Jucys-Murphy elements and Monotone Hurwitz Numbers

Mathieu Guay-Paquet

based on joint work with

Ian Goulden and Jonathan Novak

LaCIM, Université du Québec à Montréal

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Application: unitary matrix integrals

Expectations of polynomials

Question

Given a uniformly random $d \times d$ unitary matrix and a fixed polynomial in its d^2 entries (and complex conjugates), what is the expected value?

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}$$

$$\int_{\mathbf{U}(3)} u_{11}^2 u_{23} \overline{u_{11}} (\overline{u_{21}} - \overline{u_{13}})^2 \, \mathrm{d}U = (?)$$

Reductions

The problem can be reduced to:

- monomials $u_{i_1 j_1} \cdots u_{i_n j_n} \overline{u_{k_1 \ell_1}} \cdots \overline{u_{k_m \ell_m}}$, by linearity;
- balanced (with $m = n$), by translation invariance.

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}$$

A Wick's formula

(Collins 2003) and (Collins-Śniady 2006) showed that

Theorem

There is a function $\mathrm{Wg}_d \colon S_n \to \mathbb{C}$ such that

$$\int_{\mathbf{U}(d)} u_{i_1 j_1} \cdots u_{i_n j_n} \overline{u_{k_1 \ell_1}} \cdots \overline{u_{k_m \ell_m}} \, \mathrm{d}U = \sum_{\pi, \sigma \in S_n} \delta_{[\mathbf{i} = \pi(\mathbf{k})]} \delta_{[\mathbf{j} = \sigma(\boldsymbol{\ell})]} \mathrm{Wg}_d(\pi \sigma^{-1})$$

Also,

- only permutation correlators are needed;
- for fixed d, $\mathrm{Wg}_d(\pi)$ only depends on the cycle type of π;
- for fixed π, $\mathrm{Wg}_d(\pi)$ is a rational function of d.

Weingarten in the group algebra

(Collins 2003) and (Collins-Śniady 2006) showed that

Theorem

In the group algebra of S_n over $\mathbb{C}(d)$,

$$\left(\sum_{\pi \in S_n} \mathrm{Wg}_d(\pi) \cdot \pi \right)^{-1} = \left(\sum_{\pi \in S_n} d^{\#\,\mathrm{cycles}(\pi)} \cdot \pi \right) \in \mathbb{C}(d)[S_n].$$

In particular, $\mathrm{Wg}_d(\pi)$ can be obtained by a computation in the center of the symmetric group algebra.

Tool: Jucys-Murphy elements

Jucys-Murphy elements

Introduced by (Jucys 1974) and (Murphy 1981) independently.

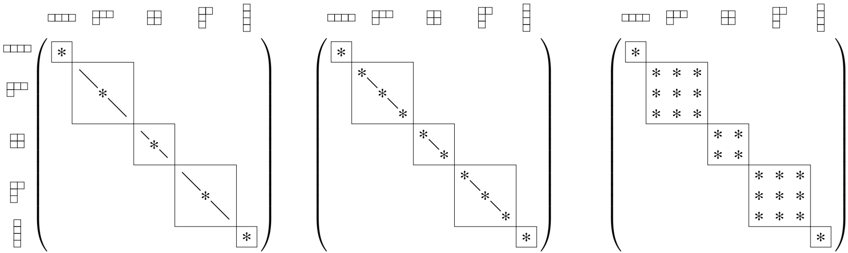
$$\begin{aligned} J_1 &= 0 \\ J_2 &= (1\,2) \\ J_3 &= (1\,3) + (2\,3) \\ J_4 &= (1\,4) + (2\,4) + (3\,4) \\ &\vdots \\ J_n &= \sum_{k=1}^{n-1} (k\,n) \in \mathbb{C}[S_n] \end{aligned}$$

They commute, but are not in the center.

Action on a Gelfand model

If $V^\lambda, \lambda \vdash n$ are irreducibles, then $\mathbb{C}[S_n] \cong \bigoplus_{\lambda \vdash n} \mathrm{End}_{\mathbb{C}}(V^\lambda)$.

$\mathcal{Z}_4$ $\quad\quad$ $\mathrm{GZ}_4 = \mathbb{C}[J_1, \ldots, J_4]$ $\quad\quad$ $\mathbb{C}[S_4]$



$$\dim \mathcal{Z}_4 = \sum_{\lambda \vdash 4} 1 \qquad \dim \mathrm{GZ}_4 = \sum_{\lambda \vdash 4} f^\lambda \qquad \dim \mathbb{C}[S_4] = \sum_{\lambda \vdash 4} \left(f^\lambda\right)^2$$

Signature of a factorization

Monomials in $J_1, \ldots, J_n$ count products of transpositions:

$$J_2 J_5 J_4 J_5 = \sum_{\substack{a<2,\, b<5 \\ c<4,\, d<5}} (a\,2)(b\,5)(c\,4)(d\,5)$$

The coefficient of π in $J_2 J_5 J_4 J_5$ is the number of factorizations of π with *signature* $(2, 5, 4, 5)$. If the signature is weakly increasing, the factorization is called *monotone*.

Symmetric functions in Jucys-Murphy elements

Lemma

The elementary symmetric polynomials in $J_1, \ldots, J_n$ are

$$\begin{aligned}(1 + J_1 t)(1 + J_2 t) \cdots (1 + J_n t) \\ &= 1 + e_1(J)t + e_2(J)t^2 + \cdots + e_n(J)t^n \\ &= \sum_{\pi \in S_n} t^{n - \#\,\text{cycles}(\pi)} \cdot \pi\end{aligned}$$

Proof

Every permutation $\pi \in S_n$ has a unique strictly monotone factorization.

Corollary

All symmetric polynomials in $J_1, \ldots, J_n$ are in the center of the symmetric group algebra.

Application to Weingarten function

(Matsumoto-Novak 2009) used Jucys-Murphy elements to get asymptotics for the Weingarten function:

$$\left(\sum_{\pi \in S_n} \mathrm{Wg}_d(\pi) \cdot \pi\right) = \left(\sum_{\pi \in S_n} d^{\#\,\mathrm{cycles}(\pi)} \cdot \pi\right)^{-1}$$

$$= \left((d + J_1)(d + J_2) \cdots (d + J_n)\right)^{-1}$$

$$= d^{-n}\left(1 - \frac{h_1(J)}{d} + \frac{h_2(J)}{d^2} - \cdots\right)$$

Complete symmetric polynomials:

$$h_k(J) = \sum_{b_1 \leq b_2 \leq \cdots \leq b_k} J_{b_1} J_{b_2} \cdots J_{b_k}.$$

Direction one: (monotone) Hurwitz theory

Goal

We can compute the complete symmetric polynomials $h_k(J)$ (so, the unitary matrix integrals) if we can count monotone factorizations.

Evolution of a factorization

Components	Permutation	Factor	Type of factor
$\{1\}, \{2\}, \{3\}$	$(1)(2)(3)$		
		$\cdot(1\,2)$	Essential join
$\{1, 2\}, \{3\}$	$(1\,2)(3)$		
		$\cdot(1\,3)$	Essential join
$\{1, 2, 3\}$	$(1\,3\,2)$		
		$\cdot(2\,3)$	Cut
$\{1, 2, 3\}$	$(1\,3)(2)$		
		$\cdot(2\,3)$	Redundant join
$\{1, 2, 3\}$	$(1\,3\,2)$		

Hurwitz numbers

Hurwitz numbers

The Hurwitz number $H^r(\lambda)$ counts the number of connected factorizations of length r of a fixed permutation of cycle type $\alpha \vdash n$.

Hurwitz generating function

$$\mathbf{H} = \sum_{r \geq 0} \sum_{n \geq 1} \sum_{\alpha \vdash n} H^r(\alpha) \frac{t^r}{r!} \frac{p_\alpha}{n!}$$

where $t, p_1, p_2, \ldots$ are indeterminates, r is the length, n is the ground set size, α is the cycle type.

Monotone Hurwitz numbers

Monotone Hurwitz numbers

The monotone Hurwitz number $\vec{H}^r(\lambda)$ counts the number of connected monotone factorizations of length r of a fixed permutation of cycle type $\alpha \vdash n$.

Monotone Hurwitz generating function

$$\vec{\mathbf{H}} = \sum_{r \geq 0} \sum_{n \geq 1} \sum_{\alpha \vdash n} \vec{H}^r(\alpha) t^r \frac{p_\alpha}{n!}$$

where $t, p_1, p_2, \ldots$ are indeterminates, r is the length, n is the ground set size, α is the cycle type.

Join-cut equations

Classical Hurwitz (Goulden-Jackson 1999)

$$\frac{\partial \mathbf{H}}{\partial t} = \frac{1}{2}\sum_{i,j\geq 1}\left((i+j)p_i p_j \frac{\partial \mathbf{H}}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial^2 \mathbf{H}}{\partial p_i \partial p_j} + ijp_{i+j}\frac{\partial \mathbf{H}}{\partial p_i}\frac{\partial \mathbf{H}}{\partial p_j}\right)$$

Monotone Hurwitz (Goulden-GP-Novak 2011)

$$\frac{\left(\sum_k \frac{kp_k \partial \vec{\mathbf{H}}}{\partial p_k}\right) - p_1}{t} = \sum_{i,j\geq 1}\left((i+j)p_i p_j \frac{\partial \vec{\mathbf{H}}}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial^2 \vec{\mathbf{H}}}{\partial p_i \partial p_j} + ijp_{i+j}\frac{\partial \vec{\mathbf{H}}}{\partial p_i}\frac{\partial \vec{\mathbf{H}}}{\partial p_j}\right)$$

Hurwitz polynomiality

Theorem (Goulden-Jackson-Vakil 2001)

For each $(g, \ell) \notin \{(0,1),(0,2)\}$, there is a polynomial $P_{g,\ell}$ such that, for all partitions $\alpha \vdash n$ with ℓ parts,

$$H_g(\alpha) = \frac{n!}{|\mathrm{aut}(\alpha)|}(n+\ell+2g-2)!\,P_{g,\ell}(\alpha_1,\ldots,\alpha_\ell)\prod_{i=1}^{\ell}\frac{\alpha_i^{\alpha_i}}{\alpha_i!}$$

Theorem (Ekedahl-Lando-Shapiro-Vainshtein 2001)

The polynomials $P_{g,\ell}$ are given by

$$P_{g,\ell}(\alpha_1,\ldots,\alpha_\ell) = \int_{\overline{\mathcal{M}}_{g,\ell}} \frac{1-\lambda_1+\cdots+(-1)^g\lambda_g}{(1-\alpha_1\psi_1)\cdots(1-\alpha_\ell\psi_\ell)}$$

Monotone Hurwitz polynomiality

Theorem (Goulden-GP-Novak 2011)

For each $(g, \ell) \notin \{(0,1),(0,2)\}$, there is a polynomial $\vec{P}_{g,\ell}$ such that, for all partitions $\alpha \vdash n$ with ℓ parts,

$$\vec{H}_g(\alpha) = \frac{n!}{|\mathrm{aut}(\alpha)|} \vec{P}_{g,\ell}(\alpha_1, \ldots, \alpha_\ell) \prod_{i=1}^{\ell} \binom{2\alpha_i}{\alpha_i}$$

Key steps

1. Guess and check the genus zero solution $\vec{\mathbf{H}}_0$.
2. Recursively solve the higher genus equation for $\vec{\mathbf{H}}_g$.
3. Check that the form of $\vec{\mathbf{H}}_g$ is preserved when solving.

Open questions

- Is there an ELSV-type geometric formula for monotone Hurwitz numbers?
- Is there a purely algebraic proof for classical Hurwitz polynomiality?
- Is there a general transfer theorem to explain the parallels between classical and monotone Hurwitz theory?

Direction two: symmetric functions

Surjectivity and injectivity

Jucys and Murphy each showed that the map

$$\begin{aligned} \mathrm{ev}_n \colon \Lambda_n &\to \mathcal{Z}_n \\ f(x_1, \ldots, x_n) &\mapsto f(J_1, \ldots, J_n) \end{aligned}$$

is surjective.

(Diaconis-Greene 1989) and (Corteel-Goupil-Schaeffer 2004) each showed that taken together, the maps are injective.

$$\begin{aligned} \mathrm{ev} \colon \Lambda &\to \prod_n \mathcal{Z}_n \\ f(x_1, x_2, \ldots) &\mapsto f(J_1, J_2, \ldots) \end{aligned}$$

Uniform formulas

In $\mathcal{Z}_6$, we have:

$$\begin{aligned} C_{3111} \cdot C_{51} = {} & 16C_{2211} + 18C_{3111} \\ & + 18C_{33} + 16C_{42} + 15C_{51} \end{aligned}$$

In $\prod_n \mathcal{Z}_n$, we have:

$$\begin{aligned} K_2 \cdot K_4 = {} & (8n - 32)K_{11} + (3n^2 - 21n + 36)K_2 \\ & + 18K_{22} + 16K_{31} + (5n - 15)K_4 \\ & + K_{42} + 7K_6 \end{aligned}$$

where, *e.g.*, $K_{31} = C_{42} + C_{421} + C_{4211} + \cdots$ is a *reduced conjugacy class*.

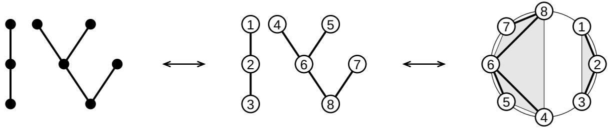
A bijection

Top terms of $m_\lambda(J)$ in terms of reduced classes K_μ computed by (Matsumoto-Novak 2009) using a bijection with parking functions.

Another bijection is with trees instead of parking functions:

Given an ordered forest, repeatedly give the rightmost root the biggest remaining label left from $\{1, 2, \ldots, n\}$. Then the edges give a minimal factorization into transpositions.

$$\underbrace{(68)(78)}_{J_8^2}\underbrace{(46)(56)}_{J_6^2}\underbrace{(23)}_{J_3}\underbrace{(12)}_{J_2} = (123)(45678)$$



The reduced class basis

Corollary

A new proof that the reduced conjugacy classes give a basis of the symmetric functions $\Lambda(J) \subset \prod_n \mathcal{Z}_n$:

$$m_\lambda(J) = K_\lambda + \sum_{\mu \succ \lambda} c_{\lambda,\mu} \cdot K_\mu$$

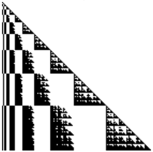
$$K_\lambda = m_\lambda(J) + \sum_{\mu \prec \lambda} c^*_{\lambda,\mu} \cdot m_\mu(J)$$

Other bases

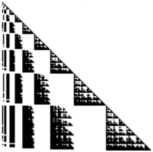
$[K_\lambda]e_\mu(J)$

$[K_\lambda]h_\mu(J)$

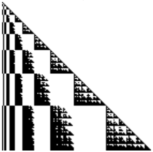
$[K_\lambda]s_\mu(J)$



$[K_\lambda]f_\mu(J)$



$[K_\lambda]m_\mu(J)$



$[K_\lambda]p_\mu(J)$

Open questions

- Are there nice bijections with trees for factorizations of all permutations, not just those with "non-crossing" cycles?
- Symmetric functions are graded. What are the homogeneous pieces of K_λ?
- Anything else about K_λ with respect to the other bases of Λ.

Direction three: sorting factorizations

Braid group action on factors

Braid group

$$B_k = \left\langle b_1,\, b_2,\, \ldots,\, b_{k-1} \,\middle|\, \begin{array}{l} b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \\ b_i b_j = b_j b_i,\ j \neq i \pm 1 \end{array} \right\rangle$$

Action on factors

B_k acts on products of k transpositions, *e.g.*

$$\begin{aligned} & (1\,3)(2\,4)(1\,2)(3\,7)(4\,7) = (1\,4)(2\,3\,7) \\ b_2 : \quad & (1\,3)(1\,2)(1\,4)(3\,7)(4\,7) = (1\,4)(2\,3\,7) \\ b_2^{-1} : \quad & (1\,3)(1\,4)(2\,4)(3\,7)(4\,7) = (1\,4)(2\,3\,7) \end{aligned}$$

Commutation of Jucys-Murphy elements

The idea of moving transpositions past other transpositions can be used to show the commutation of Jucys-Murphy elements, *e.g*, $J_3 J_4 J_2 J_7 J_7 = J_3 J_2 J_4 J_7 J_7$, if we are careful about the effect on the signature of a factorization:

$$
\begin{array}{rlcl}
 & (1\,3)(2\,4)(1\,2)(3\,7)(4\,7) & \longrightarrow & (3, 4, 2, 7, 7) \\
b_2: & (1\,3)(1\,2)(1\,4)(3\,7)(4\,7) & \longrightarrow & (3, 2, 4, 7, 7) \\
b_2^{-1}: & (1\,3)(1\,4)(2\,4)(3\,7)(4\,7) & \longrightarrow & (3, 4, 4, 7, 7)
\end{array}
$$

Proposition (Biane 2002?)

If we let s_i act by applying either b_i, b_i^{-1} or id (according to the signature), we get an action of the symmetric group:

$$
S_k = \left\langle s_1, s_2, \ldots, s_{k-1} \,\middle|\, \begin{array}{l} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ s_i s_j = s_j s_i, \; j \neq i \pm 1 \\ s_i^2 = \mathrm{id} \end{array} \right\rangle
$$

Orbits under the symmetric action

Proposition (Biane 2002?)

If we let s_i act by applying either b_i, b_i^{-1} or id (according to the signature), we get an action of the symmetric group:

$$S_k = \left\langle s_1, s_2, \ldots, s_{k-1} \,\middle|\, \begin{array}{l} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ s_i s_j = s_j s_i,\ j \neq i \pm 1 \\ s_i^2 = \text{id} \end{array} \right\rangle$$

Fact

Each orbit of this action contains exactly one monotone factorization.

Open question

Can this be used to give nicer proofs?

Thank you