F_g^{(k)}(u; v_{[1..k]}) \Bigg|_{u=v_1=\cdots=u_0(t)},$$

where k is an integer and $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_k) \in \mathbb{N}_{\geq 0}^{k+1}$ is a composition. Note that it is natural to introduce the $H_g^{k,\alpha}$, since if we let $u = u_0(t)$ in (2) (or (3)) for $k = 0$, we find:

$$\frac{u_0}{u_0 - 1} F_g^{(0)}(1) = tu_0^2 \sum_{g'=1}^{g-1} H_{g'}^{0,[0]} H_{g-g'}^{0,[0]} + u_0^3 H_{g-1}^{1,[0,1]}. \tag{4}$$

Therefore, even if we are interested in $F_g^{(0)}$ only, we cannot avoid to work with all the $H_g^{k,\alpha}$ in order to close the induction.

The rest of the proof is relatively straightforward (just apply induction on (g,k)) but a bit technical, so we just give the main ideas. First note that in view of derivating (3), we will need information on the derivatives of $K(u)$. We have:

Lemma 1. *Let $i \geq 0$ and let $K^{(i)} := \frac{\partial^i}{\partial u^i} K(u)\Big|_{u=u_0(t)}$. We have:*

$$K^{(i)} = R_i(\rho) K^{(1)} \text{ for some } R_i \in \mathbb{Q}(\rho), \tag{5}$$

where $K^{(1)} = \sqrt{\frac{\rho(\rho+2)}{3}}$.

Proof. Since everything is explicit, the expression of $K^{(1)}$ can be checked by hand or computer. The statement can then be proved by induction on i, using the (explicitely checked) fact that $(u-1)^2 K^2$ is a polynomial in u and ρ. □

Proposition 6. *For any g, and k such that $(g,k) \neq (0,0)$, for any α such that $\alpha_i > 0$ for $i =\in [0..k]$, there exists a rational function R such that*

$$H_g^{k,\alpha} = R(\rho) \left(\sqrt{\frac{\rho(\rho+2)}{3}} \right)^{k+1}.$$

Before proving the Proposition we need the lemma:

Lemma 2. *Let $k, \ell \geq 0$, and $A(u)$ be a polynomial. Then*

$$\frac{\partial^k}{\partial u^k} \frac{\partial^\ell}{\partial v^k}\Big|_{u=v=w} \frac{A(u) - A(v)}{u - v} = \frac{k!\ell!}{(k+\ell)!} \frac{\partial^{k+\ell+1}}{\partial u^{k+\ell+1}}\Big|_{u=w} A(u)$$

Proof. Exercise! □

Sketch of the proof of the proposition. We apply induction on (g,k) for the lexicographical order. We apply the operator

$$\left(\frac{\partial^{\alpha_0+1}}{\partial u^{\alpha_0+1}} \prod_i \frac{\partial^{\alpha_i}}{\partial v_i^{\alpha_i}} \right)\Bigg|_{u=v_1=\cdots=u_0(t)}$$

to the loop equation (2) multiplied by $(1-u)$. The equation involves the quantity $K^{(1)} H_g^{k,\alpha} = \sqrt{\frac{\rho(\rho+2)}{3}} H_g^{k,\alpha}$, as well as other terms that involve either a genus strictly smaller than g, or a number of faces strictly smaller than k.

From the previous lemmas, the induction hypothesis is then enough to conclude, except in the base case $(g,k) = (0,1)$. In that case, we have to check the result explicitely, which can be done by induction on α_0 from the explicit expression of $F_0(u)$. □

As promised, we now deduce the first part of Theorem 9:

Corollary 2. *For all g, the generating function $F_g^{(0)}(1)$ is a rational function in $\sqrt{1-12t}$.*

Proof. This is an immediate consequence of (4), together with the last proposition. □

3.4 Sketch of the proof of the second part of Theorem 9 (asymptotics)

The asymptotics part of Theorem 9 is proved in a similar way as the first one: the most important thing is to find the appropriate form of induction.

We first fix some notation. All the univariate series we will consider have a radius of convergence equal to $1/12$, are algebraic, and have and a unique dominant singularity at this point. For two such series $A(t)$ and $B(t)$, we will write $A \approx B$ if their Puiseux expansion near $t = 1/12$ are equal at the first non-vanishing order. Finally, in this section, c denotes a positive constant that changes from time to time.

We first state an asymptotic analogue of Lemma 1:

Lemma 3. *For $i \geq 1$ one has $K^{(i)} \approx c(1-12t)^{\frac{3-2i}{4}}$.*

Proof. Proceed by induction, as for Lemma 1. □

Using this lemma, one can then show the main point of the induction, which is an asymptotic analogue of Proposition 6:

Proposition 7. *For any g, and k and α such that $(g,k,\alpha_0) \neq (0,0,0)$, we have the asymptotic estimate:*

$$H_g^{k,\alpha} \approx (1-12t)^{-\frac{10g+5k-3+2\sum_{i\geq 0}\alpha_i}{4}}$$

for some positive constant $c = c(g,\alpha,k)$.

Proof. The proof goes by induction on (g,k). Similarly as in the proof of the last proposition, apply the operator

$$\left(\frac{\partial^{\alpha_0+1}}{\partial u^{\alpha_0+1}} \prod_i \frac{\partial^{\alpha_i}}{\partial v_i^{\alpha_i}}\right)\Bigg|_{u=v_1=\cdots=u_0(t)}$$

to the loop equation (2) multiplied by $(1-u)$. The equation involves the quantity $K^{(1)}H_g^{k,\alpha} = \sqrt{\frac{\rho(\rho+2)}{3}}H_g^{k,\alpha} \approx c(1-12t)^{1/4}H_g^{k,\alpha}$, as well as other terms that involve either a genus strictly smaller than g, or a number of faces strictly smaller than k. The induction hypothesis ensures that the remaining terms all have the same sign (hence no cancellation occurs!), and that their singularity is of the form $\approx c(1-12t)^{-\frac{10g+5k-3+2\sum_{i\geq 0}\alpha_i}{4}+\frac{1}{4}}$. So the theorem follows by induction, provided the base case $(g,k) = (0,0)$ is proved (which can be done by a simple induction on α_0 from the explicit expression of $F_0(u)$). □

4 Chapters 4/5: The Schaeffer and Marcus-Schaeffer bijections, an introduction to the bijective theory of maps

In this chapter our goal is to present combinatorial interpretations of the formulas established in the previous chapters. We will focus on the Schaeffer bijection that gives a nice proof of Tutte's formula for planar maps, and its higher genus version (known as the Marcus-Schaeffer bijection) that enables a (partial) combinatorial interpretation of Bender and Canfield's result on higher genus maps.

4.1 Main statements

In this chapter our goal is to enumerate rooted maps by genus and number of edges, bijectively. Thanks to Tutte's bijection described in Chapter 1, we will rather focus on *bipartite quadrangulations.* We start with a few definitions.

NOTATION: *In this chapter, all maps are rooted.*

Definition 4. A map is *pointed* if it is equipped with a distinguished vertex

Definition 5. A *one-face map* is a map having only one face. A *labelled* one-face map is a one-face map equipped with a labelling of its vertices by integers $\ell : V \to \mathbb{Z}$, in such a way that if $u \sim v$ are two incident vertices, $\ell(u) - \ell(v) \in \{-1, 0, 1\}$.

Labelled one-face maps are considered up to common translation of all the labels. The *normal representative* of a labelled one-face map is obtained by translating all labels in such a way that the minimum label is equal to 1. The *centered representative* of a labelled one-face map is the one for which the root vertex receives label 0.

We will abusively talk about the *label* of a corner, for the label of the unique vertex incident to it.

Theorem 11 ($g = 0$: Cori-Vauquelin 1981, Schaeffer 1998; $g > 0$: Marcus-Schaeffer 1999)**.** *There is a 1-to-2 mapping between bipartite quadrangulations of genus g with n faces which are both rooted and pointed, and labelled one-face maps of genus g with n edges.*
The bijection sends a quadrangulation with $n+2-2g$ vertices to a one-face map with $n+1-2g$ vertices, in such a way that for $i > 0$, the number of vertices in the quadrangulation that are at graph-distance i from the pointed vertex is equal to the number of vertices of label i in the labelled one-face map (in the normal representation).

Corollary 3 (Bijective proof of Tutte's formula)**.** *The number of rooted planar maps with n edges is $\frac{2 \cdot 3^n}{n+2}\mathrm{Cat}(n)$.*

Proof of the corollary. First, by Tutte's bijection (Chapter 1), rooted planar maps with n edges are equinumerous with rooted (bipartite) planar quadrangulations with n faces. Now, since a planar one-face map is just a plane tree, the number of labelled one-face maps of genus 0 is clearly $3^n\mathrm{Cat}(n)$ (think of the centered representation).

Since a planar quadrangulation with n faces has $n + 2$ vertices (from Euler's formula), the last theorem thus gives:

$$(n + 2)m_0(n) = 2 \cdot 3^n\mathrm{Cat}(n). \qquad \square$$

Remark 9. In genus $g > 0$, the counting of labelled one-face maps is less obvious than in the plane. But it is still doable in a convincingly combinatorial way. In section 4.4 we will thus obtain a combinatorial proof of Bender and Canfield's asymptotics:

$$m_g(n) \sim c \cdot n^{\mathbf{\frac{5}{2}(g-1)}} 12^n,$$

and in particular an interpretation of the counting exponent $\frac{5}{2}(g-1)$. We will also prove combinatorially a weaker form of Bender and Canfield's rationality statement for the generating function of maps of genus g (giving a combinatorial interpretation of the full statement is still an open problem).

4.2 The bijection

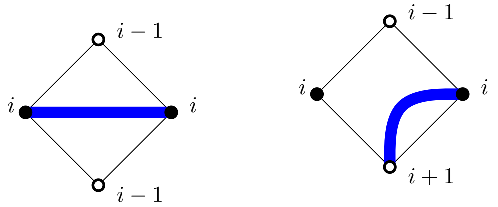
We now start describing the bijection, à la Schaeffer. Let $\mathfrak{q}$ be a pointed bipartite quadrangulation of genus g with n faces, with pointed vertex v_0. The bijection is extremely easy to describe and consists in two steps:

• **Step 1.** We first label all the vertices of $\mathfrak{q}$ by their graph-distance to v_0. So v_0 has label 0, its neighbours have label 1, etc... We note $\ell(u)$ the label of the vertex u. Notice that since $\mathfrak{q}$ is bipartite, if u and v are two vertices of $\mathfrak{q}$ linked by an edge, their distances to v_0 satisfy:

$$|\ell(v) - \ell(u)| = 1.$$

Indeed, this (integral) quantity is at most one by the triangle inequality, and it is congruent to 1 mod 2 since by asumption the quadrangulation is bipartite (all its cycles are even).

• **Step 2.** The last equation implies that, in $\mathfrak{q}$, the there are only two kinds of faces: either the labels of its corners form a sequence $(i, i+1, i, i+1)$ or $(i, i+1, i+2, i+1)$, for some $i \geq 0$:



Then, in each face of $\mathfrak{q}$, we add a new (thick) edge inside the face, according to the rules displayed on the previous figure. We let $\mathfrak{t}$ be the "map" consisting of all the vertices of $\mathfrak{q}$ except from the pointed vertex v_0, and all the newly created edges. See Figure 6.

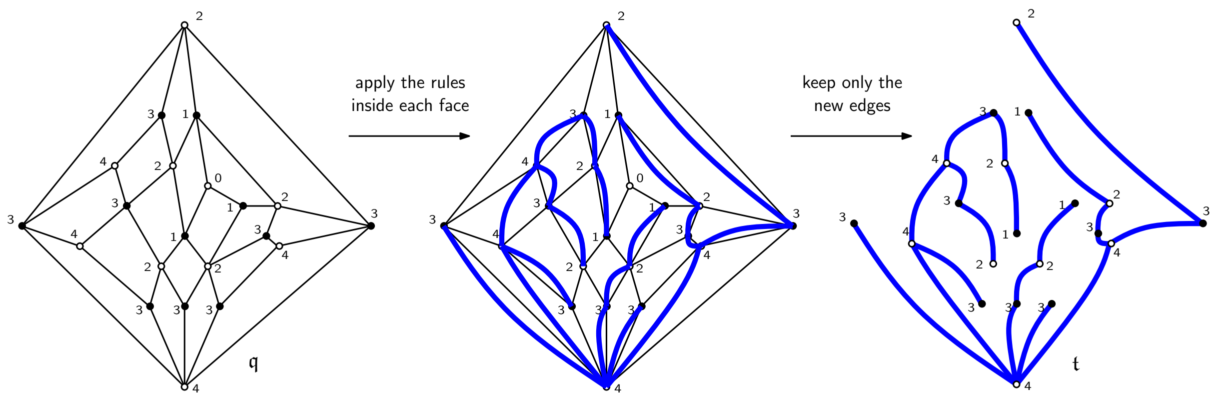


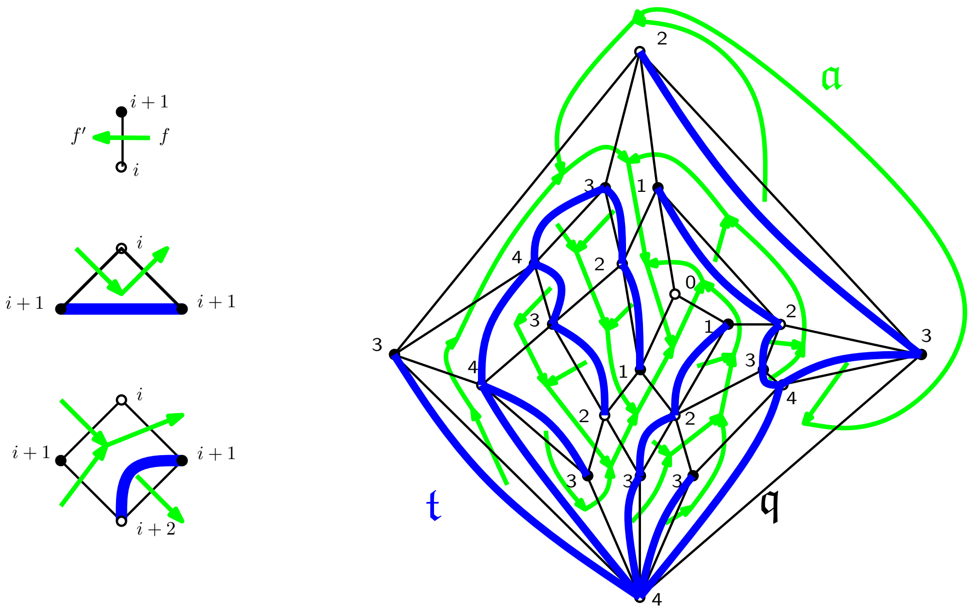
Figure 6: An example of the Cori-Vauquelin-Schaeffer bijection, in genus 0.

Lemma 4. $\mathfrak{t}$ *is a well-defined map, and it has only one face, genus g, and n edges.*

Proof. Note that several things are unclear: first, it is not clear that $\mathfrak{t}$ is connected. Second, it is not clear that it has genus g. We will solve these two problem at once by showing that $\mathfrak{t}$ is the "dual submap" of a suitably constructed tree, in a suitably constructed larger map.

• *First step of the proof: constructing the larger map* $\mathfrak{m}$ *and the dual submap* $\mathfrak{a}$

We let $\mathfrak{m}$ be the (larger map) consisting of all the original vertices and edges of $\mathfrak{q}$, and of all the (newly added) edges of $\mathfrak{t}$. Being a subdivision of $\mathfrak{q}$, $\mathfrak{m}$ is clearly a well defined map of genus g. We now let $\mathfrak{m}'$ be the dual map of $\mathfrak{m}$, and we look at the submap $\mathfrak{a}$ of $\mathfrak{m}'$ formed by dual edges of edges of $\mathfrak{q}$, as on the following figure:



Note that, for future use in the proof, we have oriented the edges of $\mathfrak{a}$, in such a way that each edge sees a larger label on its right than on its left.

• *Second step of the proof:* $\mathfrak{a}$ *is a unicyclic graph, whose only cycle "encircles" the vertex v_0.* First, $\mathfrak{a}$ has has many edges as $\mathfrak{q}$, hence $2n$, and by construction it has 2 vertices per face of $\mathfrak{q}$, hence $2n$ vertices.

Therefore the only thing to show is that the only cycle of $\mathfrak{a}$ is the one that encircles the vertex v_0. Let $c = (e_1, e_2, \ldots, e_k)$ be a cycle of $\mathfrak{a}$. No the key point is to observe that: (1) with our orientation of the edges of $\mathfrak{a}$, each vertex has outdegree exactly one, so each cycle in in fact a directed cycle; (2) when we follow an oriented edge of $\mathfrak{a}$, the label we see on its right cannot decrease, so along any cycle the label we see on the rigt is constant; (2') the only way for a cycle to see a constant label is to enclose a single vertex, say of label i, whose all neighbors have label $i+1$; (3) the only such vertex is the pointed vertex, v_0.

• *Third step of the proof.* We are now almost done. Observe that by construction $\mathfrak{t} \cup \{v_0\}$ (a map with two connected components formed by the trivial map v_0 and the map $\mathfrak{t}$) is the dual map of $\mathfrak{a}$. This implies that $\mathfrak{t}$ is a unicellular map of genus g (in particular, $\mathfrak{t}$ is connected and well-defined).

□

Remark 10. Note that it is clear by construction that the unicellular map $\mathfrak{t}$ satisfies the constraints of a labelling (variation of ± 1 or 0 along edges), and that it is also clear by construction that the labels in $\mathfrak{t}$ correspond to distances to v_0 in $\mathfrak{q}$.

4.3 The reverse bijection and the proof

So far we have described the bijection and proved it is well-defined. We now desribe the converse bijection (which in particular will prove the bijectivity).

Start with a labelled unicellular map $\mathfrak{t}$ of genus g with n edges. Since $\mathfrak{t}$ is rooted and has only one face, its $2n$ corners are naturally indexed $(c_1, c_2, \ldots, c_{2n})$ with c_1 being the root corner, and c_{i+1} being the corner following c_i when walking along the unique face in clockwise direction.

The converse bijection consists in constructing a new edge for each corner of $\mathfrak{t}$, as follows:

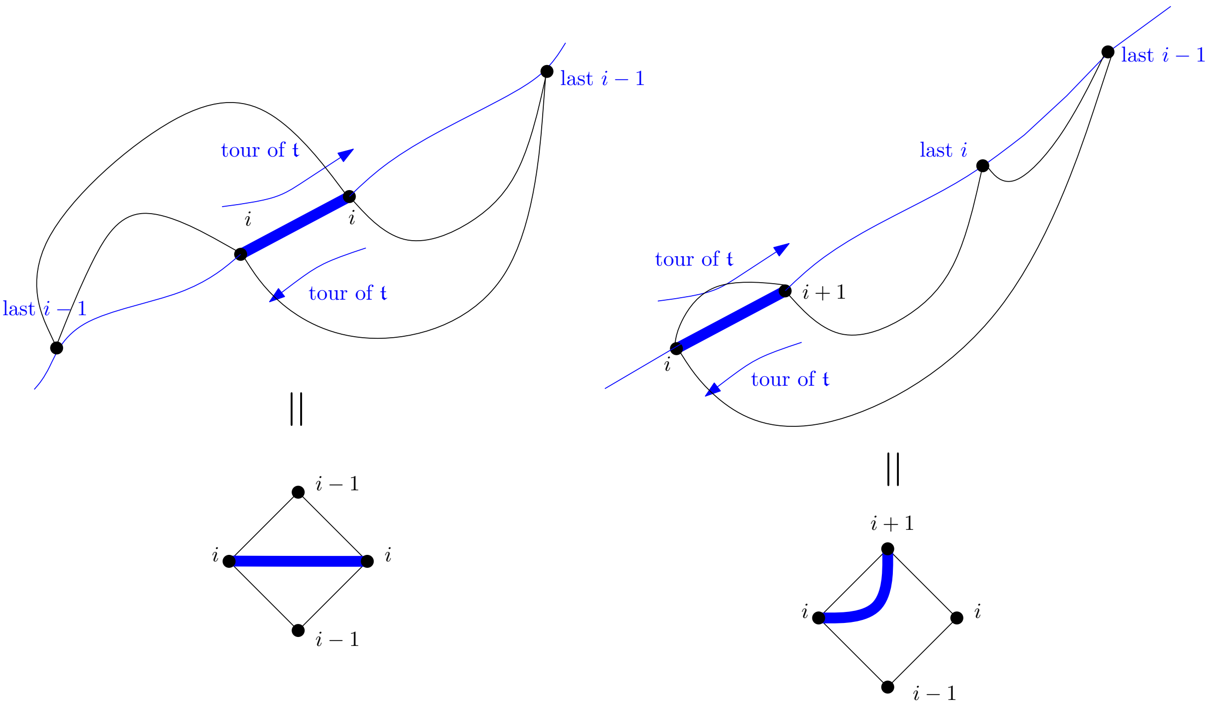
- Start by adding a new vertex (call it v_0) inside the unique face of $\mathfrak{t}$, and link it by a new edge to every corner of $\mathfrak{t}$ labelled 1;
- Draw an edge from each corner c of $\mathfrak{t}$ of label $i > 1$ to its *successor*, defined as the first corner of labelled $i-1$ encountered counterclockwise arount the face of $\mathfrak{t}$, starting from c.

We let $\mathfrak{q}$ be the map consisting of all the vertices of $\mathfrak{t}$ plus the new vertex v_0, and of all the newly added edges.

Proposition 8. *$\mathfrak{q}$ is a well-defined bipartite quadrangulation of genus g with n edges.*

Proof. First, the underlying graph of $\mathfrak{q}$ is bipartite by construction, since we only draw edges between even and odd labelled vertices.

Now observe that, at the end of the construction, each edge of $\mathfrak{t}$ is bordered by a face of degree 4:



Therefore $\mathfrak{q}$ is a bipartite quadrangulation. By construction it has n faces, $2n$ edges, and genus $n + 2 - 2g$ vertices, so it has genus g. □

End of the proof that the two constructions are inverse from each other. First it is clear that when we construct $\mathfrak{q}$ from $\mathfrak{t}$, the labels of the vertices in $\mathfrak{t}$ become their distance to v_0 in $\mathfrak{q}$ (since edges always connect a vertex of label i to one of label $i-1$, and since v_0 is the only vertex with label 0).

Looking at the previous picture once again, it is then clear that by applying the Schaeffer rules to $\mathfrak{q}$, we get back the unicellular map $\mathfrak{m}$. □

4.4 Combinatorial interpretation of Tutte's formula

The combinatorial interpretation of the Tutte'scounting formula for planar maps was already given in the proof of Corollary 3... but this deserved a section!

In passing, let us mention that the generating function of rooted labelled treesis obviously solution of the equation:

$$T = 1 + 3tT^2,$$

which is a simple variant of the equation for plane trees (we have now three types of edges, since the increment of the label when we go away from the root of the tree can be -1, 0, or $+1$).

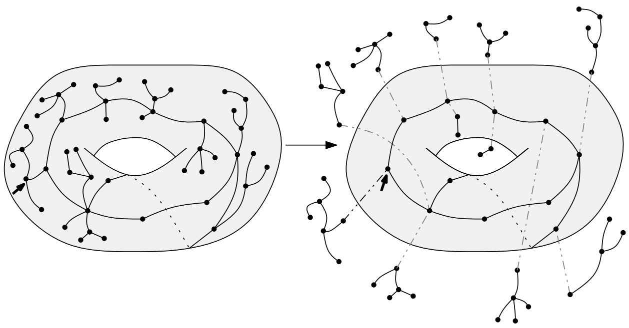
4.5 Schemes, and combinatorial interpretation of the counting exponent $\frac{5(g-1)}{2}$

In order to make use of the Marcus-Schaeffer bijection in positive genus, we need to be able to enumerate one-face maps. We are going to do that with generating functions[4]. We are going to build a small number of "basic generating functions" that will serve us as building blocks, and we are going to play Lego with them.

4.5.1 The scheme decomposition

Let $\mathfrak{t}$ be a one-face map of genus g with n edges. Since $\mathfrak{t}$ has $n + 1 - 2g$ edges, its *cyclomatic number* (i.e. the number of edges that complement any spanning tree) is $2g$. So we are going to apply to $\mathfrak{t}$ a type of decomposition that is classical in graph theory for graphs with finitely many cycles.

The construction has two step: first erase all the leaves (vertices of degree 1 of $\mathfrak{t}$), and do this recursively until there are no more such vertices left; then replace each maximal chain of vertices of degree 2 by a unique edge. We obtain a map $\mathfrak{s}$ called *the scheme* of $\mathfrak{t}$:



Note that the scheme naturally inherits a root corner from the root corner of $\mathfrak{t}$.

Lemma 5. *The $\mathcal{S}_g$ of schemes of genus g (that is to say, the set of rooted one-face maps of genus g with no vertices of degree* 1 *nor* 2 *is finite.*

Proof. Let d_i be the number of degree i in a given scheme. By Euler formula we have:

$$\sum_i d_i + 1 = \frac{1}{2}\sum_i i d_i + 2 - 2g,$$

so that $\sum_i (i-2)d_i = 4g - 2$. Since $(i-2) > 0$ for $i \geq 3$, there are only finitely possible degree sequences (d_i), hence finitely possible maps. □

Note that the parts of the one-face maps that are removed during the construction of its scheme are essentially "tree-like". Therefore we have a strategy to enumerate one-face maps: start with one of the finitely many schemes of genus g, and re-attach tree like parts to the scheme. That would work in a straightforward way for one-face maps[5] (without labels), but in

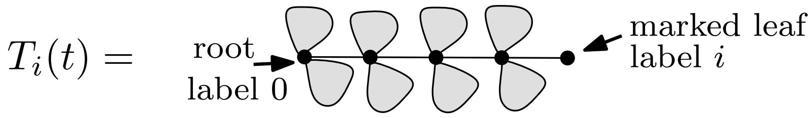
[4]One may ask: if we count one-face maps with generating functions, then why would this be more combinatorial than Bender and Canfield's approach? There are two answers to this: first, the counting of labelled one-face maps with generating functions will not work by solving equations (as with the Bender-Canfield approach) but by directly constructing the objects. So we will have a much better understanding of where the various "pieces" of these generating functions come from and how their singularities arise. Second, the approach with labelled one-face maps gives tons of information about the distances in the map, and are at the basis of the study of their scaling limit.

[5]See the exercises

order to apply it to the case of *labelled* one-face maps, we first need to study a few building blocks – they will enable us to reconstruct the ”tree-like parts” with sufficient control on the labelling.

4.5.2 The basic building blocks

The basic generating function we will need is the generating function $M_i(t)$ of rooted trees with a distinguished leaf (different from the root vertex), in which the root vertex has label 0 and the marked one has label i.



First note that this series can be written

$$M_i(t) = N_i(s)$$

where $s = tT(t)^2$, and $N_i(s)$ is the generating function of Motzkin walks[6] ending at position i, by the number of steps. In order to compute this generating function, we first introduce the generating function $M(s)$ of Motzkin walks that start at 0, end at 0, and always stay nonnegative. By decomposing such walks at their last passage to 0 we obtain:

$$M = 1 + +sM + s^2M^2.$$

The generating function of *bridges* (Motzkin walks ending at 0, without sign constraint) follows by decomposing a bridge at its passages to 0:

$$B = N_0 = \frac{1}{1 - s - 2s^2M}.$$

Now, $i \geq 0$, a walk ending at i can be decomposed as a first bridge leading to its last passage at 0, followed by a series of excursions above each integer $j \in [1..i]$. We thus obtain:

$$N_i = BU^{|i|},$$

where $U = tM(t) = t(1 + U + U^2)$.

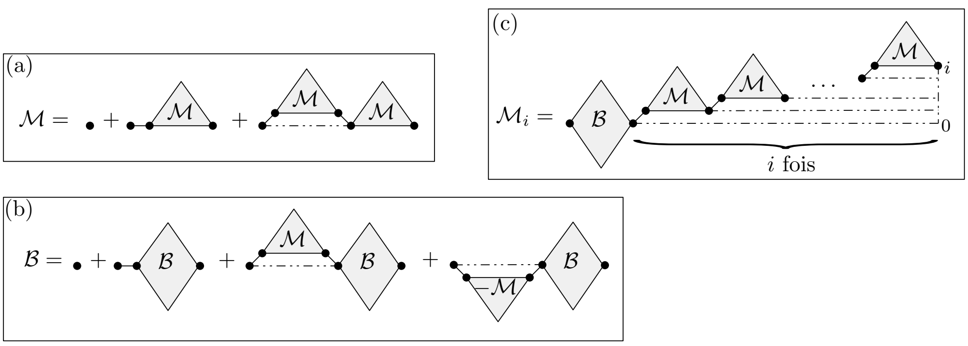
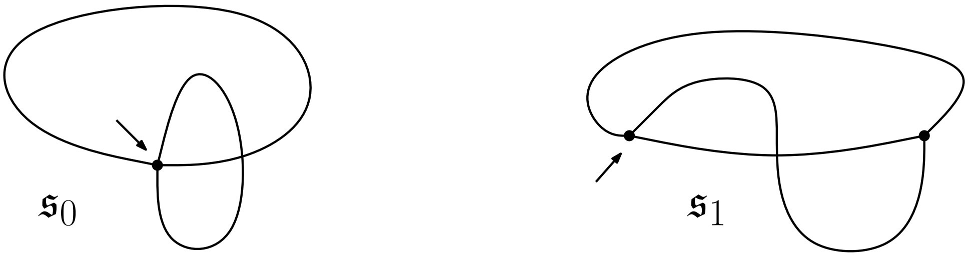


Figure 7: The decompositions of Motzkin walks that enable to determine the generating functions $U(s), B(s)$, and $N_i(s)$.

[6]For us, a Motzkin walk is a sequence of (positive or not) integers starting at 0 and taking steps in $\{-1, +1, 0\}$.

4.5.3 Example of the torus

Let us now start using our building blocks, in the case of the torus ($g = 1$). In this case we have two schemes $\mathfrak{s}_0$ and $\mathfrak{s}_1$:



The generating function for the first case is clearly given by:

$$F_{\mathfrak{s}_1} = \frac{1}{4}\frac{2td}{dt}(M_0 - 1)^2,$$

Indeed we can reconstruct any map of scheme $\mathfrak{s}_0$ by replacing each edge of $\mathfrak{s}_0$ by a (nonempty) bridge, and then choosing a root – and we obtain each in 4 different ways.

Counting maps with scheme $\mathfrak{s}_1$ is more complicated. Since we consider labelled one-face maps up to translation of the labels, we can assume that, in each map $\mathfrak{m}$ of scheme $\mathfrak{s}_1$, the label of the root vertex of $\mathfrak{s}_1$ is 0. We obtain that the generating function of maps of scheme $\mathfrak{s}_1$ is given by:

$$F_{\mathfrak{s}_2} = \frac{1}{6}\frac{2td}{dt}\sum_{i\neq 0} N_i(t)^3 + \frac{1}{6}\frac{2td}{dt}(N_0(t) - 1)^3$$

In the above formula, the integer i denotes the label in $\mathfrak{m}$ of the other vertex of $\mathfrak{s}_1$: it is clear that for a given i all maps can be obtained by gluing three branches of trees of increment i together, hence the term $M_i(t)^3$. The operator $\frac{1}{6}\frac{td}{dt}$ accounts for the rooting. The last term accounts for the case $i = 0$, for which we need to avoid the empty bridge, hence the factor $(M_0 - 1)$ instead of M_0.

Note that since $M_i = BU^i$ for $i > 0$ and $M_i = BU^{-i}$ for $i < 0$ the infinite sum in the above expression rewrites as a rational function in U:

$$F_{\mathfrak{s}_2} = \frac{1}{3}\frac{td}{dt}B^3\left(\frac{U^3}{1-U^3} + \frac{U^{-3}}{1-U^{-3}}\right) + \frac{1}{3}\frac{td}{dt}(B-1)^3.$$

Finally, observe t is a rational function in U, and so are $B = \frac{t(1+2U)}{1-t(1+2U)}$ and $\frac{d}{dt}U(t)$. Therefore all the above quantities are rational functions in U.

Proposition 9 (Combinatorial counting of maps on the torus, C-Marcus-Schaeffer 2009)**.** *The generating function*

$$F_1^\bullet(t) = \sum_{n\geq 0} n m_1(n) t^n$$

of rooted maps on the torus with one additional pointed vertex or face, which by the Marcus-Schaeffer bijection is twice the generating function of labelled one-face maps on the torus, is a rational function of $U = U(tT^2)$, where $T = 1 + 3tT^2$ and $U(s) = s(1 + U + U^2)$, given by:

$$F_1(t) = \frac{td}{dt}\frac{U^{-1} + 4 + U}{2(1-U)^2(1-U^{-1})^2(1+U)(1+U)^{-1}}.$$

Remark 11. The rationality of F_1 in U is weaker than the rationality in T, that is stated in Bender and Canfield's theorem. So far, it is an open problem to give a combinatorial proof of the rationality in T. Observe that the scheme decomposition may not be a correct tool for this stronger statement: indeed, the rationality in T is a fact that we observe on the final expression, but that was not true for the generating functions corresponding to each scheme.

Remark 12. The important thing in the "combinatorial interpretation" is not really the exact expression of $F_1^\bullet$ in terms of U, but the philosophy behind this rationality: labelled one-face maps can be built from "pieces" counted by U, arranged in a "rational" way.

4.5.4 The case of any genus.

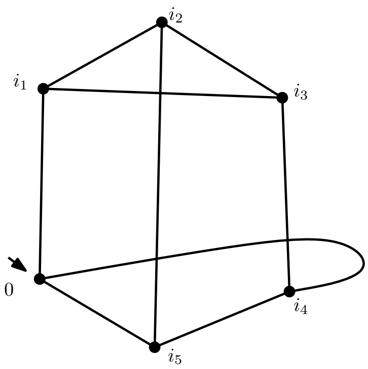
In the case of arbitrary genus, we will do the same as in the plane. I don't write up all the details (see my paper with Marcus and Schaeffer for the full computation), but try to explain where the rationality comes from.

Because the Marcus-Schaeffer bijection works for pointed quadrangulations, we introduce the series:

$$F_g^\bullet(t) := \sum_{n\geq 0}(n+2-2g)m_g(n)t^n.$$

Proposition 10 (Combinatorial counting of maps of genus g, C-Marcus-Schaeffer 2009)**.** *The generating function $F_g^\bullet$ of rooted maps of genus g with one additional pointed vertex or face, is a rational function of $U = U(tT^2)$, where $T = 1+3tT^2$ and $U(s) = s(1+U+U^2)$, that can be (in theory) computed as a sum over all schemes of genus g.*

To illustrate the proof, let us imagine computing the contribution of the following scheme $\mathfrak{s}$ (of genus 2):



The contribution of this scheme to the generating function will look like:

$$\frac{1}{18}\frac{2d}{dt}\sum_{i_1,i_2,\ldots,i_5\in\mathbb{Z}}\prod_{e\in E(\mathfrak{s})}\left(M_{l(e_-)-l(e_+)} - \mathbf{1}_{l(e_-)=l(e_+)}\right)$$

where the indicator function is there to avoid replacing an edge of $\mathfrak{s}$ by an empty path, the derivative is there for the rooting, and the factor $\frac{1}{18}$ is there to forget the rooting of the scheme.

Because of the annoying indicator function, let us first compute the contribution of the sum over the set of indices (i_j) that are all distinct (and distinct from 0). Because we also want to get rid of the absolute values, we can then decompose this sum furthermore into a sum over cones, where each cone corresponds to a relative ordering of the (i_j). Hence in each cone we will have to evaluate a sum of the form:

$$\sum_{(i_1,i_2,\ldots,i_5)\in\text{cone}}\prod_{e\in E(\mathfrak{s})} BU^{a_1i_1+a_2b_2+\ldots a_5b_5}$$

where $a_1, \dots a_5 \in \mathbb{Z}$ are fixed integers depending on the cone (they come from expanding the absolute values explicitely, given the fact that the sign of each $l(e_-) - l(e_+)$ is fixed in each cone). It is easy to see (and standard, the g.f. of integer points on a cone) that this will resum as:

$$B^{|E(\mathfrak{s})|} \times \mathrm{Rational}(U),$$

which gives a rational function in U (for the reasons already given just before Proposition 9. This is all there is in the above theorem (up also to the fact that we have to distinguish separately each facet bounding the different cones, i.e. consider a different summation for each possible set of equalities between the i_j's, because of the indicator function at the beginning that has to be treated separately; this is complicated notationwise but straightforward conceptually).

4.5.5 The combinatorial interpretation of the counting exponent $\frac{5}{2}(g-1)$.

One can check from the equations that the quantities U and B have a unique dominant singularity at $t = \frac{1}{12}$, of the form:

$$U = 1 + c_1(1-12t)^{1/4} + \dots, c_1 > 0$$

$$B = \frac{c_2}{(1-12t)^{1/4}} + \dots, c_2 > 0.$$

Therefore, in the previous calculation, the main singularity come from two phenomena

- Because of the factor $B^{|E(\mathfrak{s})|}$, each edge of the scheme $\mathfrak{s}$ contributes a factor $(1-12t)^{-1/4}$ to the main singularity, hence leading to a factor of $(1-12t)^{-|E(\mathfrak{s})|/4}$ alltogether.
- The summation over a cone gives a maximal contribution to the singularity when no $(i_j)'s$ are distinct, and in this case the fraction resulting from the summation has a pole at $U = 1$ of order equal to the number of summation indices. Since we fix the label of the root vertex of the scheme to be 0 in the summation (because labelled maps are considered up to translation), this second source of singularity gives a contribution equal to $(1-12t)^{-\frac{V(\mathfrak{s})-1}{4}}$.

Finally, taking into account the derivative that gives another factor $(1-12t)^{-1}$ to the singularity, we obtain:

Lemma 6. *The contribution of a given scheme $\mathfrak{s}$ to the generating function $F_g^\bullet$ has a singularity of the form:*

$$c(1-12t)^{-\frac{|E(\mathfrak{s})|+|V(\mathfrak{s})|+3}{4}}$$

for some $c > 0$.

Corollary 4 (Bijective interpretation of the counting exponent $\frac{5}{2}(g-1)$ for maps of genus g)**.** *For fixed g, the number of maps of genus g with n edges satisfies:*

$$m_g(n) \sim t_g n^{\frac{5}{2}(g-1)} 12^n,$$

for some $t_g > 0$.

Proof. From the above estimate, the maximal contribution to the generating function is given by schemes for which $|E(\mathfrak{s})| + |V(\mathfrak{s})|$ is maximal. It is not hard to see that this is the case for cubic schemes, i.e. schemes whsoe all vertices have degree 3. Applying Euler's formula, we find that such a scheme has $6g-3$ edges and $4g-2$ vertices. Hence the contribution to the singularity of $F^{\bullet}$ is:

$$c(1-12t)^{-\frac{10g-2}{4}} = c(1-12t)^{-\frac{5g-1}{2}}.$$

Applying standard transfer theorem (e.g. Flajolet and Odlyzko) we find that the coefficient $(n+2-2g)m_g(n)$ of t^n in $F^{\bullet}(t)$ is equivalent to:

$$n^{\frac{5g-3}{2}}12^n,$$

and dividing by $(n+2-2g)$ we get the desired result. □