An introduction to map enumeration

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What this course is about (I)

A map is a graph embedded in a surface:

Maps appear (almost?) everywhere in mathematics. Map enumeration alone is an enormous area.

At least three reasons to be interested in it:

- because you would like to start working seriously on the subject.
- because map enumeration contains powerful tools that can be useful to other parts of combinatorics (functional equations, bijective tricks, algebraic tools...).
- because it is likely that your favorite subject is linked to map enumeration in at least some special case: looking at where this problem appears in the world of maps is a very good source of new questions.
What this course is about (II)

Topics not covered: - link with algebraic geometry
                     matrix integrals
                     string theory
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SEE E.G. SURVEY BY MIERMONT
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Topics covered:  I – Maps
   II – Tutte equation, counting planar maps
   III – Tutte equation, counting maps on general surfaces
   IV – Bijective counting of maps

The exercises contain entry points to other subjects
   (one-face maps, link with the symmetric group)
Lecture I – What is a map? (the oral tradition)
Rough definition

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A surface is a connected, compact, oriented, 2-manifold considered up to oriented homeomorphism.

Example: \( S_g := \) the \( g \)-torus = the sphere with \( g \) handles attached

\[
\begin{align*}
S_0 & \quad S_1 & \quad S_2 & \quad S_3 \ldots
\end{align*}
\]

Theorem of classification: every surface is one of the \( S_g \) for some \( g \geq 0 \) called the genus.
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Our graphs are unlabelled, connected, and may have loops or multiple edges.
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**Examples:**

![Graph Example](image-url)
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Examples:

The degree of a vertex is the number of half-edges incident to it.
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A proper embedding of a graph in a surface is a continuous drawing of the graph on the surface without edge-crossings.
Topological definition of a map

A map is a proper embedding of a graph $G$ in a surface $S$ such that the connected components of $G \setminus S$ (called faces) are topological disks.

(a) not a map

(b) valid map of genus 1
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Maps are considered up to oriented homeomorphisms.
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**Important notion:** a corner is an angular sector delimited by two consecutive half-edges in the neighborhood of a vertex.

There is a canonical bijection between corners and half-edges.
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The **degree** of a vertex (or face) is the number of corners incident to it.

If a map has $n$ edges then:

$$2n = \# \text{ corners} = \# \text{ half-edges} = \sum \text{ face degrees} = \sum \text{ vertex degrees}.$$
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A rooted map is a map with a distinguished corner (or half-edge)
Example 1: planar maps

Convention The infinite face is taken to be the root face.
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Maps of genus 0 are called planar rather than spherical...

Convention The infinite face is taken to be the root face.

Advertisement for the next lecture:

There are \( \frac{2 \cdot 3^n}{n + 2} \text{Cat}(n) \) rooted planar maps with \( n \) edges

(a nice number)
Maps as polygon gluings

An easy way to construct a map:

start with a family of polygons with $2n$ sides in total and glue them according to your favorite matching – just be careful to obtain something connected.
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\[
\begin{array}{ccc}
8 & & 10 \\
3 & & 4 \\
2 & 7 & 1 \\
9 & 5 & 6 \\
\end{array}
\]

Claim: provided it is connected, the object we construct is a map.

Proof: We clearly build a surface with a graph on it, and by construction the faces are our polygons – hence topological disks.

Proposition: any map can be obtained in this way.

Heuristic proof: to go from right to left, just cut the surface along the edges of the graph.
Maps as rotation systems

A rotation system on a graph is the data of a cyclic order of the half-edges around each vertex.
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**Fact:** There is a natural mapping:

Maps \(\longrightarrow\) Graphs equipped with a rotation system

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Reminder: At this stage I should draw more examples on the board
Combinatorial definition with permutations

A labelled map of size $n$ is a triple of permutations $(\sigma, \alpha, \phi)$ in $\mathfrak{S}_{2n}$ such that
- $\alpha \sigma = \phi$
- $\alpha$ has cycle type $(2, 2, \ldots, 2)$.
- $\langle \sigma, \alpha, \phi \rangle$ acts transitively on $[1..2n]$. 
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\( \text{???)} \)
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**Thm:** There is a bijection between labelled maps of size \( n \) and graphs with rotation systems whose half-edges are labelled from 1 to \( 2n \).

\[
\begin{align*}
\sigma &= (1, 8, 10, 5, 2)(3, 9, 6, 7, 4) \\
\alpha &= (1, 10)(2, 6)(3, 7)(4, 8)(5, 9) \\
\phi &= \alpha \sigma = (1, 4, 7, 8)(2, 10, 9)(3, 5, 6)
\end{align*}
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**Note:**
Combinatorial definition with permutations

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- vertices = cycles of $\sigma$
- edges = cycles of $\alpha$
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A rooted map is an equivalence class of labelled maps under renumbering of $[2..2n]$.

labelled map “=” $(2n - 1)! \times$ rooted map
Duality

A labelled map of size $n$ is a triple of permutations $(\sigma, \alpha, \phi)$ in $\mathfrak{S}_{2n}$ such that
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The mapping $(\sigma, \alpha, \phi) \rightarrow (\phi, \alpha, \sigma)$ is an involution on maps called duality. It exchanges vertices and faces.
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There is also a well-known graphical version:
**Duality**

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Duality II – dual submap

If \( m \) is a map with underlying graph \( G \) then any subgraph \( H \subset G \) induces a submap of \( G \), with same vertex set, by restricting the cyclic ordering to \( H \).

Note that the submap is not necessarily connected (and can have a different genus).

The dual submap is the submap of \( m^* \) the formed by edges whose dual is not in \( H \).

**Proposition:** The total number of faces of a submap and its dual submap are equal.
Euler’s formula

For a map of genus $g$ with $n$ edges, $f$ faces, $v$ vertices, we have:

$$v + f = n + 2 - 2g$$

In particular we can recover the genus from the combinatorics (we don’t need to “see” the surface...)
Example II: one-face maps

What is a one-face map? Clear in the “polygon gluing viewpoint”.

Start with a $2n$-gon and glue the edges together according to some matching.
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**Proposition:** The number of one-face maps with $n$ edges is $(2n - 1)!! = \frac{(2n)!}{2^n n!}$. 
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Much harder: control the genus! (see the exercises)
Example III: planar one-face maps (=plane trees, Catalan trees)

Euler formula: \[ v + f = n + 2 - 2g \]

\[ f = 1, \; g = 0 \] gives \[ v = n + 1 \]

this is a tree!

Tree + root corner + rotation system = plane tree (a.k.a. ordered tree)

**Proposition:** The number of rooted plane trees with \( n \) edges is \( \text{Cat}(n) = \frac{1}{n+1} \binom{2n}{n} \).
Tomorrow: we start counting!

everything will be planar (no strange surface yet so don’t be afraid)

if you don't know what to do tonight, try exercise 0 from the webpage.