An introduction to map enumeration

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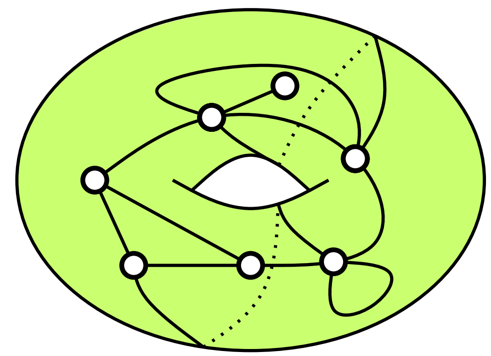
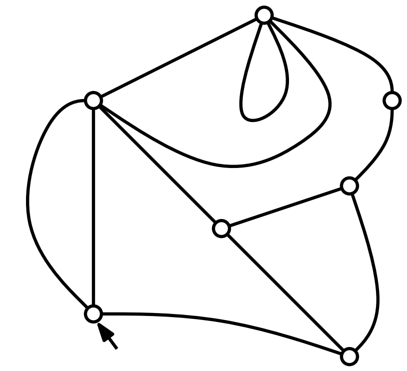
AEC Summer School 2014, Hagenberg.

What this course is about (I)

A map is a graph embedded in a surface:



or

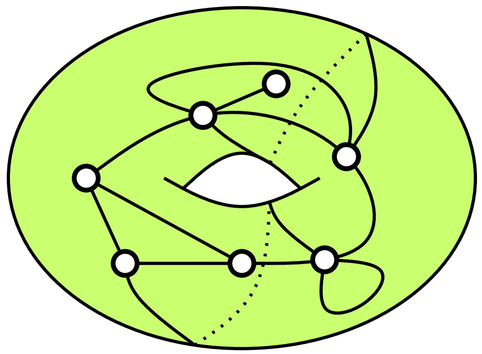


Maps appear (almost?) everywhere in mathematics.
Map enumeration alone is an enormous area.

At least three reasons to be interested in it:

- because you would like to start working seriously on the subject.
- because map enumeration contains powerful tools that can be useful to other parts of combinatorics (functional equations, bijective tricks, algebraic tools...).
- because it is likely that your favorite subject is linked to map enumeration in at least some special case: looking at where this problem appears in the world of maps is a very good source of new questions.

What this course is about (II)



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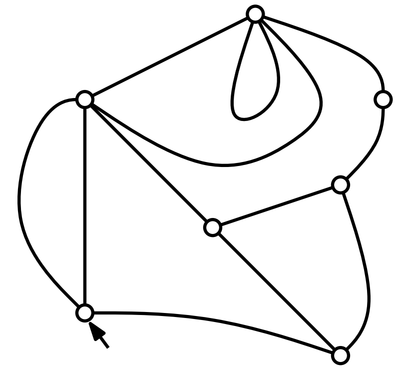

Topics not covered: - link with algebraic geometry

matrix integrals

string theory

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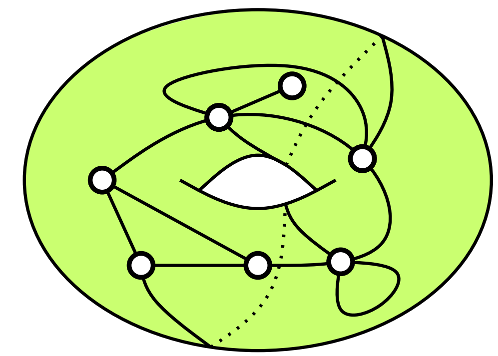

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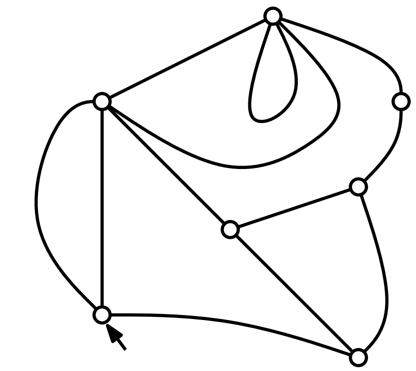

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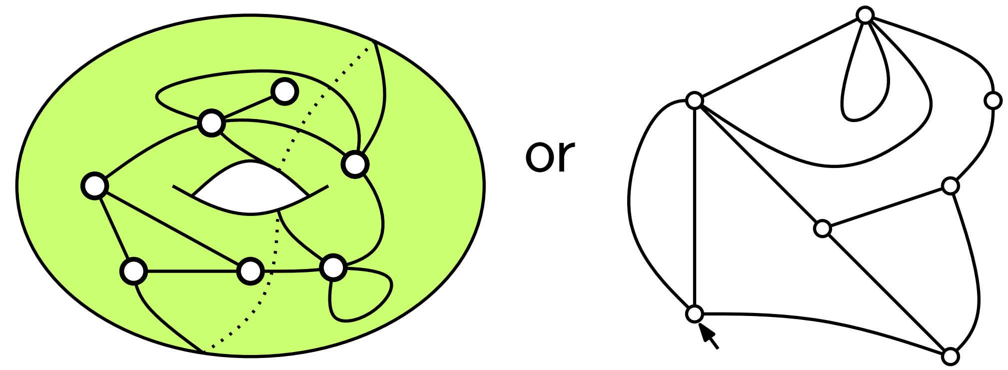

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SEE THE BOOK BY LANDO-ZVONKIN

SEE E.G. SURVEY BY MIERMONT

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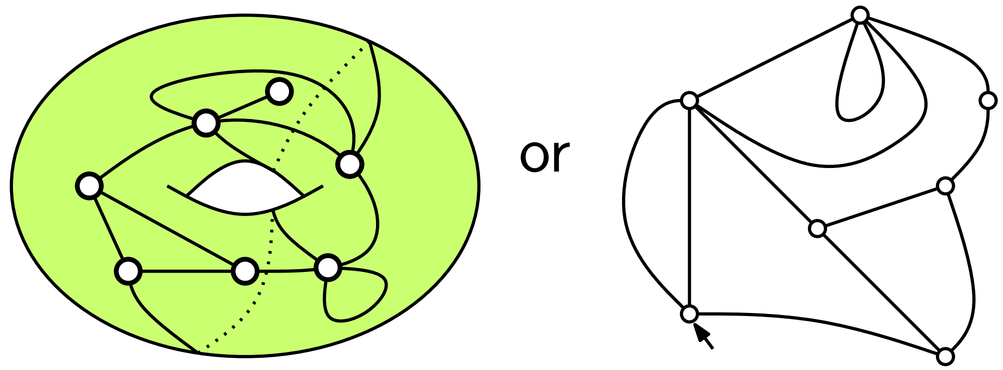

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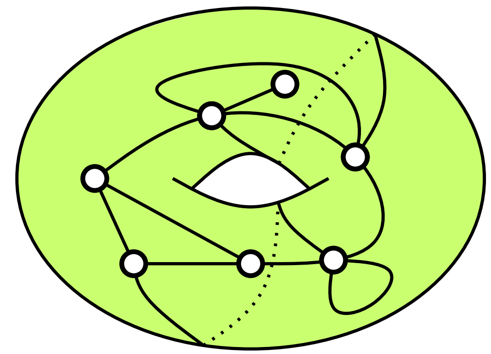
Topics covered: I – Maps
II – Tutte equation, counting planar maps
III – Tutte equation, counting maps on general surfaces
IV – Bijective counting of maps

The exercises contain entry points to other subjects
(one-face maps, link with the symmetric group)

Lecture I – What is a map?
(the oral tradition)

Rough definition

A map is a graph drawn on a surface.



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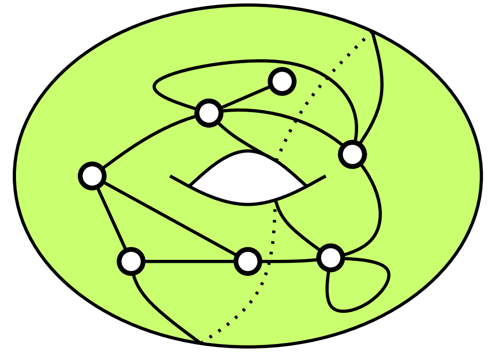

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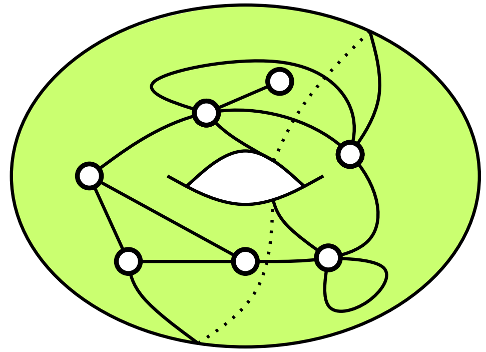

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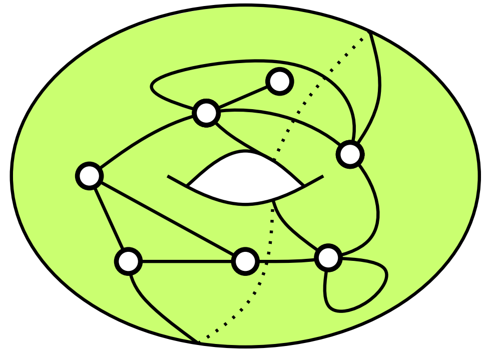

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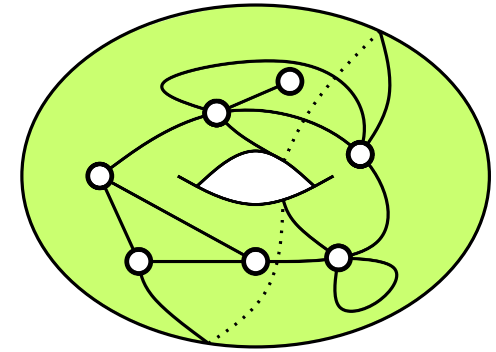
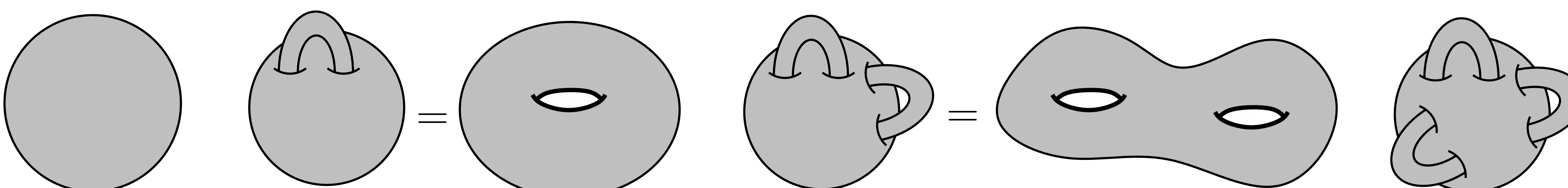


A surface is a connected, compact, oriented, 2-manifold
considered up to oriented homeomorphism.

Example: $\mathbb{S}_g$:= the g-torus = the sphere with g handles attached



$\mathbb{S}_0$ $\mathbb{S}_1$ $\mathbb{S}_2$ $\mathbb{S}_3 \ldots$

Theorem of classification: every surface is one of the $\mathbb{S}_g$ for some $g \geq 0$
called the genus.

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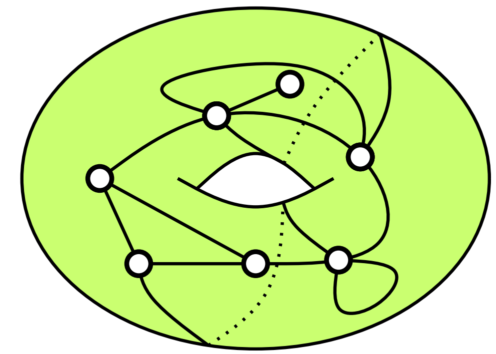

Our graphs are unlabelled, connected, and may have loops or multiple edges.

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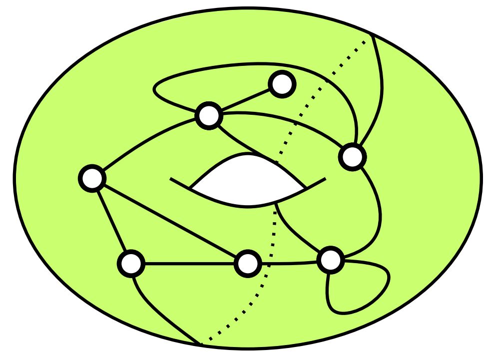

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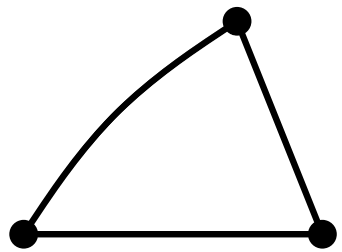

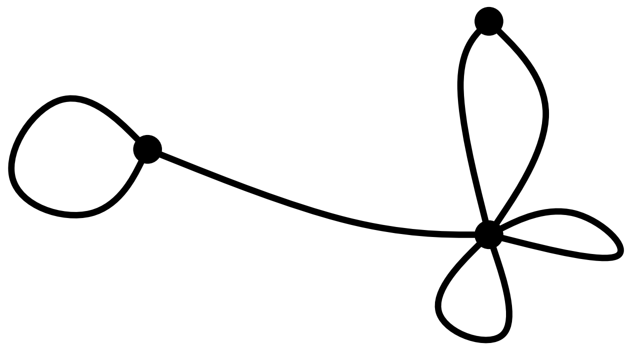

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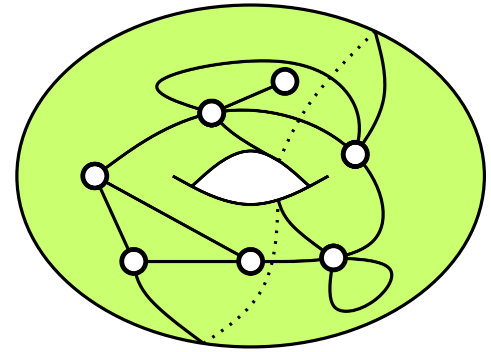

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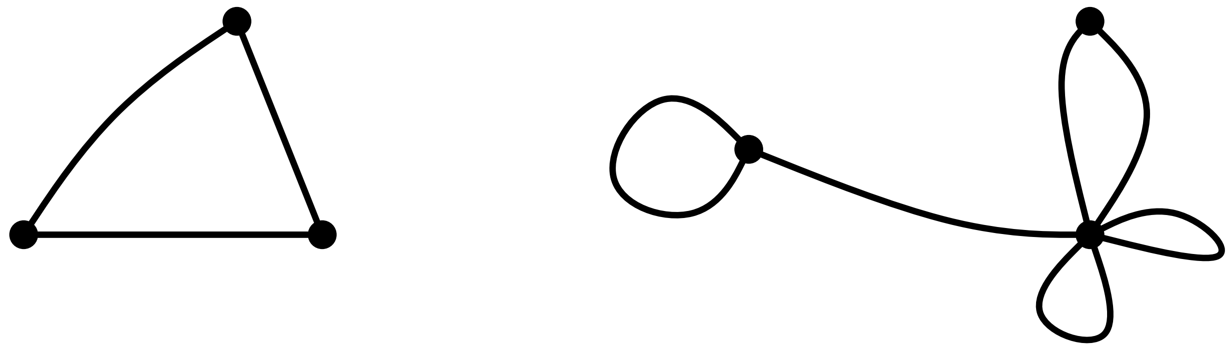

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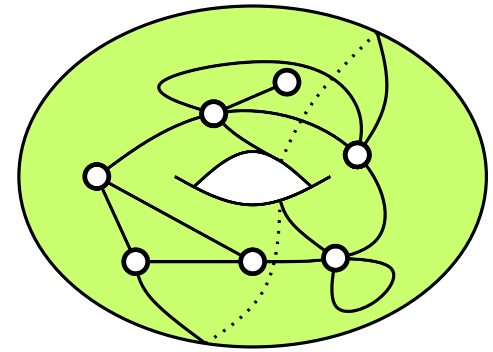

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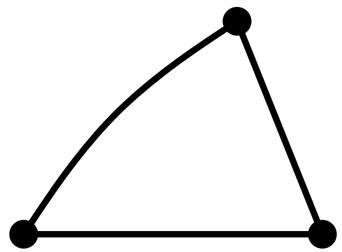
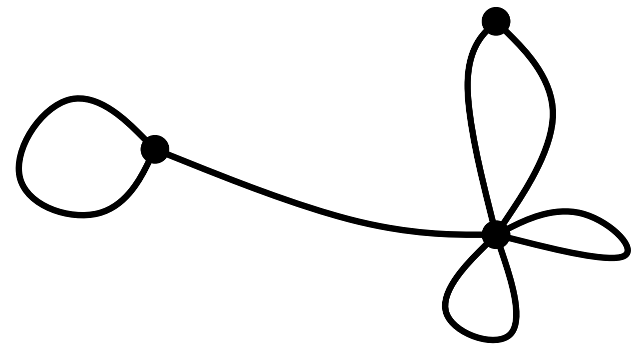




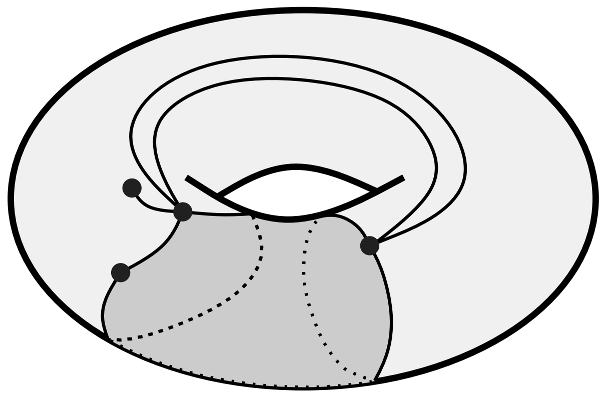
The degree of a vertex is the number of half-edges incident to it.

A proper embedding of a graph in a surface is a continuous drawing of the graph on the surface without edge-crossings.

Topological definition of a map

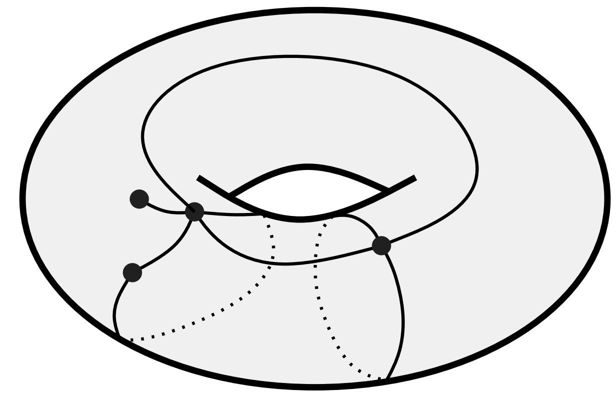
A map is a proper embedding of a graph G in a surface $\mathbb{S}$ such the connected components of $G \setminus S$ (called faces) are topological disks.

(a)



not a map

(b)



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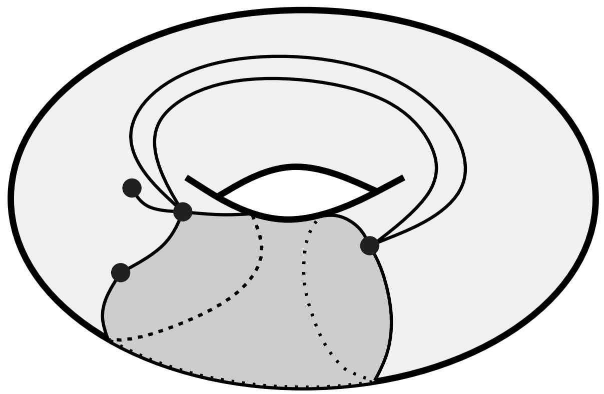

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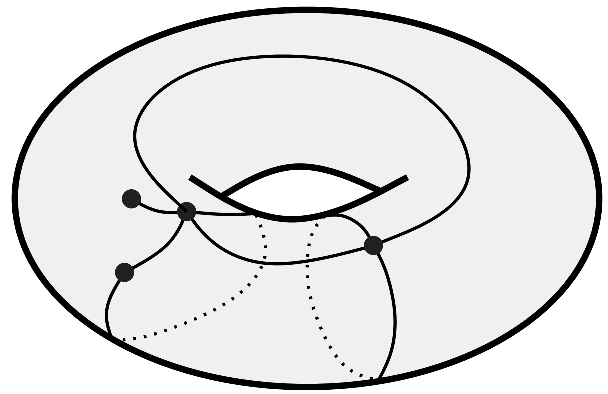

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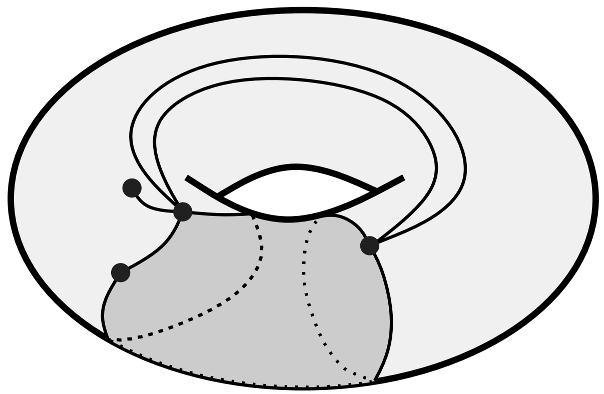
(a) 

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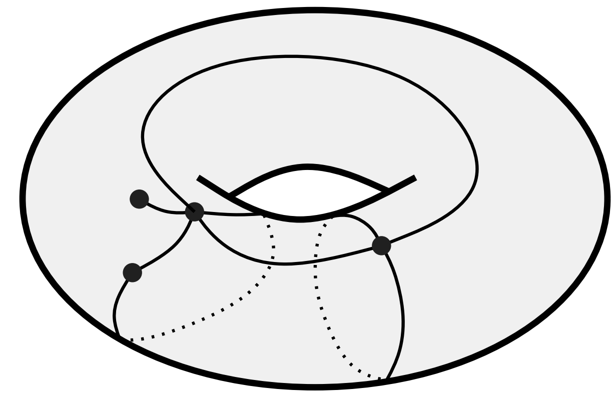
(b) 

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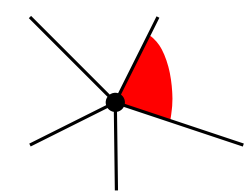
Important notion: a corner is an angular sector delimited by two consecutive half-edges in the neighborhood of a vertex.



There is a canonical bijection between corners and half-edges.

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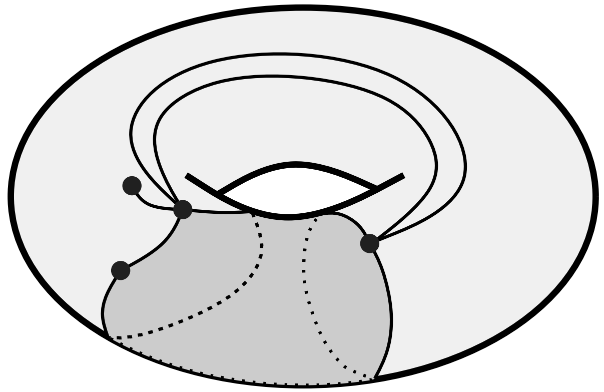

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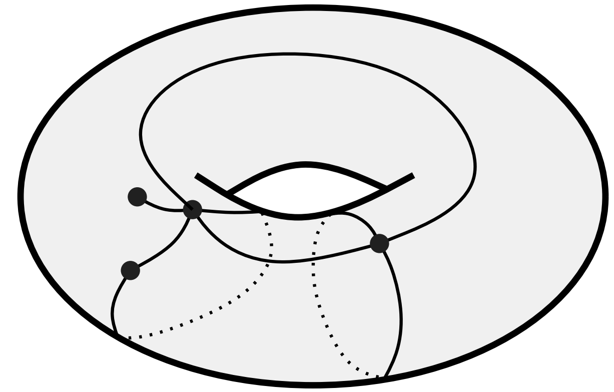

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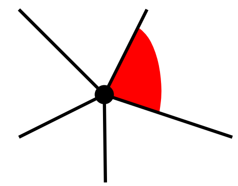

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The degree of a vertex (or face) is the number of corners incident to it.

If a map has n edges then:

$$2n = \# \text{ corners} = \# \text{ half-edges} = \sum \text{ face degrees} = \sum \text{ vertex degrees.}$$

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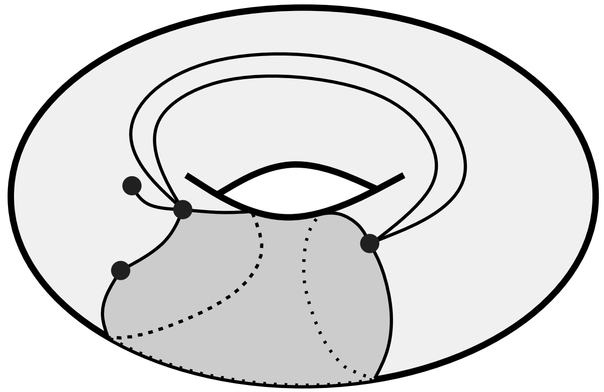
(a) 

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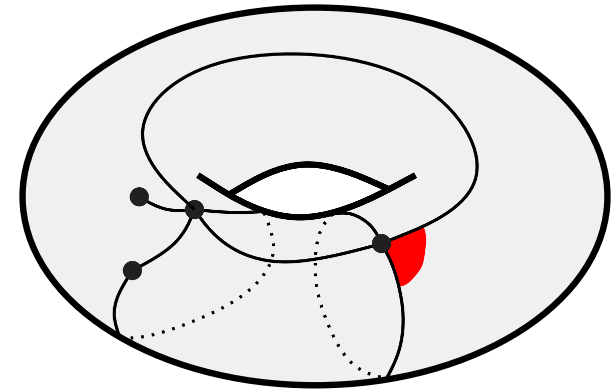
(b) 

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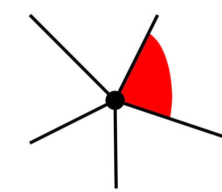

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A rooted map is a map with a distinguished corner (or half-edge)

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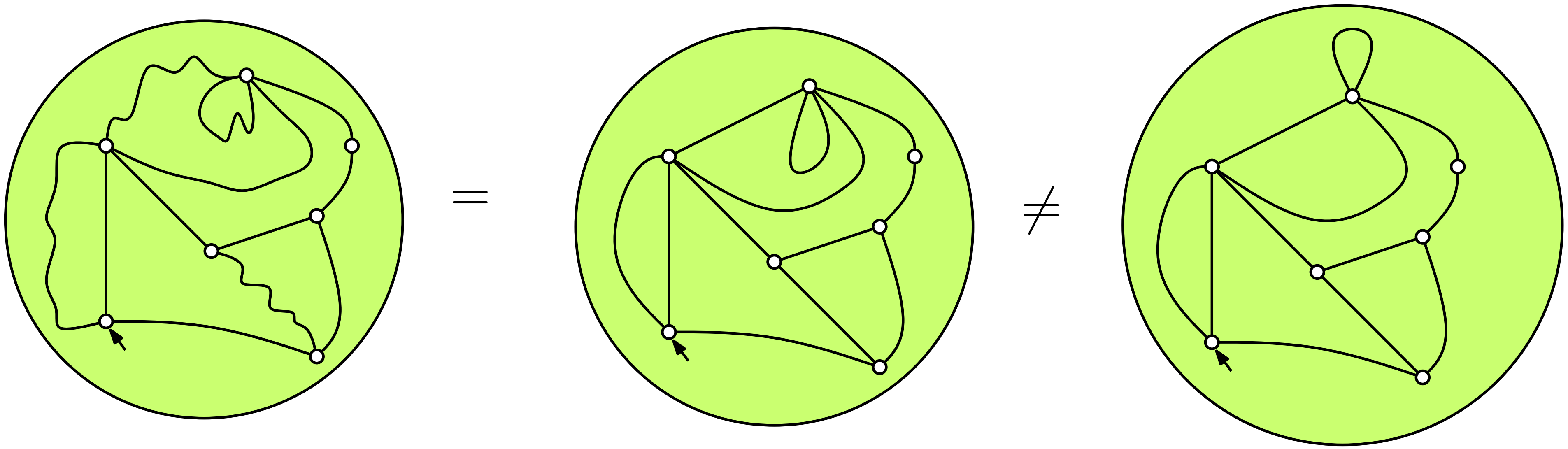

Convention The infinite face is taken to be the root face.

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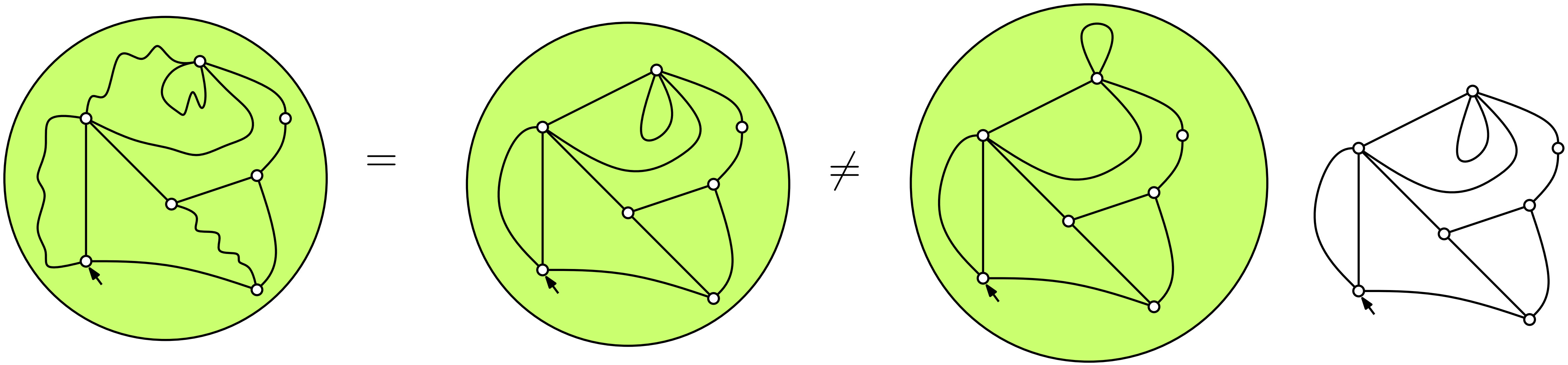


Maps of genus 0 are called planar rather than spherical...

Convention The infinite face is taken to be the root face.

Advertisement for the next lecture:

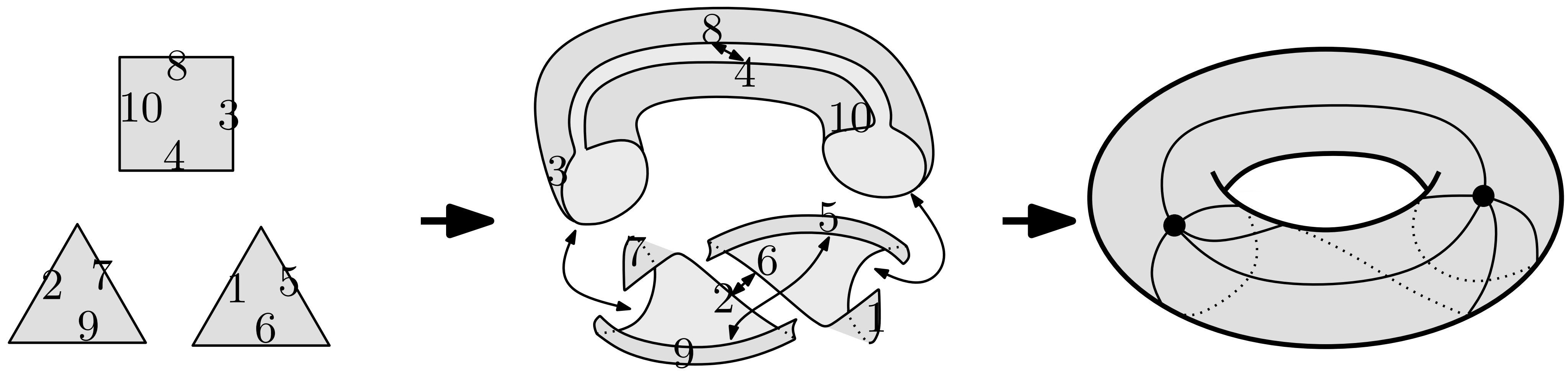
There are $$\frac{2 \cdot 3^n}{n+2}\mathrm{Cat}(n)$$ rooted planar maps with n edges

(a nice number)

Maps as polygon gluings

An easy way to construct a map:

start with a family of polygons with $2n$ sides in total and glue them according to your favorite matching – just be careful to obtain something connected.



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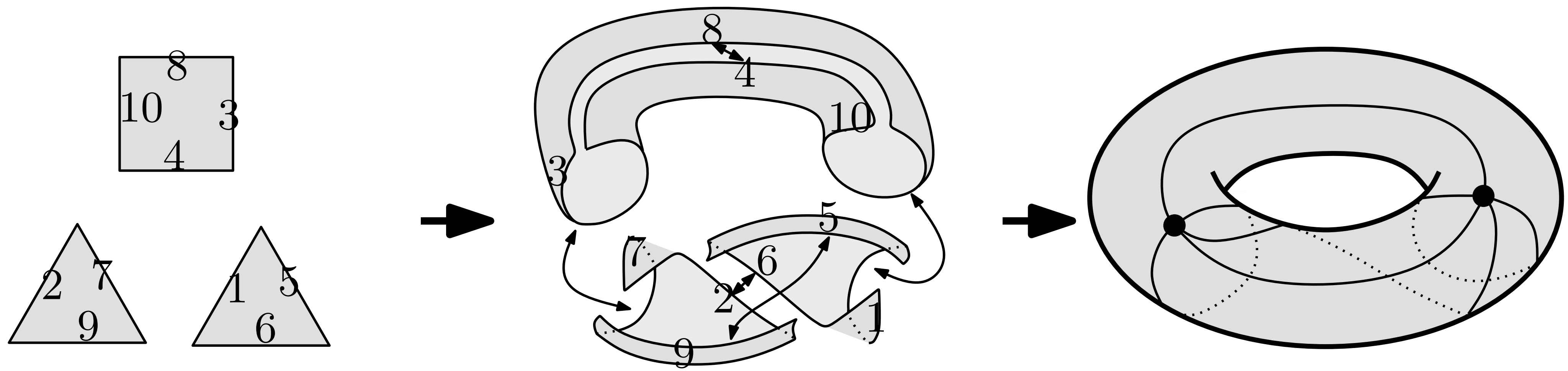

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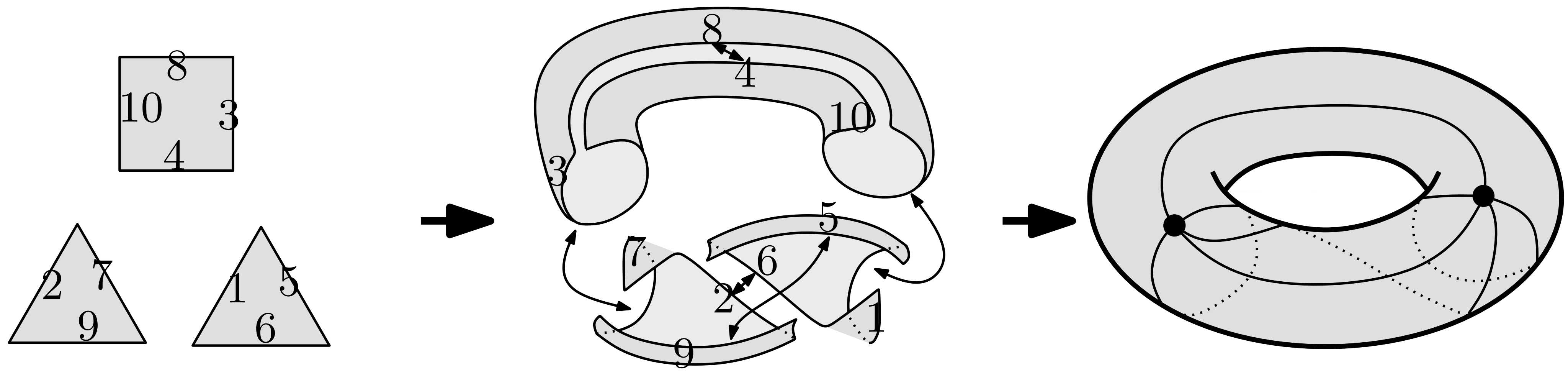

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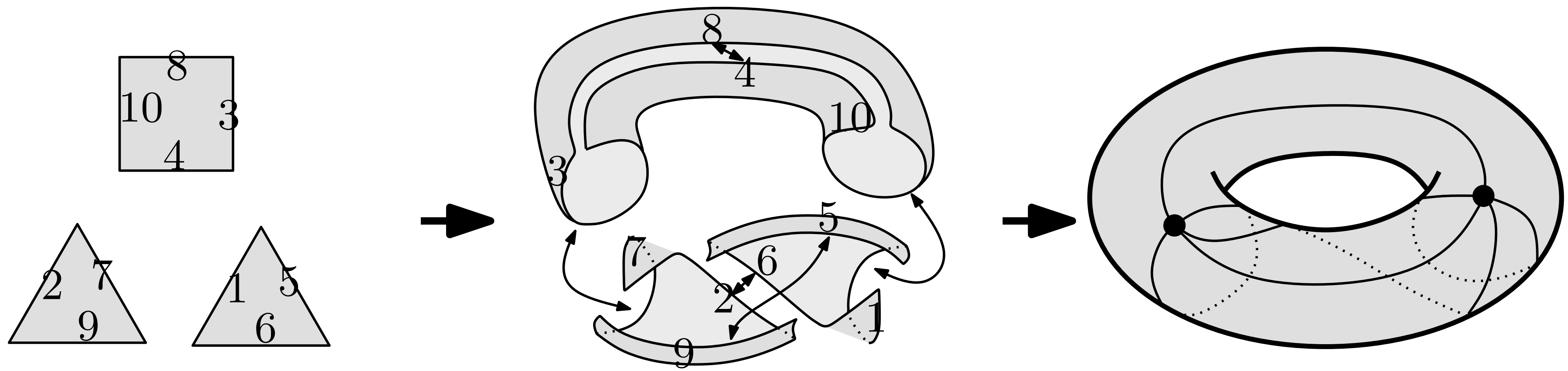

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Proposition: any map can be obtained in this way.

Heuristic proof: to go from right to left, just cut the surface along the edges of the graph.

Maps as rotation systems

A rotation system on a graph is the data of a cyclic order of the half-edges around each vertex.

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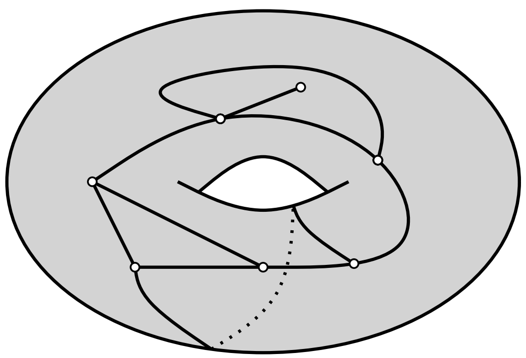
Fact: There is a natural mapping:

Maps $\longrightarrow$ Graphs equipped with a rotation system

...given by the local counterclockwise ordering!



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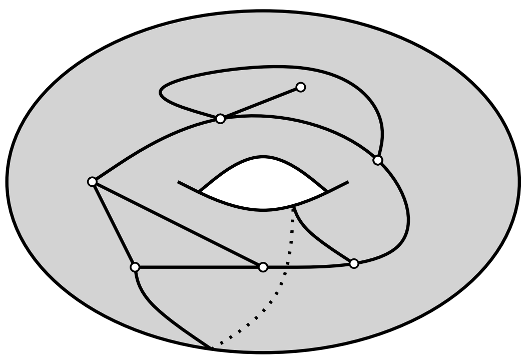

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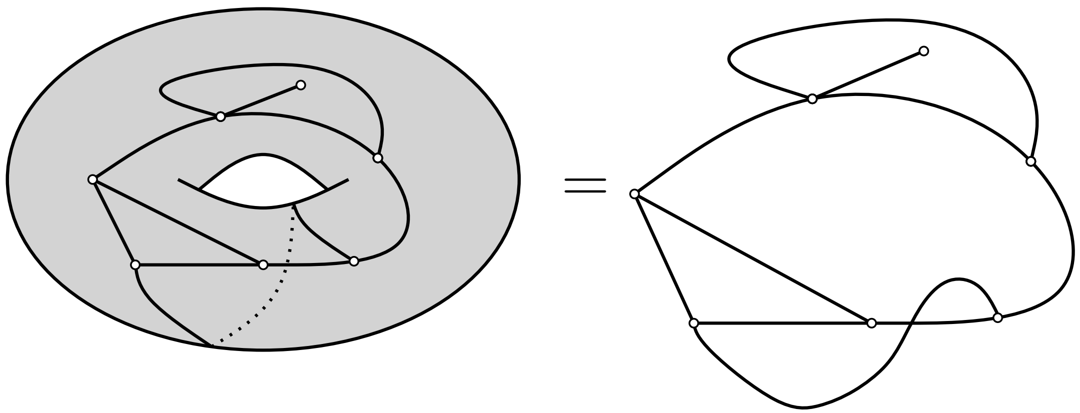
Corollary: We can easily draw a map on a sheet of paper by representing the "surface clockwise order" as the "paper clockwise" order:



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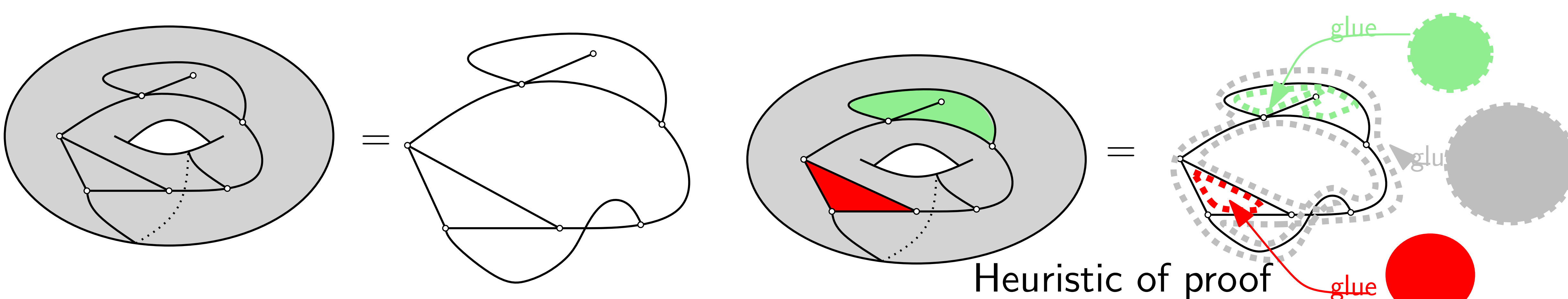

Reminder: At this stage I should draw more examples on the board

Combinatorial definition with permutations

A labelled map of size n is a triple of permutations (σ, α, ϕ) in $\mathfrak{S}_{2n}$ such that

- $\alpha\sigma = \phi$
- α has cycle type $(2, 2, \ldots, 2)$.
- $\langle \sigma, \alpha, \phi \rangle$ acts transitively on $[1..2n]$.

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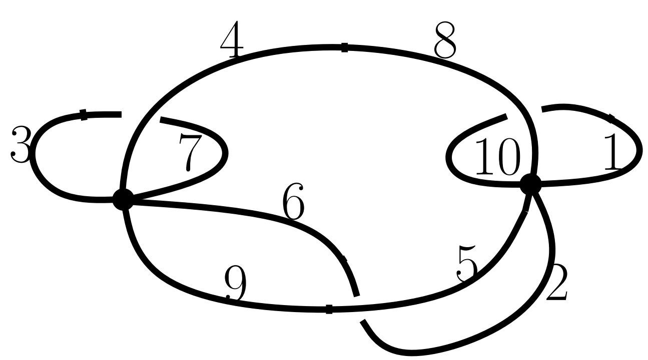
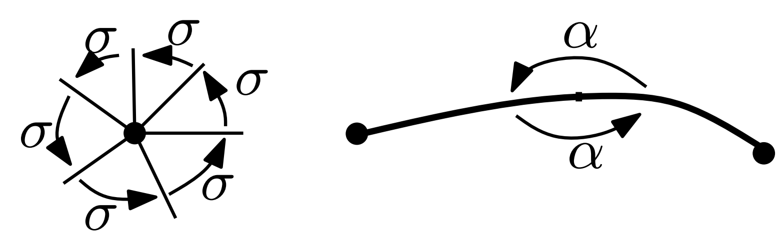
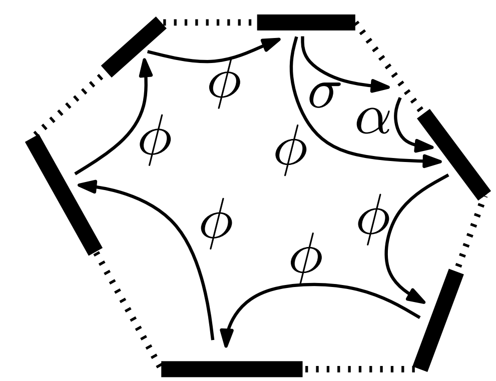
Thm: There is a bijection between labelled maps of size n and graphs with rotation systems whose half-edges are labelled from 1 to $2n$.





$$\sigma = (1, 8, 10, 5, 2)(3, 9, 6, 7, 4)$$
$$\alpha = (1, 10)(2, 6)(3, 7)(4, 8)(5, 9)$$
$$\phi = \alpha\sigma = (1, 4, 7, 8)(2, 10, 9)(3, 5, 6)$$



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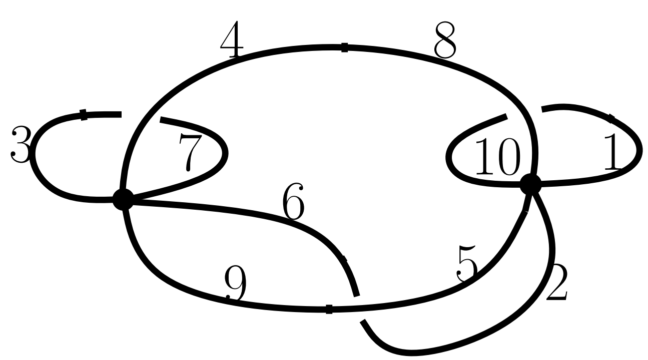

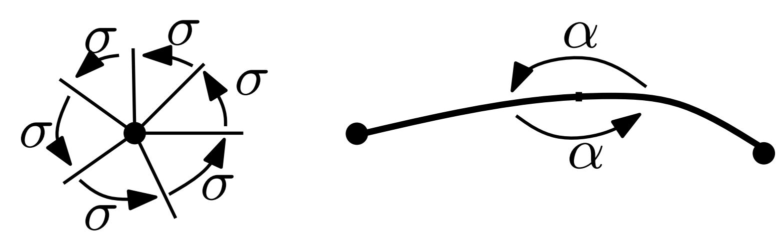

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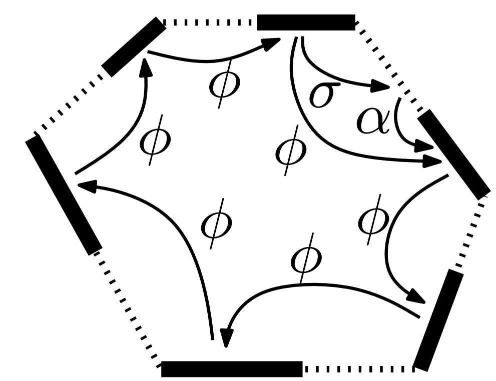

Note: vertices $=$ cycles of σ

edges $=$ cycles of α

faces $=$ cycles of ϕ

Combinatorial definition with permutations

A labelled map of size n is a triple of permutations (σ, α, ϕ) in $\mathfrak{S}_{2n}$ such that

- $\alpha\sigma = \phi$
- α has cycle type $(2, 2, \ldots, 2)$.
- $\langle \sigma, \alpha, \phi \rangle$ acts transitively on $[1..2n]$.

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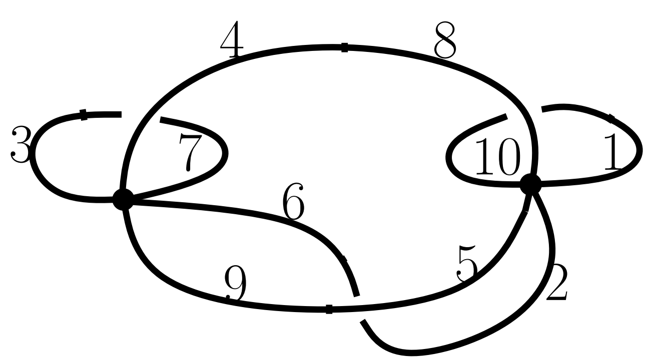

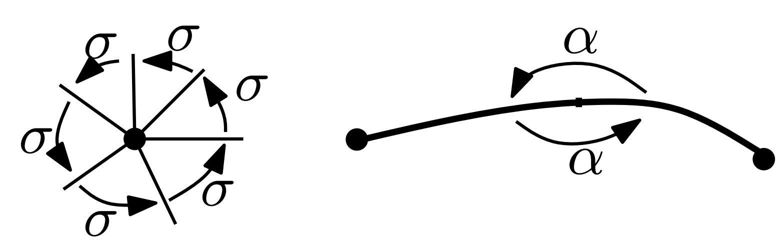

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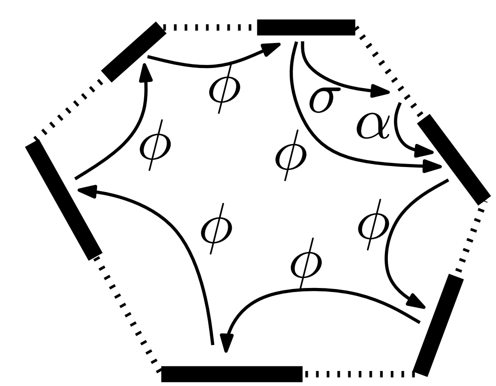

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A rooted map is an equivalence class of labelled maps under renumbering of $[2..2n]$.

labelled map "$=$" $(2n-1)!\times$ rooted map

Duality

A labelled map of size n is a triple of permutations (σ, α, ϕ) in $\mathfrak{S}_{2n}$ such that

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The mapping $(\sigma, \alpha, \phi) \to (\phi, \alpha, \sigma)$ is an involution on maps called duality. It exchanges vertices and faces.

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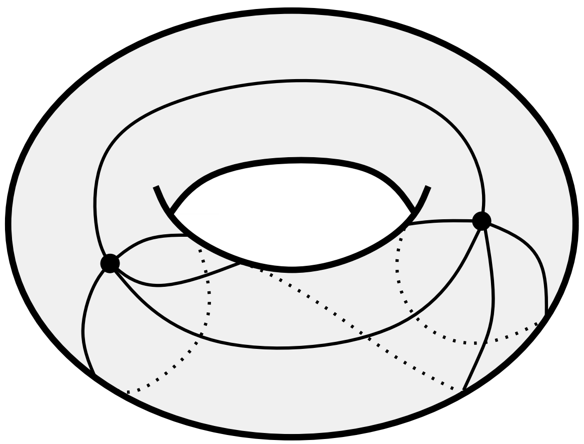
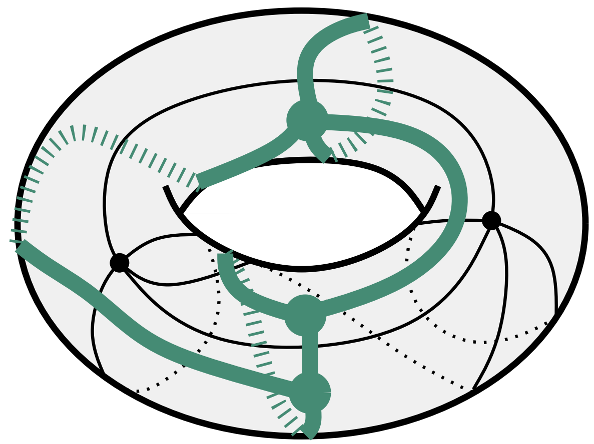
There is also a well-known graphical version:





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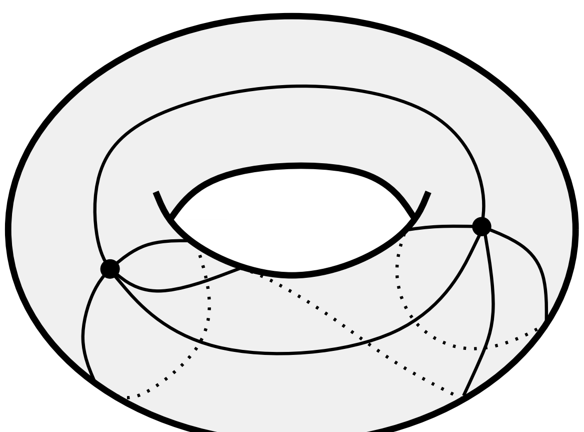
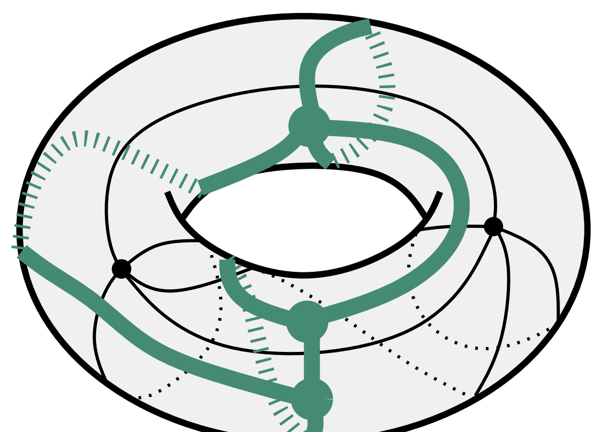


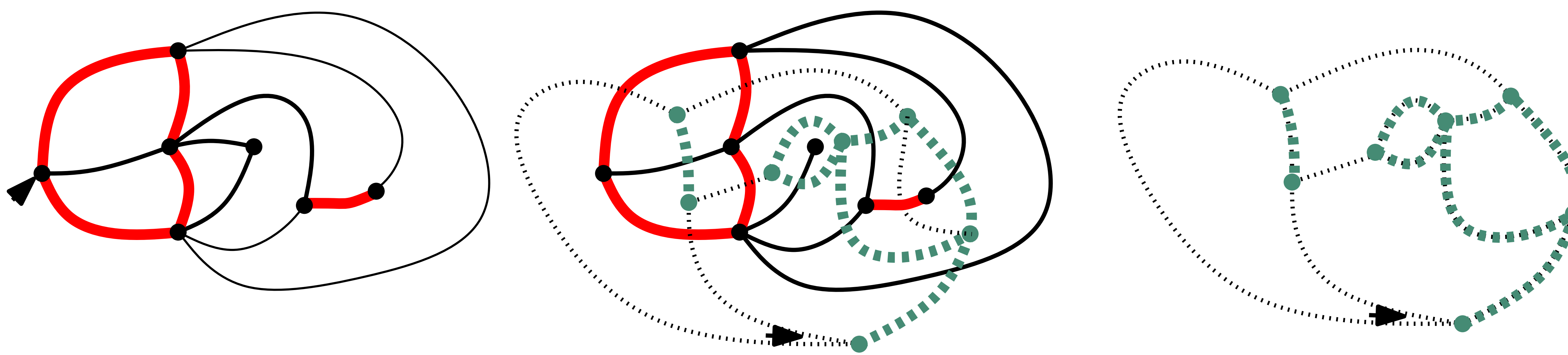


	duality ⟷
edge	edge
vertex of degree i	face of degree i
face of degree i	vertex of degree i
"polygon gluings viewpoint"	"rotation systems viewpoint"

Duality II – dual submap

If $\mathbf{m}$ is a map with underlying graph G then any subgraph $H \subset G$ induces a submap of G, with same vertex set, by restricting the cyclic ordering to H.

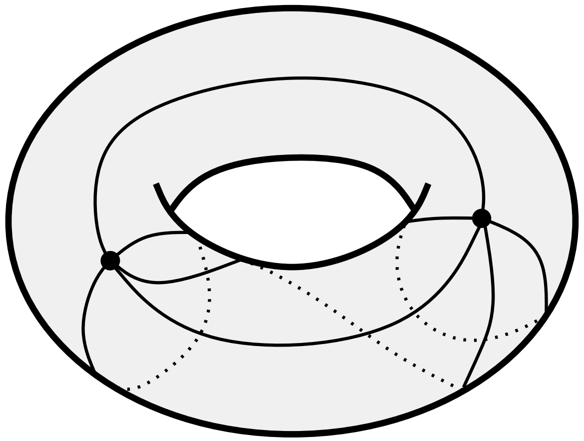
Note that the submap is not necessarily connected (and can have a different genus)



The dual submap is the submap of $\mathbf{m}*$ the formed by edges whose dual is not in H

Proposition: The total number of faces of a submap and its dual submap are equal.

Euler's formula



For a map of genus g with n edges, f faces, v vertices, we have:

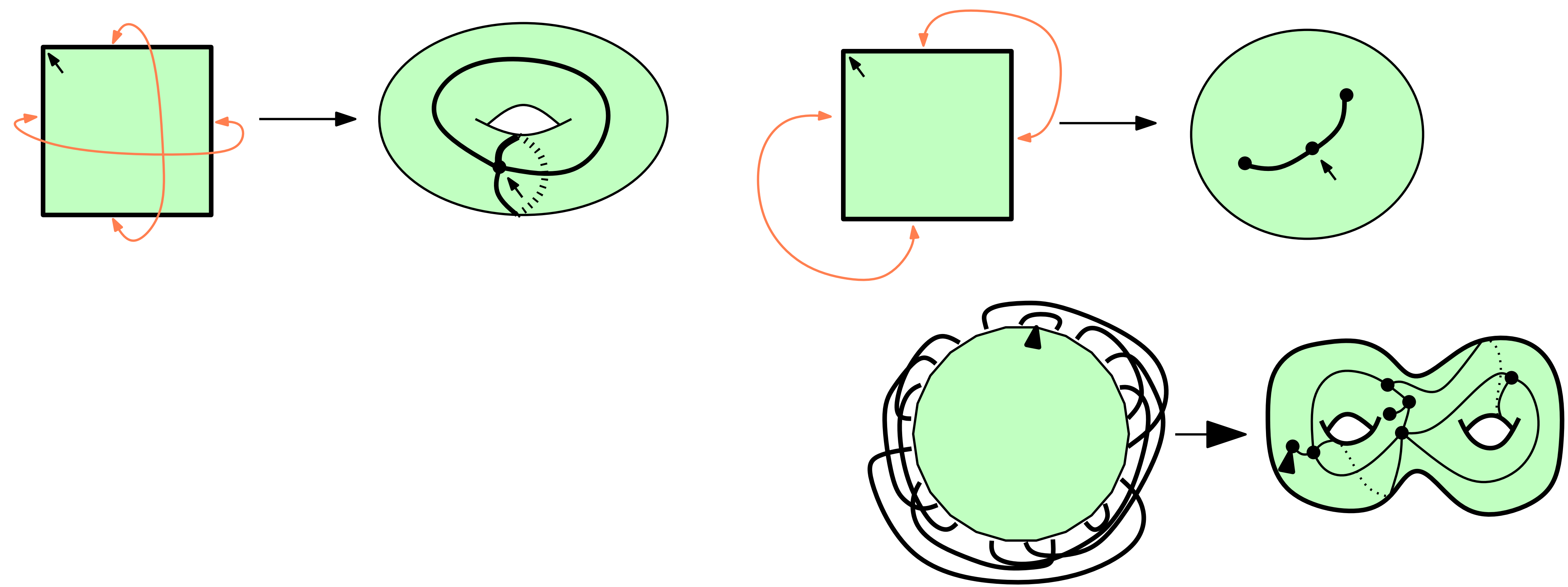
$$v + f = n + 2 - 2g$$

In particular we can recover the genus from the combinatorics
(we don't need to "see" the surface...)

Example II: one-face maps

What is a one-face map? Clear in the “polygon gluing viewpoint”.

Start with a $2n$-gon and glue the edges together according to some matching.



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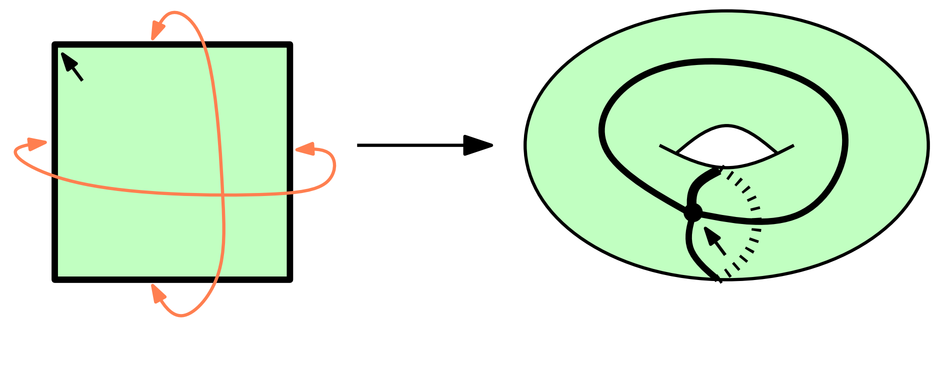
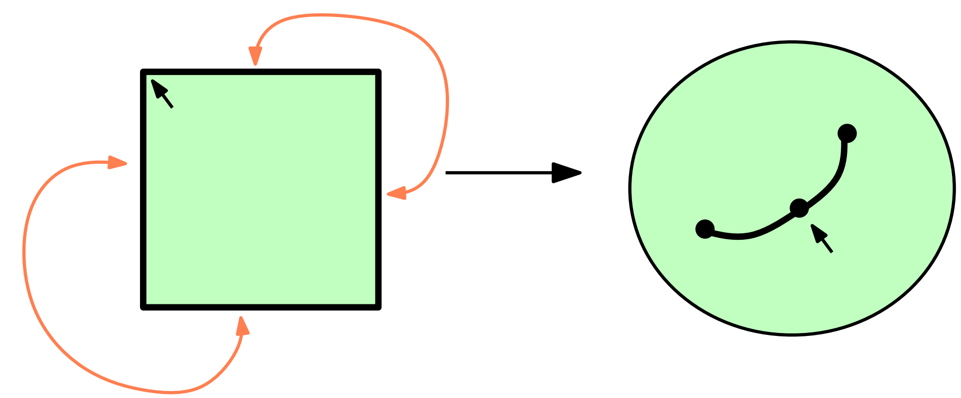
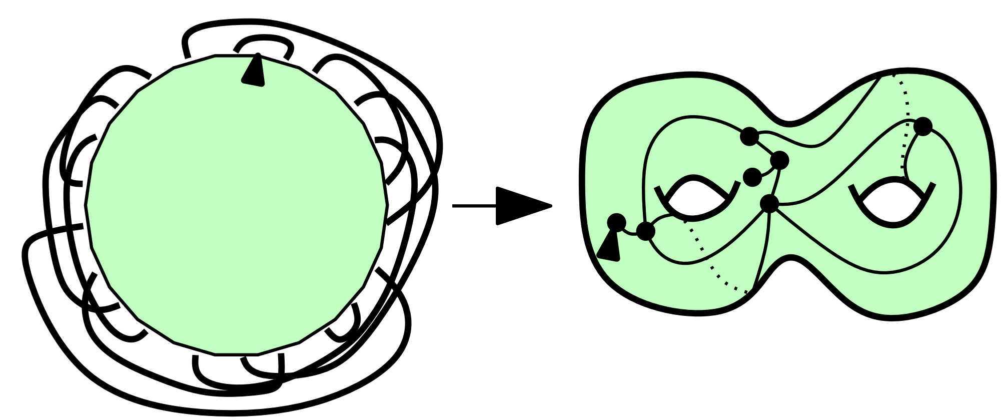




Proposition: The number of one-face maps with n edges is $(2n-1)!! = \frac{(2n)!}{2^n n!}$.



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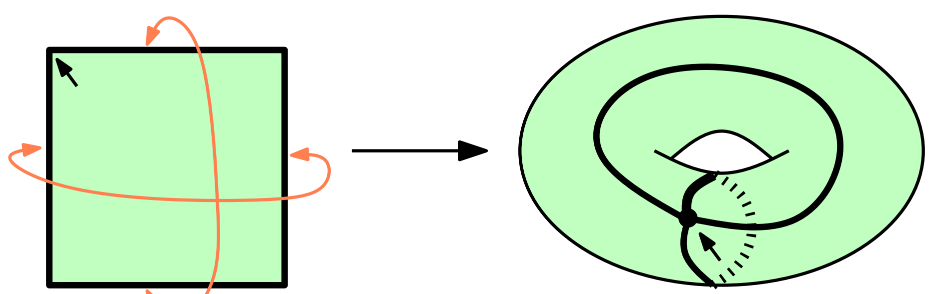

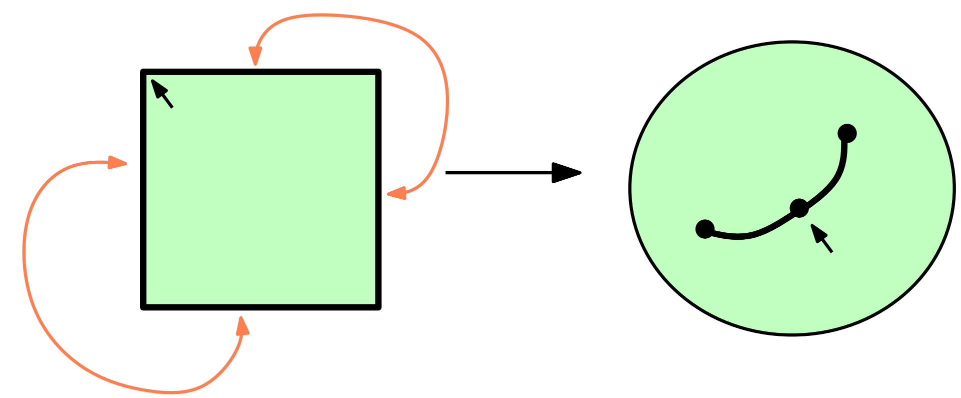

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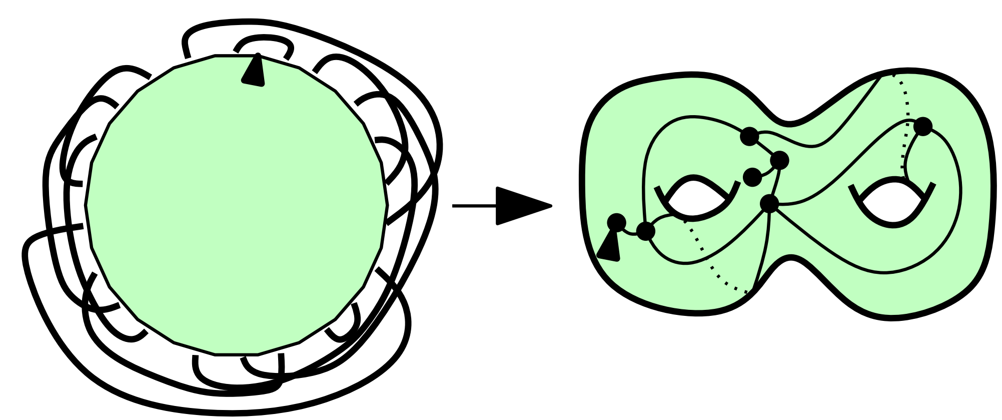

Other viewpoint: factorisation $\alpha\sigma = (1, 2, \ldots, 2n)$ where α is some matching.

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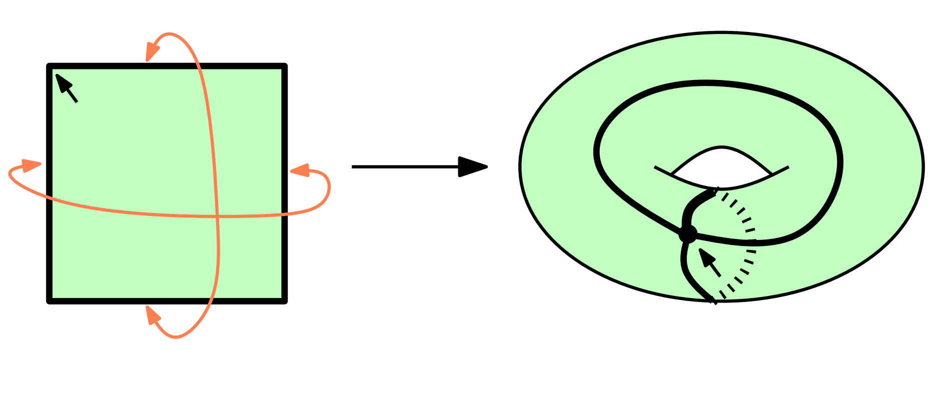

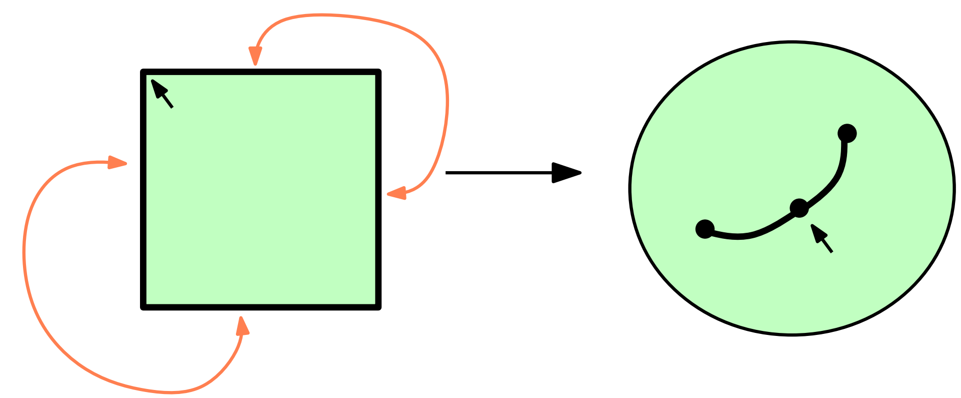

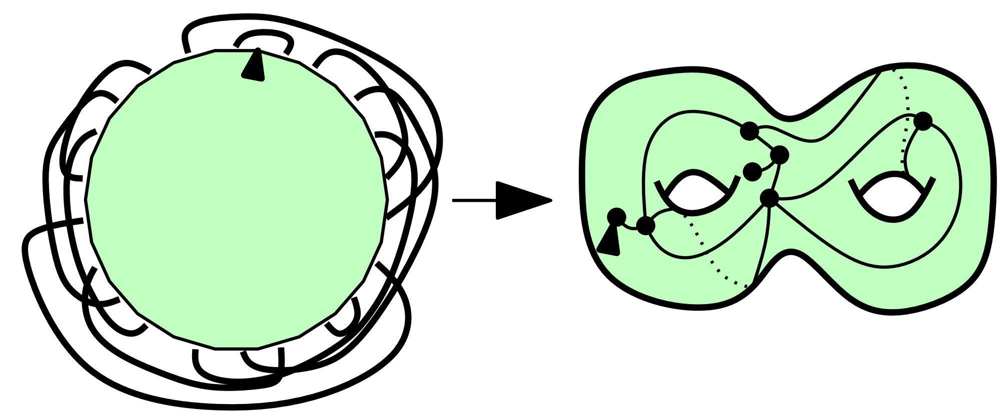

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Much harder: control the genus! (see the exercises)

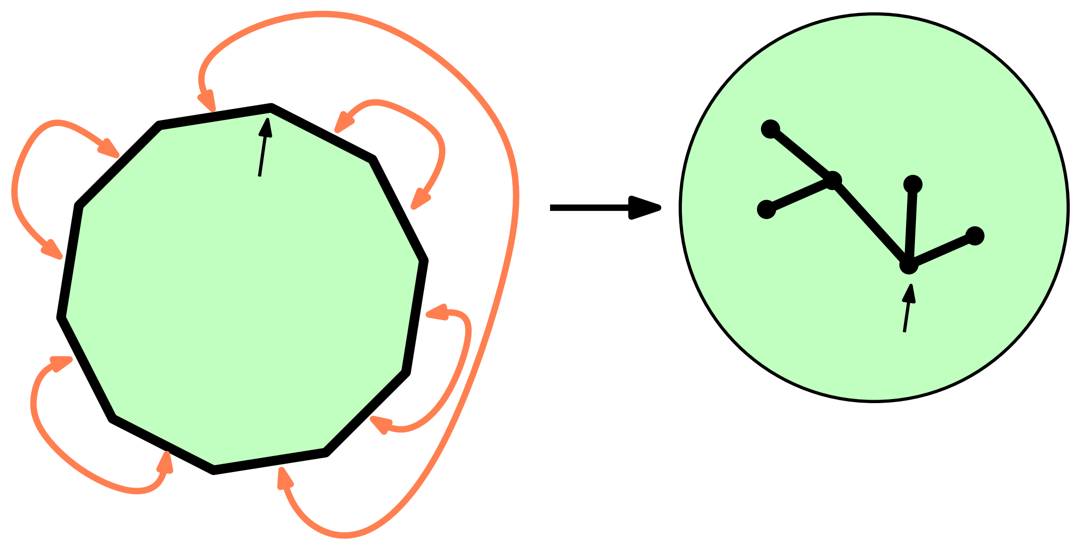
Example III: planar one-face maps (=plane trees, Catalan trees)

Euler formula: $v + f = n + 2 - 2g$

$f = 1$, $g = 0$ gives $v = n + 1$

this is a tree!

Tree+root corner+rotation system = plane tree (a.k.a. ordered tree)



Proposition: The number of rooted plane trees with n edges is $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$.

Tomorrow: we start counting!

everything will be planar (no strange surface yet so don't be afraid)

if you don't know what to do tonight, try exercise 0 from the webpage.