THE COMBINATORICS OF NON-COMMUTATIVE DISCRETE INTEGRABLE SYSTEMS

(P. Di Francesco, R. Kedem)

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0. Positive Laurent Phenomenon & Discrete Integrability
1. Discrete Integrable systems: warmup example
2. The non-commutative A_1 Q-system
   integrability, continued fractions, path solution
3. Friezes and A_1 T-system
   integrability, GL_2 connection and solution
4. Non-commutative A_1 T-system
   definition, GL_2 connection and solution

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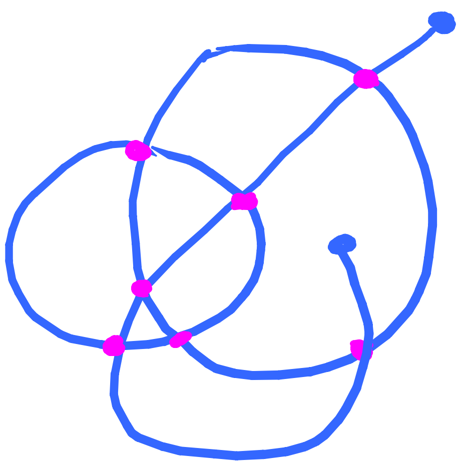
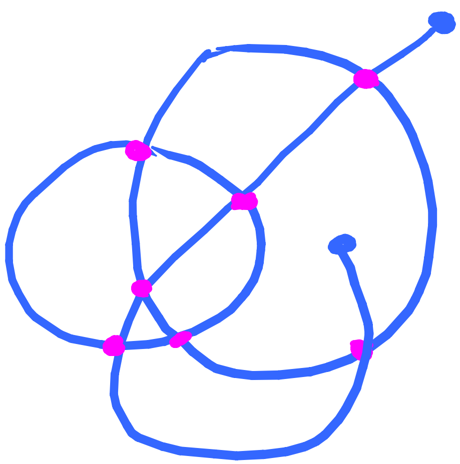
0. Discrete Integrability [BDG 04]



2-leg 4-valent planar maps
wt g/inner vertex
connected

- R_n = g.f. with dist(• •) $\leq n$
- $\boxed{R_n = 1 + g R_n (R_{n+1} + R_n + R_{n-1})}$ (*)
- Boundary conditions $\begin{cases} R_{-1} = 0 \\ R_\infty = R = \frac{1-\sqrt{1-12g}}{6g} \end{cases}$

0. Discrete Integrability [BDG 04]



2-leg 4-valent planar maps

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- R_n = g.f. with dist$(\circ\ \bullet) \leq n$
- $\boxed{R_n = 1 + g R_n (R_{n+1} + R_n + R_{n-1})}$ $(*)$
- Boundary conditions $\begin{cases} R_{-1} = 0 \\ R_\infty = R = \frac{1-\sqrt{1-12g}}{6g} \end{cases}$
- Discrete evolution n = time $\in \mathbb{Z}$.
- Integrable: $\boxed{\varphi(R_n, R_{n+1}) = \varphi(R_{n-1}, R_n)}$

  modulo the equation $(*)$

  $\varphi(x,y) = xy(1-g(x+y)) - x - y$

  = conservation of charge in tree picture
- Exact solution $R_n = R \frac{1-x^{n+1}}{1-x^{n+2}} \frac{1-x^{n+4}}{1-x^{n+3}}$

0 bis. Laurent Phenomenon

- Discrete evolution $R_n = f(R_{n-1}, R_{n-2}, \dots R_{n-k})$
- (Cauchy) initial data $(R_0, R_1, \dots, R_{k-1})$
- Laurent phenomenon if R_n = polynomial of $(R_0^{\pm 1}, \dots, R_{k-1}^{\pm 1})$

0 bis. Laurent Phenomenon

- Discrete evolution $R_n = f(R_{n-1}, R_{n-2}, \ldots R_{n-k})$
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- "LP machine" = Cluster Algebras [FZ00]
  - multi-dimensional evolution
  - rule = given by a quiver, that evolves too

$$\boxed{x_k\, x'_k = \prod_{i \to k} x_i + \prod_{k \to j} x_j}$$

Conjecture
LP is >0!

1. Discrete Integrable Systems: Warmup

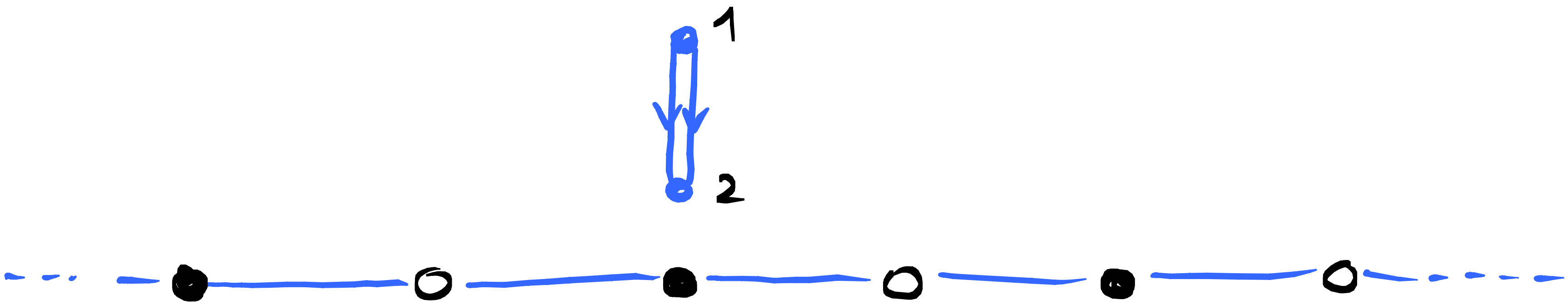
- evolution in d+1 dimension; discrete time $n \in \mathbb{Z}$
- conserved quantities ($D/2$ for D-space of solutions)
- exactly solvable (action/angle variables)

Basic Example: A_1 Q-system

$$\boxed{\begin{gathered} R_{n+1} R_{n-1} = R_n^2 + 1 \\ (R_0, R_1) \text{ initial data} \end{gathered}}$$

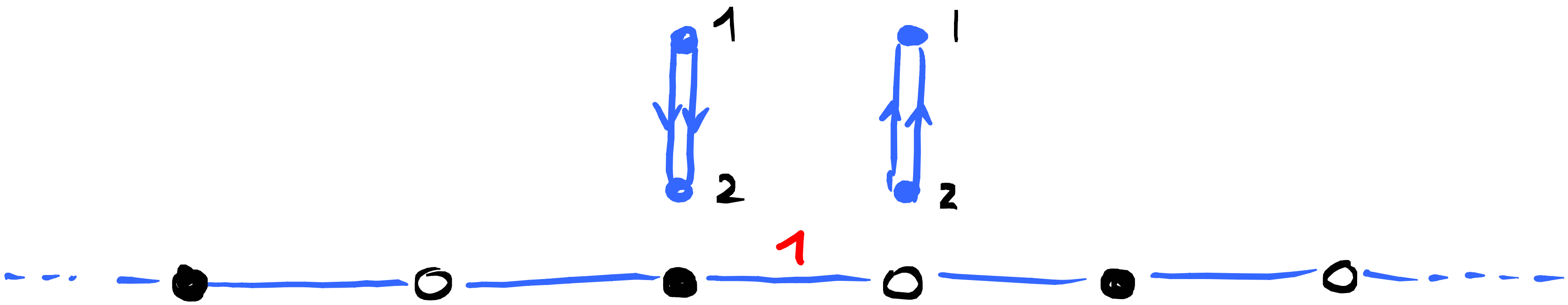
(
- reps thy sl_2
- fibonacci seq
- Chebyshev 2nd kind
)

Cluster Algebra formulation



$$\begin{pmatrix} R_0 \\ R_1 \end{pmatrix}$$

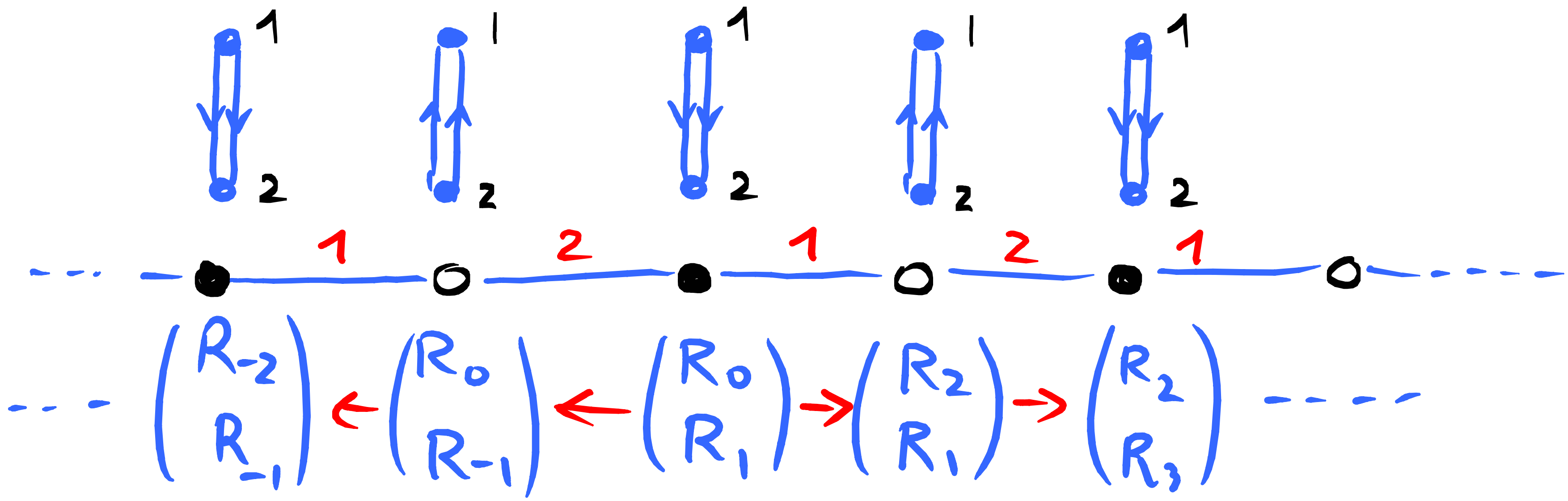
Cluster Algebra formulation



$$\begin{pmatrix} R_0 \\ R_1 \end{pmatrix} \rightarrow \begin{pmatrix} R_2 \\ R_1 \end{pmatrix}$$

$$R_0 R_2 = R_1^2 + 1$$

Cluster Algebra formulation



$$R_{n+1} R_{n-1} = R_n^2 + 1$$

$$\boxed{\text{THM} \quad R_n(R_0, R_1) = \text{Laurent Polynomial}}$$

LP is > 0

INTEGRABILITY

$$R_{n+1} R_{n-1} = R_n^2 + 1 \quad (*)$$

Lemma $K_n = \dfrac{R_{n+1} + R_{n-1}}{R_n} = K$ is independent of $n \in \mathbb{Z}$ for R_n solution of $(*)$

Proof: compute

$$R_n K_n R_{n+1} = R_{n+1}^2 + R_n^2 + 1$$

$$R_n K_{n+1} R_{n+1} = R_{n+1}^2 + 1 + R_n^2$$

qed

Cor R_n satisfies a linear recursion relation

$$R_{n+1} - K R_n + R_{n-1} = 0$$

$$K = \frac{R_1}{R_0} + \frac{1}{R_0 R_1} + \frac{R_0}{R_1}$$

NB: action-angle: $R_n = a_+ \lambda_+^n + a_- \lambda_-^n$

EXACT SOLUTION

Def: 1. $F(t) = \sum_{n=0}^{\infty} t^n R_n$

2. $y_1 = \frac{R_1}{R_0}$ $y_2 = \frac{1}{R_0 R_1}$ $y_3 = \frac{R_0}{R_1}$ $\left(\begin{array}{l} K = y_1 + y_2 + y_3 \\ 1 = y_1 y_3 \end{array} \right)$

We have:

$$F(t) = R_0 \frac{1 - t(y_2 + y_3)}{1 - t(y_1 + y_2 + y_3) + t^2}$$

EXACT SOLUTION

Def: 1. $F(t) = \sum_{n=0}^{\infty} t^n R_n$

2. $y_1 = \frac{R_1}{R_0} \quad y_2 = \frac{1}{R_0 R_1} \quad y_3 = \frac{R_0}{R_1}$ $\left(\begin{matrix} K = y_1 + y_2 + y_3 \\ 1 = y_1 y_3 \end{matrix} \right)$

We have:

$$F(t) = R_0 \frac{1 - t(y_2 + y_3)}{1 - t(y_1 + y_2 + y_3) + y_1 y_3 t^2} = \cfrac{R_0}{1 - t y_1 \cfrac{}{1 - t y_2 \cfrac{}{1 - t y_3}}}$$

EXACT SOLUTION

Def: 1. $F(t) = \sum_{n=0}^{\infty} t^n R_n$

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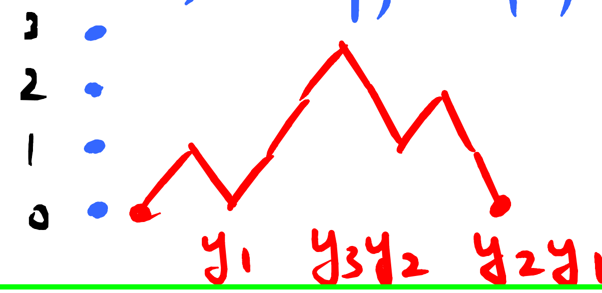
We have:

$$F(t) = R_0 \frac{1 - t(y_2 + y_3)}{1 - t(y_1 + y_2 + y_3) + y_1 y_3 t^2} = \cfrac{R_0}{1 - \cfrac{t y_1}{1 - \cfrac{t y_2}{1 - t y_3}}}$$

POSITIVE LAURENT PHENOMENON:

$\frac{R_n}{R_0}$ = partition function of paths $0 \to 0$, $2n$ steps, on $\{0,1,2,3\}$

with weights $\begin{cases} w_{i \to i+1} = 1 \\ w_{i \to i-1} = y_i \end{cases}$



2. Non-Commutative A_1 Q-system

$$R_{n+1} R_n^{-1} R_{n-1} = R_n + R_n^{-1} \quad (**) \qquad \text{(Kontsevich)}$$

(R_0, R_1) initial data

INTEGRABILITY:

Lemma The are 2 conserved quantities modulo $(**)$

(1) $C_n = R_{n+1}^{-1} R_n R_{n+1} R_n^{-1} = C$

(2) $K_n = (R_{n+1} + C R_{n-1}) R_n^{-1} = K$

$= R_n^{-1} (R_{n+1} C + R_{n-1})$

NB: $C = R_1^{-1} R_0 R_1 R_0^{-1} \qquad K = R_1 R_0^{-1} + R_1^{-1} R_0^{-1} + R_0 R_1^{-1}$

Proofs: $\qquad R_{n+1}R_n^{-1}R_{n-1} = R_n + R_n^{-1} \quad (**)$

$$(1)\quad C_n = R_{n+1}^{-1} R_n \underbrace{R_{n+1} R_n^{-1}} \qquad C_{n-1} = \underbrace{R_n^{-1} R_{n-1}} R_n R_{n-1}^{-1}$$

$$R_{n+1} C_n R_{n-1} = R_n\left(R_n + R_n^{-1}\right) = R_n^2 + 1$$

$$R_{n+1} C_{n-1} R_{n-1} = \left(R_n + R_n^{-1}\right) R_n = R_n^2 + 1$$

qed

$$C_n = C_{n-1} = C \Rightarrow (1)\ R_n R_{n+1} = R_{n+1} C R_n \Rightarrow R_{n+1} R_n^{-1} = R_n^{-1} R_{n+1} C$$

$$(2)\quad R_{n+1} C R_{n-1} = R_n^2 + 1$$

$$(2)\quad K_n = \left(R_{n+1} + C R_{n-1}\right) R_n^{-1} = L_n = R_n^{-1}\left(R_{n+1} C + R_{n-1}\right)$$

$$R_{n+1} K_n R_n = R_{n+1}^2 + R_n^2 + 1$$

$$R_{n+1} L_{n+1} R_n = \underbrace{R_{n+2} C R_n}_{R_{n+1}^2 + 1} + R_n^2$$

qed

(1) gives quasi-commutation relations

$$R_n R_{n+1} = R_{n+1} C R_n$$

(2) gives linear recursion relations:

$$R_{n+1} - K R_n + C R_{n-1} = 0$$

The initial eqn (**) can be recast into

$$R_{n+1} C R_{n-1} = R_n^2 + 1$$

Pb = no action-angle variables. Solution?
(no notion of a root of a NC polynomial).

EXACT SOLUTION

Def: 1. $F(t) = \sum_{0}^{\infty} t^n R_n$

2. $y_1 = R_1 R_0^{-1}$ $y_2 = R_1^{-1} R_0^{-1}$ $y_3 = R_1^{-1} R_0$

$$\boxed{\begin{aligned} C &= y_3 y_1 \\ K &= y_1 + y_2 + y_3 \end{aligned}}$$

$$F(t) R_0^{-1} = \left(1 - t(y_1 + y_2 + y_3) + y_3 y_1 t^2\right)^{-1} \left(1 - t(y_2 + y_3)\right)$$

NC continued fraction expansion!

EXACT SOLUTION

Def: 1. $F(t)=\sum_{0}^{\infty} t^n R_n$

2. $y_1=R_1R_0^{-1}$ $\quad y_2=R_1^{-1}R_0^{-1}$ $\quad y_3=R_1^{-1}R_0$

$$\left(\begin{array}{l} C=y_3y_1 \\ K=y_1+y_2+y_3 \end{array}\right)$$

$$F(t)=\left(1-t(y_1+y_2+y_3)+y_3y_1t^2\right)^{-1}\left(1-t(y_2+y_3)\right)$$

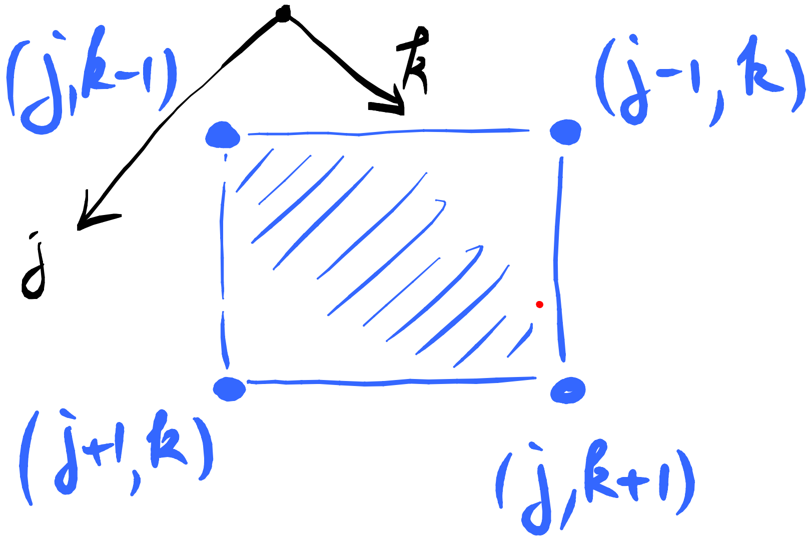
$$=\left(1-t\left(1-t\left(1-t\,y_3\right)^{-1}y_2\right)^{-1}y_1\right)^{-1}R_0$$

POSITIVE NC LAURENT PHENOMENON

$\Rightarrow R_nR_0^{-1}$ = partition function of paths $0\to 0$, $2n$ steps, on $\{0,1,2,3\}$ with NC weights $\begin{cases} w_{i\to i+1}=1 \\ w_{i\to i-1}=y_i \end{cases}$

3. Friezes and A_1 T-system

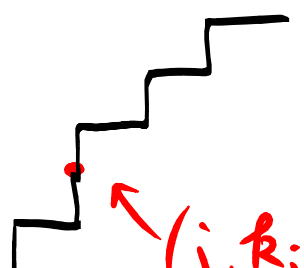
- quantum spin chains
- Coxeter-Conway



$$\boxed{\begin{vmatrix} T_{j,k-1} & T_{j-1,k} \\ T_{j+1,k} & T_{j,k+1} \end{vmatrix} = 1}$$

$$\left(T_{j,k-1} T_{j,k+1} = 1 + T_{j+1,k} T_{j-1,k} \quad (***) \right)$$

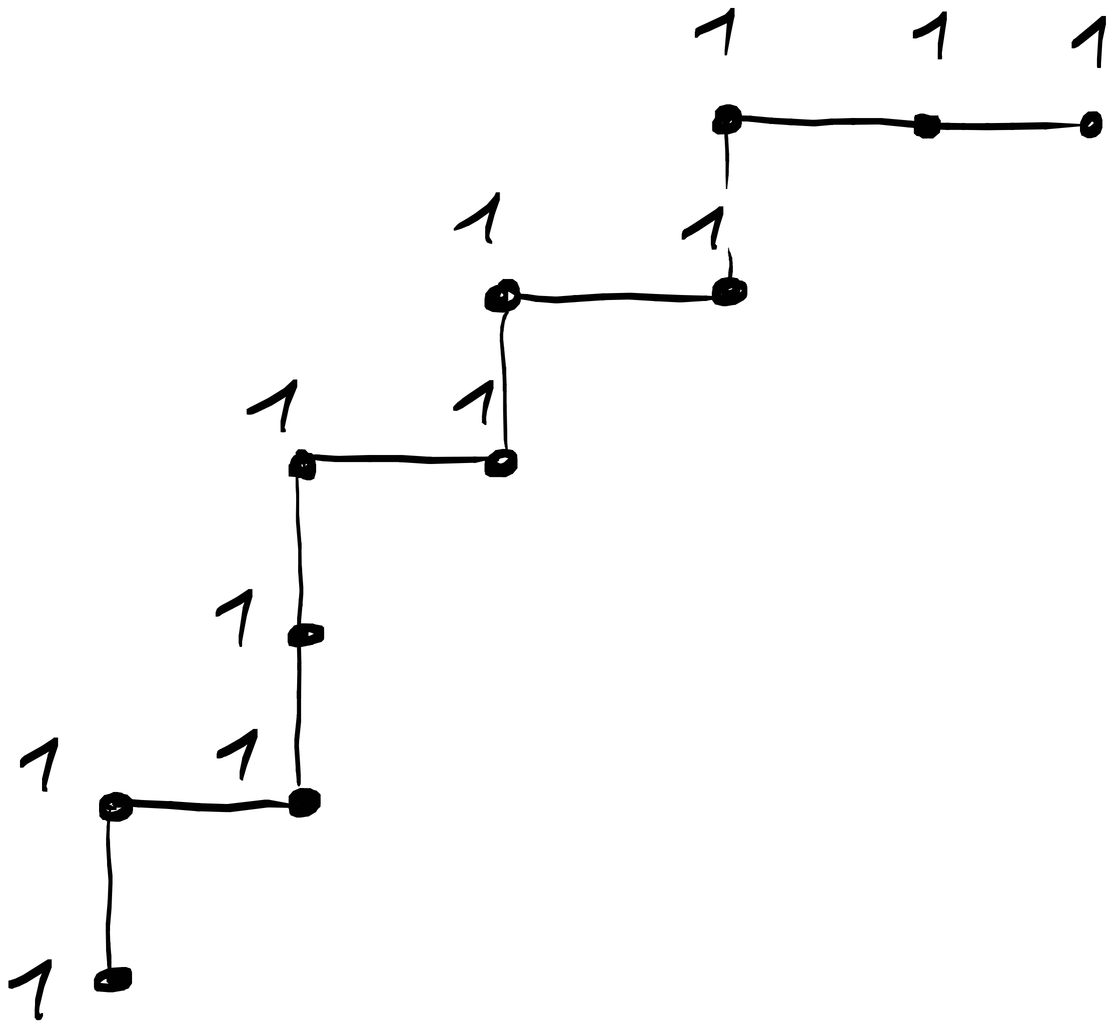
initial data



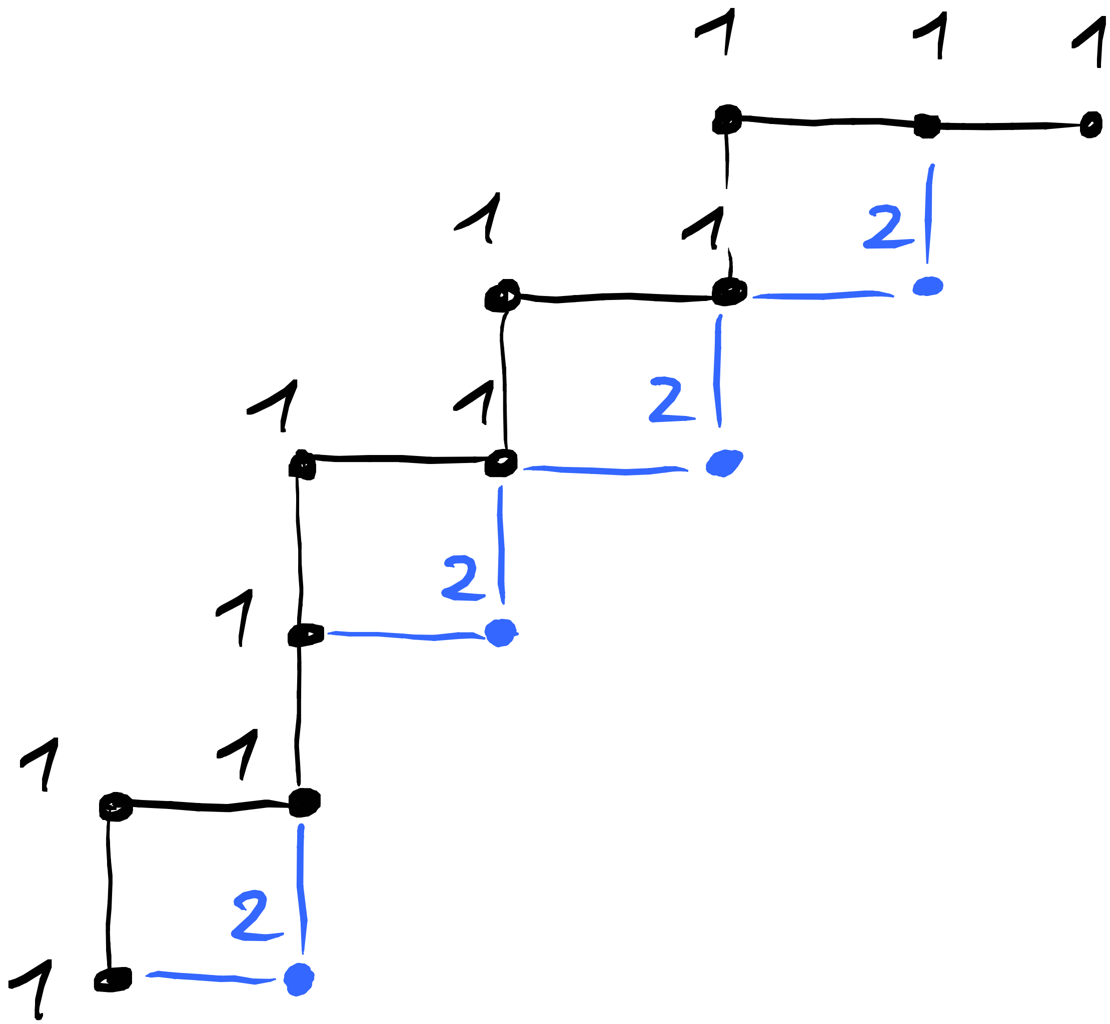
$(j,k_j)_{j\in\mathbb{Z}}$ path

$T_{j,k_j} = t_j \quad (j\in\mathbb{Z})$

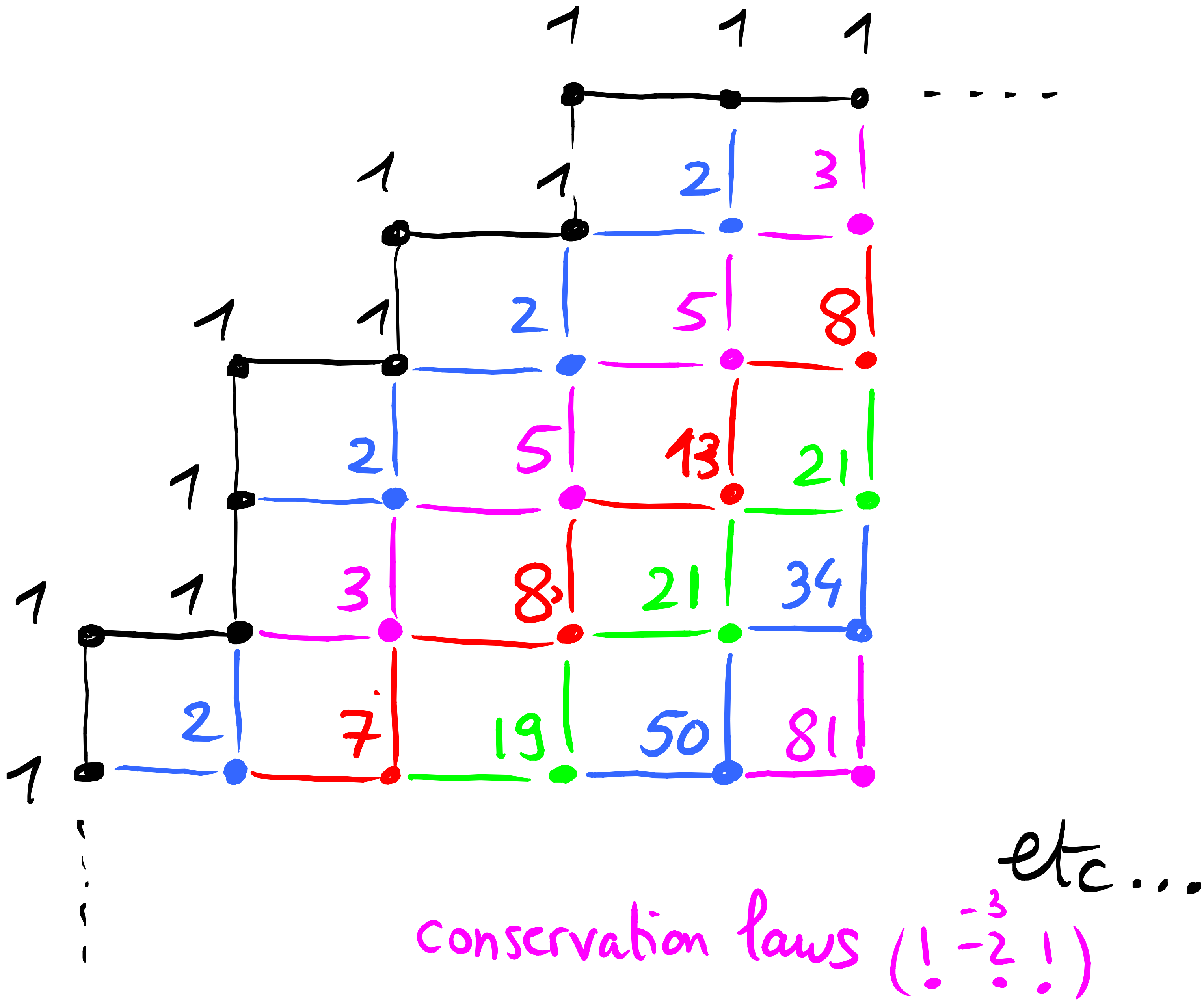
Ex



Ex



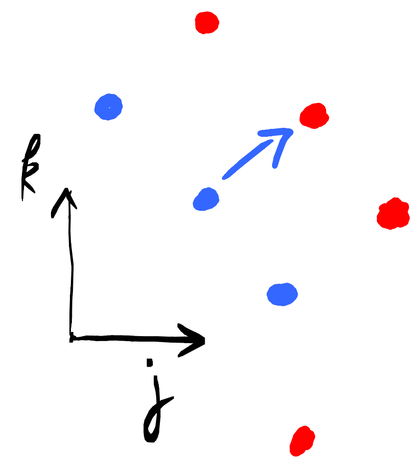
Ex


1 1 1
1 1 2 3
1 1 2 5 8
1 2 5 13 21
1 1 3 8 21 34
1 2 7 19 50 81
1
etc...
conservation laws (! -3 -2 !)

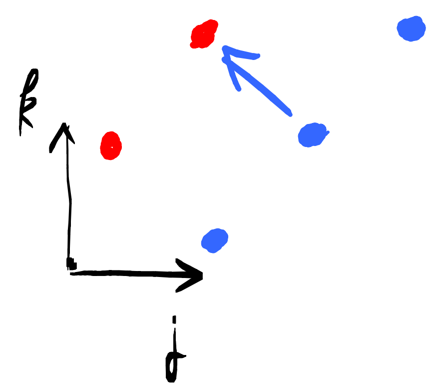
INTEGRABILITY:

Two families of conservation laws:

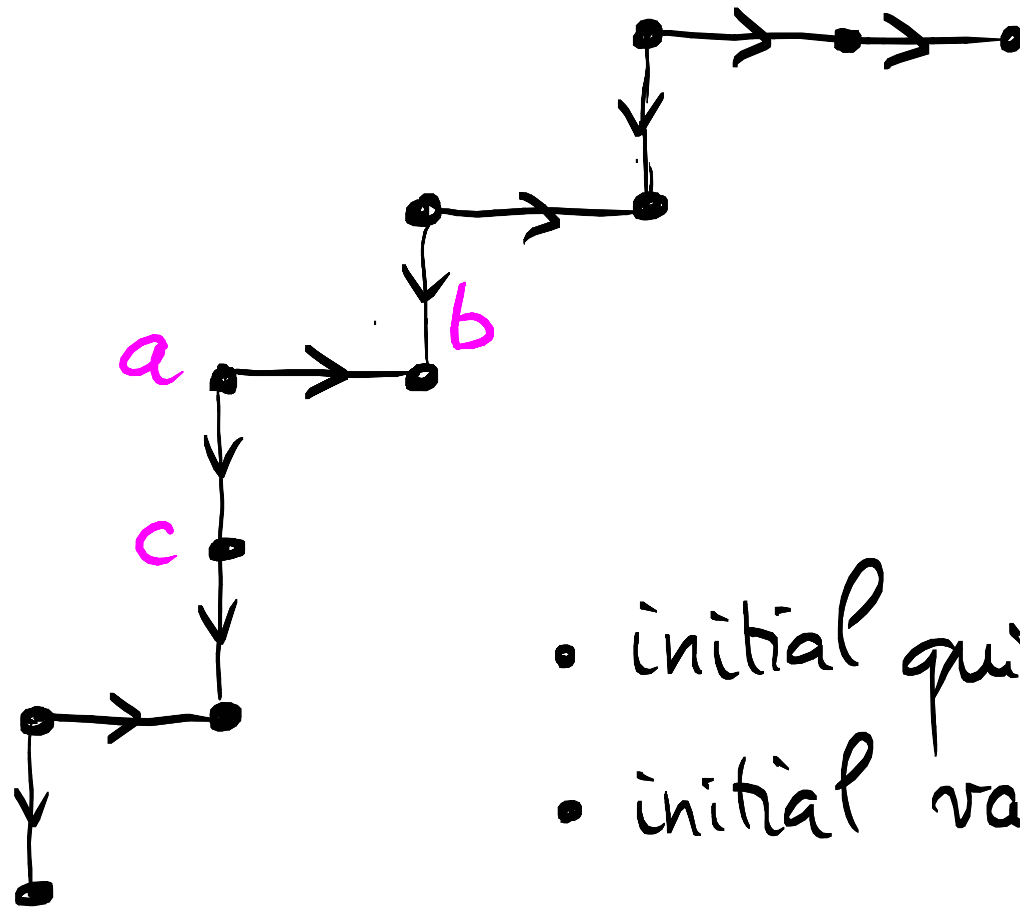
$$\boxed{T_{j-1,k+1} - \gamma_{j-k} T_{j,k} + T_{j+1,k-1} = 0}$$



$$\boxed{T_{j-1,k-1} - \delta_{j+k} T_{j,k} + T_{j+1,k+1} = 0}$$

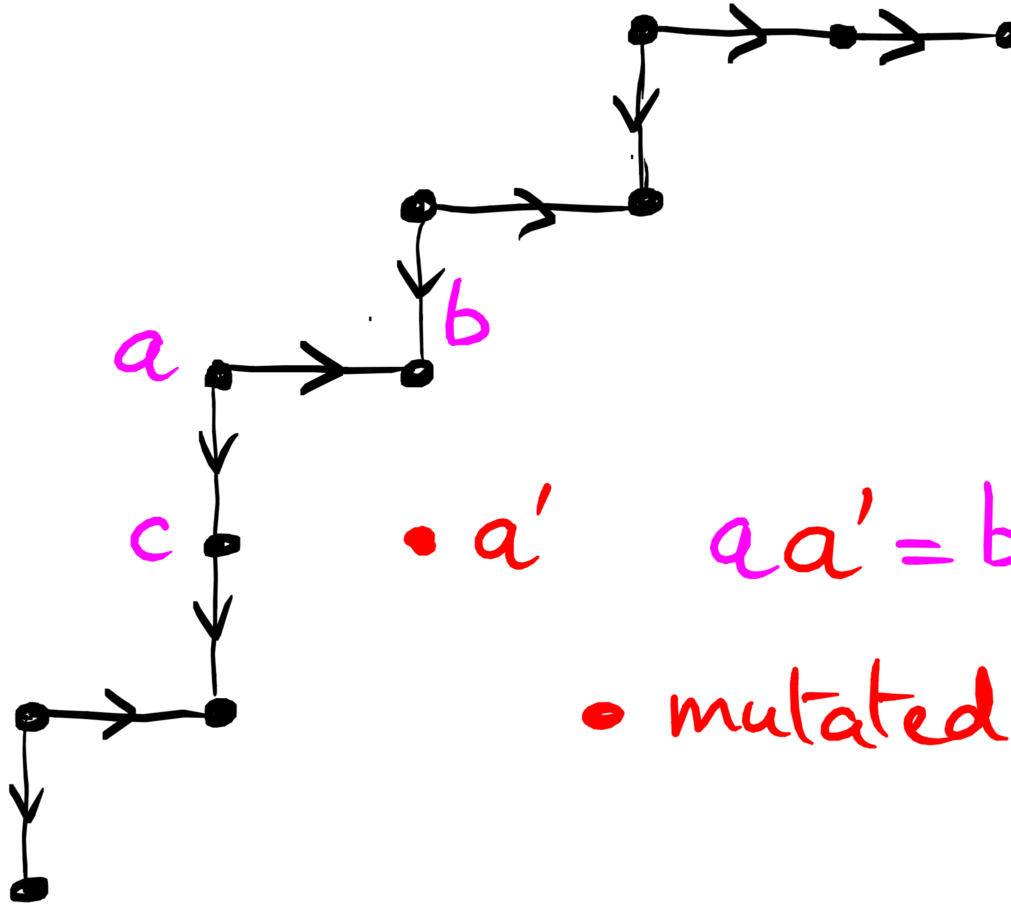


Cluster Algebra formulation



- initial quiver
- initial variables

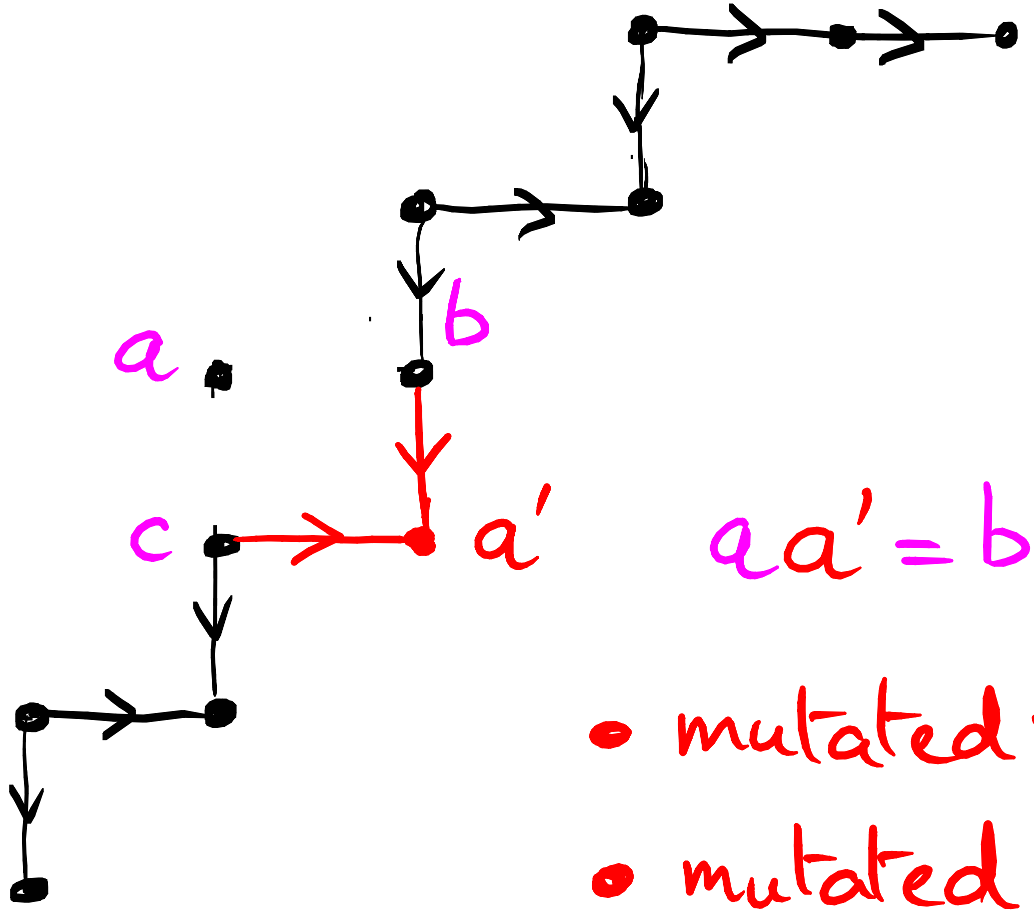
Cluster Algebra formulation



$$aa' = bc + 1$$

• mutated variables

Cluster Algebra formulation



$$aa' = bc + 1$$

- mutated variables
- mutated quiver

4. Non-Commutative Frieze/A_1-T-system

Def

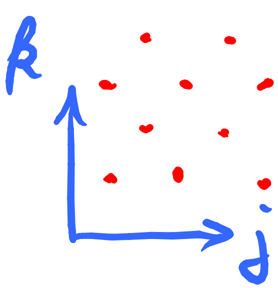
(1) Two sets of variables T_{jk}, $T^{\bullet}_{jk} \in \mathcal{A}$

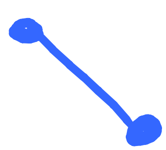
$\bullet$ = antiautomorphism of $\mathcal{A}$, involution.

(2) System:

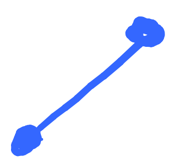
$$\boxed{T^{\bullet}_{j,k-1} \begin{vmatrix} T_{j,k-1} & T_{j-1,k} \\ T_{j+1,k} & \boxed{T_{j,k+1}} \end{vmatrix} = 1 \qquad (****)}$$

(3) Relations:





$$T_{j-1,k+1}\, T_{jk}^{-1} = \left(T^{\bullet}_{jk}\right)^{-1} \cdot T^{\bullet}_{j-1,k+1}$$



$$T_{j-1,k-1}^{-1}\, T_{jk} = T^{\bullet}_{jk} \left(T^{\bullet}_{j-1,k-1}\right)^{-1}$$

$$\begin{vmatrix} a & c \\ b & \boxed{d} \end{vmatrix} = \text{quasideterminant of [Gelfand-Retakh]}$$
$$= d - ca^{-1}b$$

$\Longleftrightarrow$ $\boxed{\text{NC } A_1 \text{ T-SYSTEM}}$

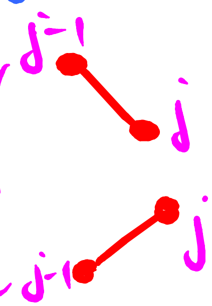
$$\boxed{T_{j,k+1} = (T^{\bullet}_{j,k-1})^{-1} + T_{j-1,k}\, T_{j,k-1}^{-1}\, T_{j+1,k}}$$

(****)

INITIAL DATA (doubled!)

(1) path $(j,k_j)_{j\in\mathbb{Z}}$; $|k_{j+1}-k_j| = 1$.

(2) $T_{j,k_j} = t_j$



$$\begin{cases} t_{j-1}\, t_j^{-1} = (t_j^{\bullet})^{-1}\, t_{j-1}^{\bullet} \\ t_{j-1}^{-1}\, t_j = t_j^{\bullet}\, (t_{j-1}^{\bullet})^{-1} \end{cases}$$

SUMMARY : elementary move

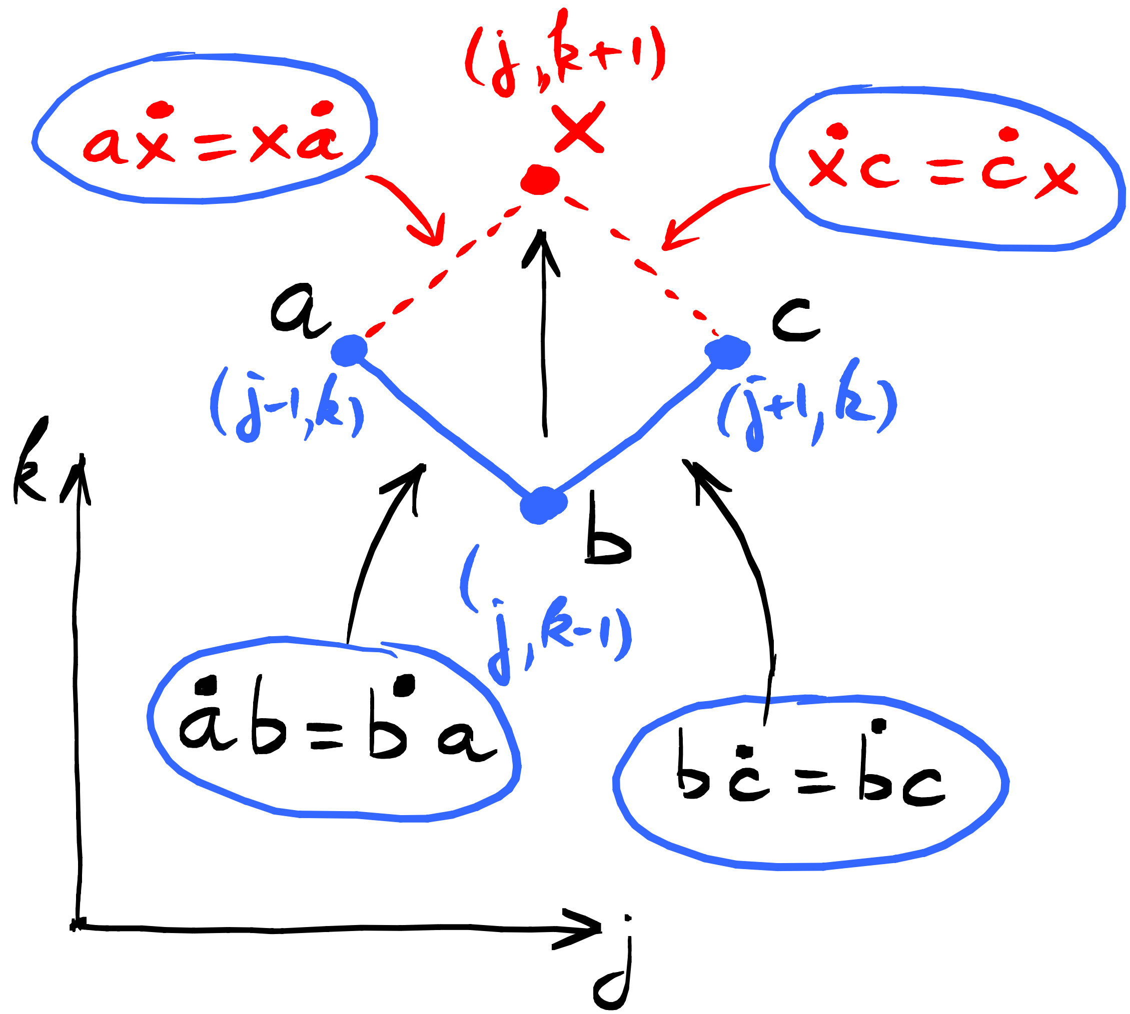
$$a b^{-1} = \dot{b}^{-1} \dot{a} \,;\quad b^{-1} c = \dot{c}\, \dot{b}^{-1}$$

$$x = (\dot{b})^{-1} + a b^{-1} c$$

$$\Updownarrow$$

$$\boxed{x \dot{b} = 1 + a \dot{c}}$$

$$a^{-1} x = \dot{x} \dot{a}^{-1} \,;\quad x c^{-1} = \dot{c}^{-1} \dot{x}$$



INTEGRABILITY

THM There are 2 families of "conserved quantities" modulo

(****): $\Gamma_{jk} = T_{j+1,k+1} T_{jk}^{-1} + (T_{jk}^{\bullet})^{-1} T^{\bullet}_{j+1,k-1} = \Gamma_{j-k}$

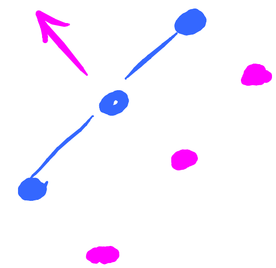
$$\Delta_{jk} = T_{jk}^{-1} T_{j+1,k+1} + T^{\bullet}_{j-1,k-1} (T^{\bullet}_{jk})^{-1} = \Delta_{j+k}$$

POSITIVE LAURENT PROPERTY

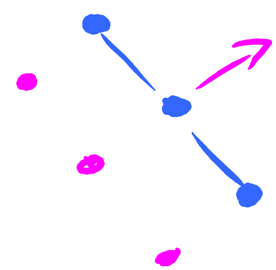
THM The solution T_{jk} of (****) with arbitrary initial data is a positive Laurent Polynomial of the $\{t_j, t^{\bullet}_j\}_{j\in\mathbb{Z}}$

NB The conserved quantities imply linear recursion relations:

$$\boxed{T_{j+1,k+1} - \square_{j-k}\, T_{j,k} + T_{j-1,k-1} = 0}$$

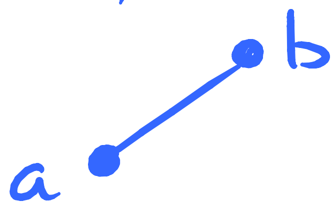


$$\boxed{T_{j+1,k-1} - T_{j,k}\, \Delta_{j+k} + T_{j-1,k+1} = 0}$$

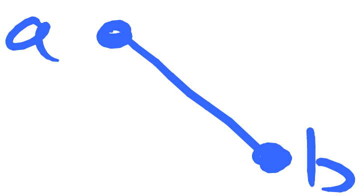


EXACT SOLUTION I: Connexion

Def $a,b \in \mathcal{A}$; $U,V \in GL_2(\mathcal{A})$:

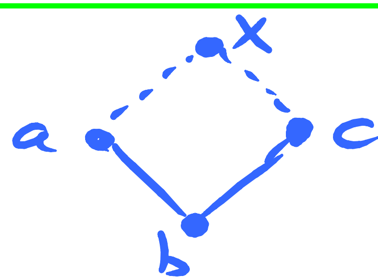


$$U(a,b) = \begin{pmatrix} 1 & 0 \\ b^{-1} & a^{\bullet}(b^{\bullet})^{-1} \end{pmatrix}$$



$$V(a,b) = \begin{pmatrix} ab^{-1} & (b^{\bullet})^{-1} \\ 0 & 1 \end{pmatrix}$$

Lemma

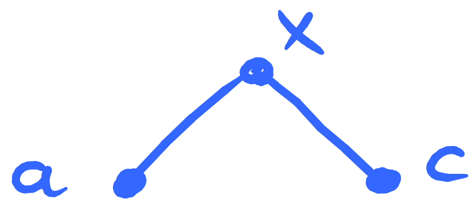


if $ab^{-1} = (b^{\bullet})^{-1}a^{\bullet}$ and $b^{-1}c = c^{\bullet}(b^{\bullet})^{-1}$

then $V(a,b)\,U(b,c) = U(a,x)\,V(x,c)$

$\Longleftrightarrow \quad x = (b^{\bullet})^{-1} + ab^{-1}c \;\&\; b^{\bullet} = x^{-1} + a^{\bullet}(x^{\bullet})^{-1}c^{\bullet}$

and moreover:



$$a^{-1}x = \dot{x}\,\dot{a}^{-1} \quad \text{and} \quad x\,c^{-1} = (\dot{c})^{-1}x^{\bullet}$$

$\boxed{\begin{array}{l}\textbf{Lemma} \quad \text{if } ab^{-1} = (b^{\bullet})^{-1}\dot{a} \text{ and } b^{-1}c = \dot{c}(b^{\bullet})^{-1} \text{ then} \\ \left\{ x = (b^{\bullet})^{-1} + ab^{-1}c \ \ \& \ \ b^{\bullet} = x^{-1} + a^{\bullet}(x^{\bullet})^{-1}c^{\bullet} \right\} \\ \iff \quad x\,b^{\bullet} = 1 + a\,\dot{c} \\ \text{and moreover} \quad a^{-1}x = x^{\bullet}(\dot{a})^{-1} \text{ and } x\,c^{-1} = (\dot{c})^{-1}\dot{x}\end{array}}$

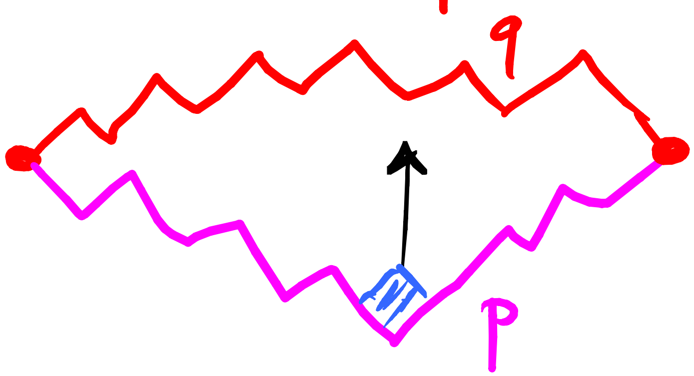
Cor With the usual / & \ $T\,T^{\bullet}$ relations, the NC A_1 T-system is equivalent to:

$$\boxed{T_{j,k+1}\,T^{\bullet}_{j,k-1} = 1 + T_{j-1,k}\,T^{\bullet}_{j+1,k}}$$

SUMMARY

• U, V form a flat $GL_2(\mathcal{A})$ connexion on the solutions of (****)

• products are independent of paths w/ fixed ends

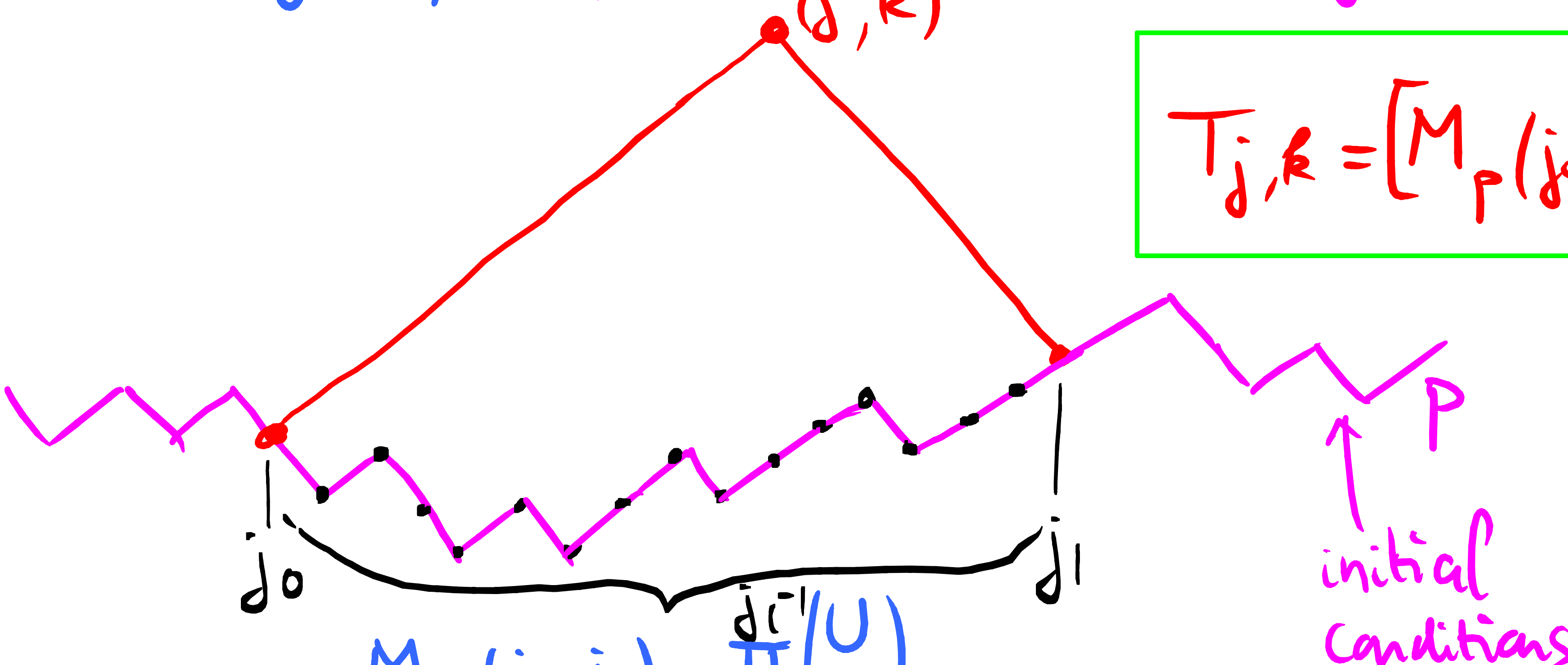


go from p to q by iteration of $VU \to UV$

"box addition/subtraction".

SOLUTION

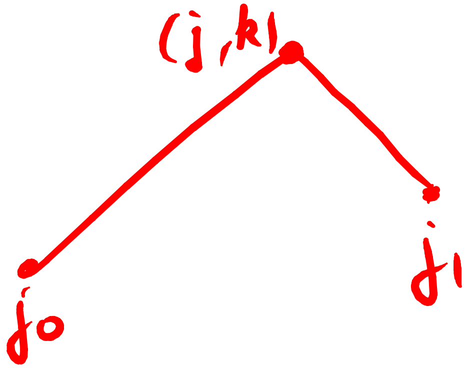
Let T_{jk} = solution of (∗∗∗∗) w/ fixed initial data along a path p, with assignments (t_j)



$$\boxed{T_{j,k} = [M_p(j_0,j_1)]_{1,1}\, t_{j_1}}$$

$$M_p(j_0,j_1) = \prod_{j_0}^{j_1-1} \begin{pmatrix}U\\V\end{pmatrix}$$

Proof We just have to check the formula on the red path



$$\to (UU\cdots U\,VV\cdots V)_{1,1}\; t_{j_1}$$

U lower triang. V upper triang.

$$= \underbrace{(UU\cdots U)_{11}}_{=1}\;\underbrace{(VV\cdots V)_{11}}_{V_{11}V_{11}\cdots V_{11}}\; t_{j_1}$$

$$\left(U = \begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix}\right) \qquad \underbrace{V_{11}V_{11}\cdots V_{11}}_{T_{j,k}\,t_{j_1}^{-1}}\; t_{j_1}$$

$$T_{j,k}\, t_{j_1}^{-1}\, t_{j_1}$$

$$= T_{j,k}$$

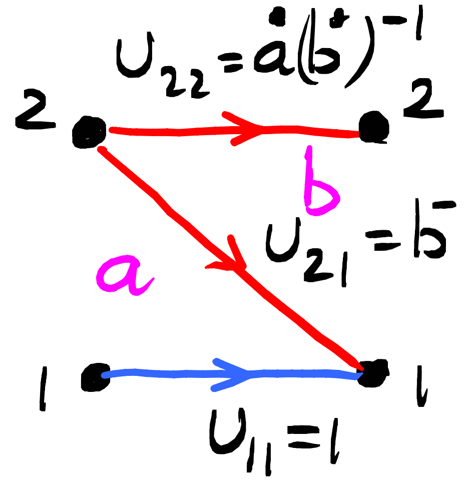
qed.

EXACT SOLUTION II: NC Networks

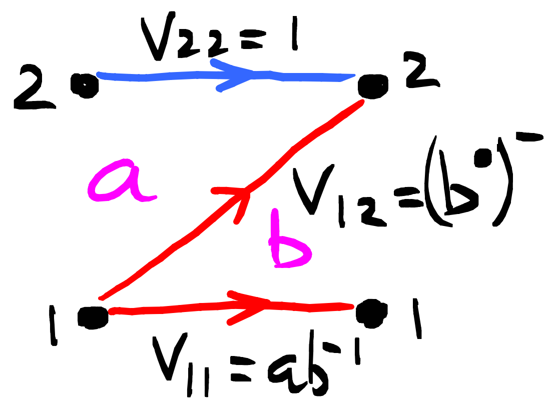
associate "chips" of directed graph

matrix element $\equiv$ edge weight

$U(a,b) \longleftrightarrow$



$V(a,b) \longleftrightarrow$



Product of U, V = concatenation of chips with compatible face labels $\equiv$ NC NETWORK

Lemma The matrix element

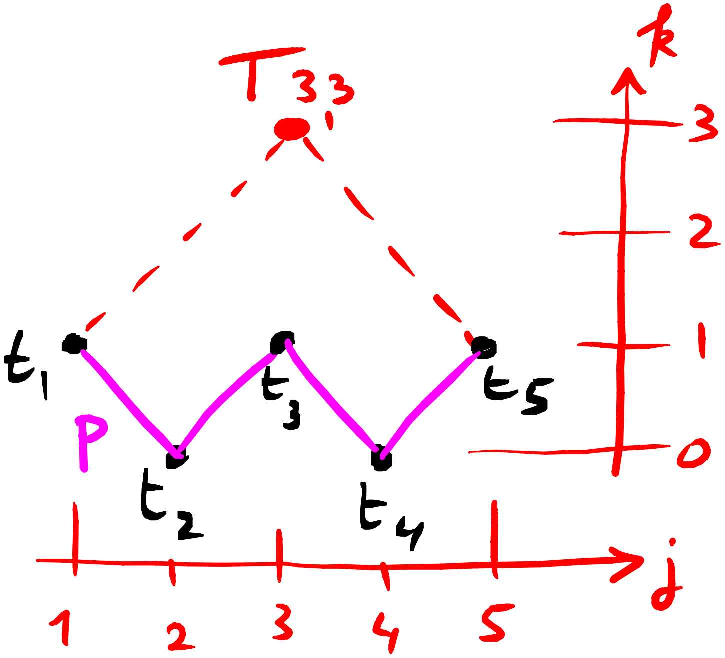
$$\left[U(a_1 a_2)\, U(a_2 a_3) \cdots V(a_j\, a_{j+1}) \cdots \right]_{\ell, m}$$

= partition function of paths $\ell \to m$ on the associated NC network. (

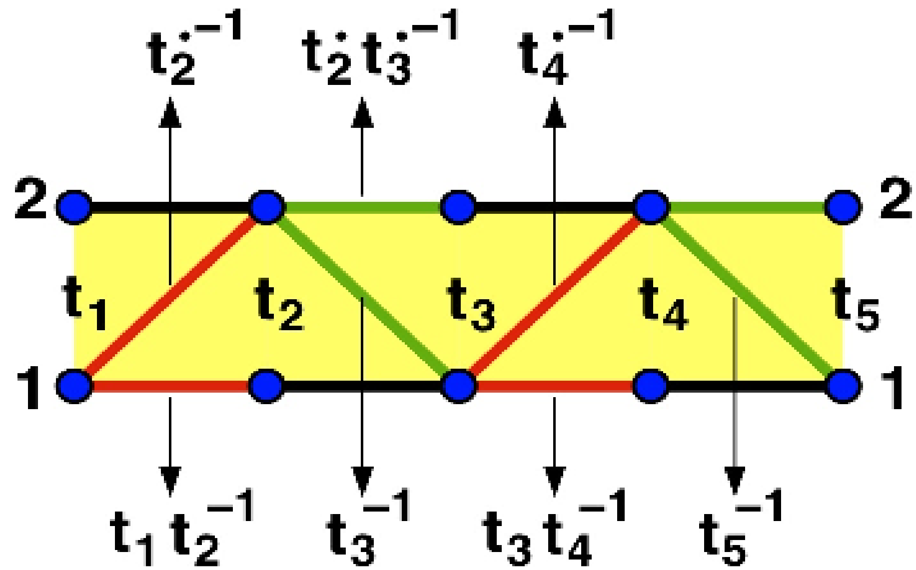
)

Cor $T_{j,k} \cdot t_{j'}^{-1}$ = partition function of paths $1 \longrightarrow 1$ on the concatenation graph corresponding to the projection of (jk) onto p.

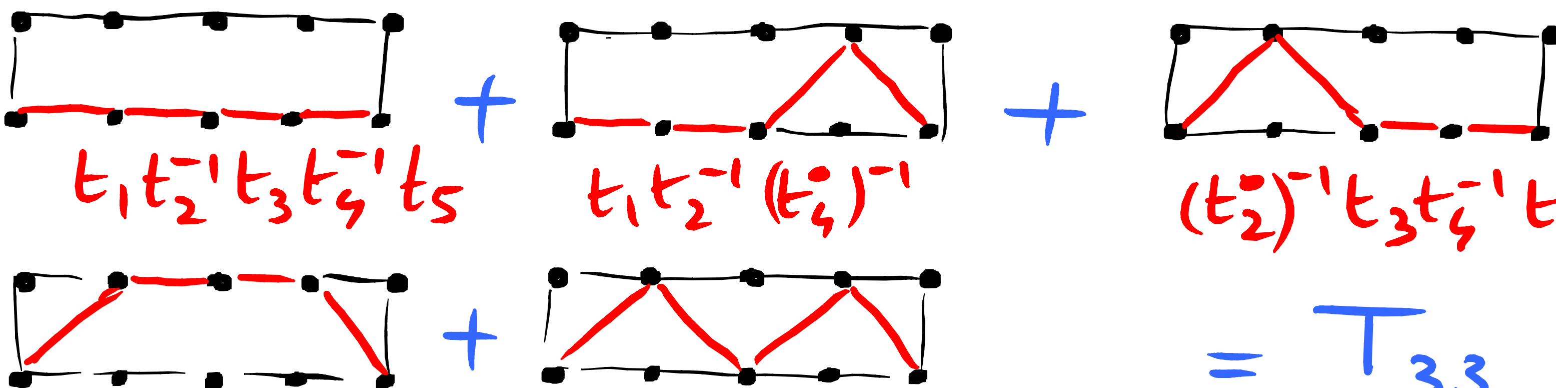
EXAMPLE



$\longleftrightarrow$



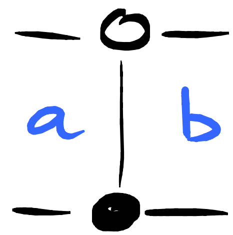
NC NETWORK



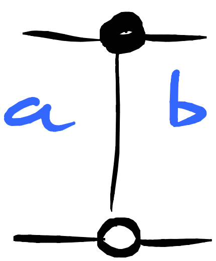
$$t_1 t_2^{-1} t_3 t_4^{-1} t_5 + t_1 t_2^{-1} (t_4^{\bullet})^{-1} + (t_2^{\bullet})^{-1} t_3 t_4^{-1} t_5 + (t_3^{\bullet})^{-1} + (t_2^{\bullet})^{-1} t_3^{-1} (t_4^{\bullet})^{-1} = T_{3,3}$$

EXACT SOLUTION III: NC DIMERS

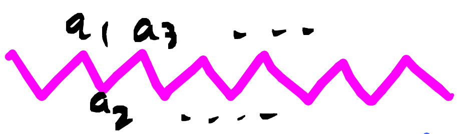
$V(a,b) \longrightarrow$



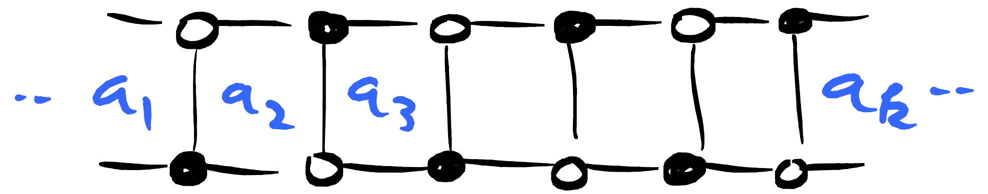
$U(a,b) \longrightarrow$



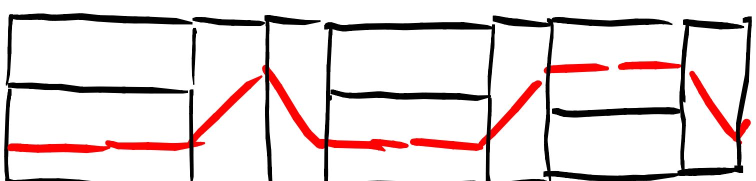
$P =$



$\cdots VUVUVU \cdots \longrightarrow$

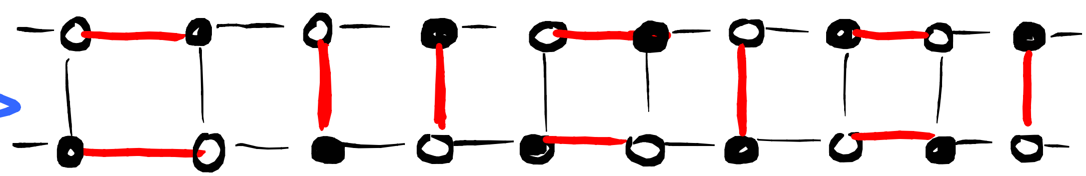


LADDER GRAPH



NC NETWORK PATH

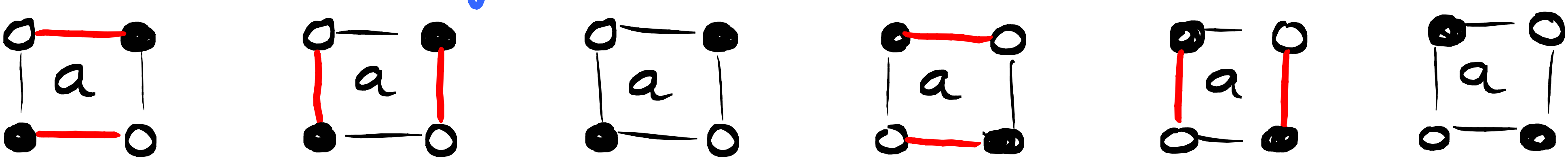
$\underset{1-1}{\longleftrightarrow}$



NC DIMER CONFIG.

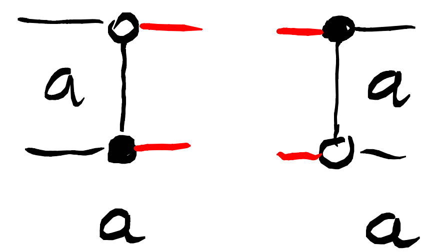
Collecting weights:

THM $T_{jk}\, t_{j'}^{-1}$ = partition function for NC dimers on the ladder graph of the projection of (jk) onto p w/ face weights $= 1$ unless:

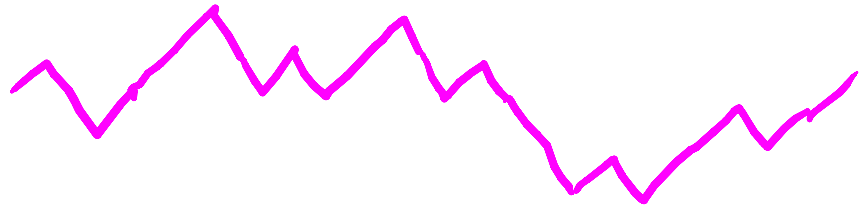


a^{-1}	$(\dot{a})^{-1}$	$a^{\bullet}$	$(a^{\bullet})^{-1}$	a^{-1}	a

+ boundaries

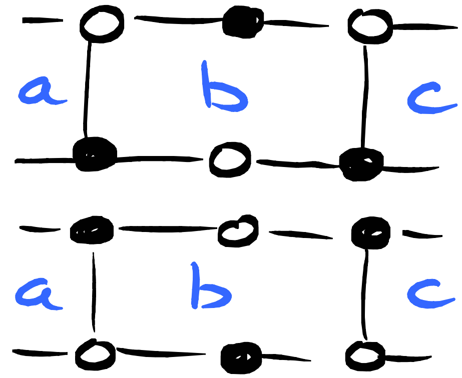


Rem: bijective weights.

Arbitrary $P =$  ...

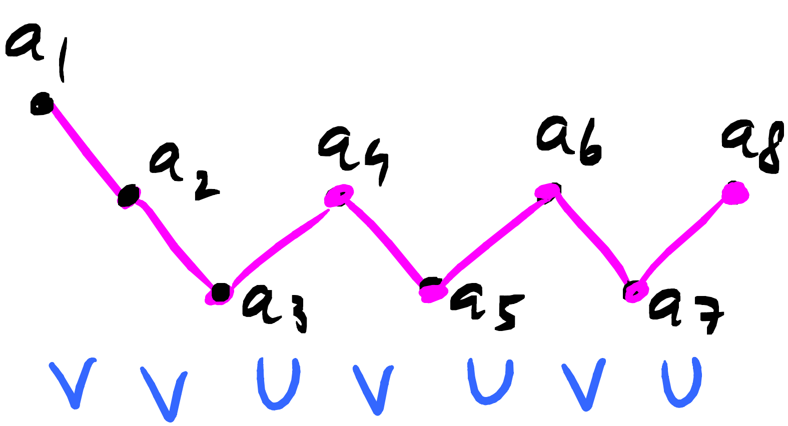
$V(a,b)V(b,c) \rightarrow$

$U(a,b)U(b,c) \rightarrow$

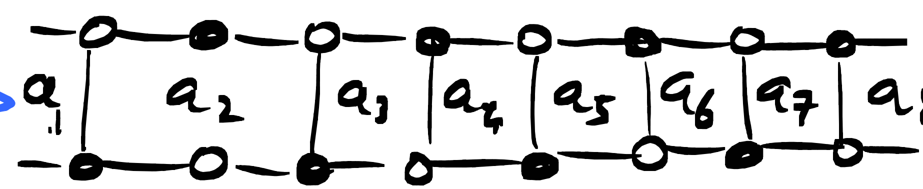


new hexagonal faces in the ladder graph

NC Network paths $\longleftrightarrow$ NC DIMERS on square/hexagon ladder graph.



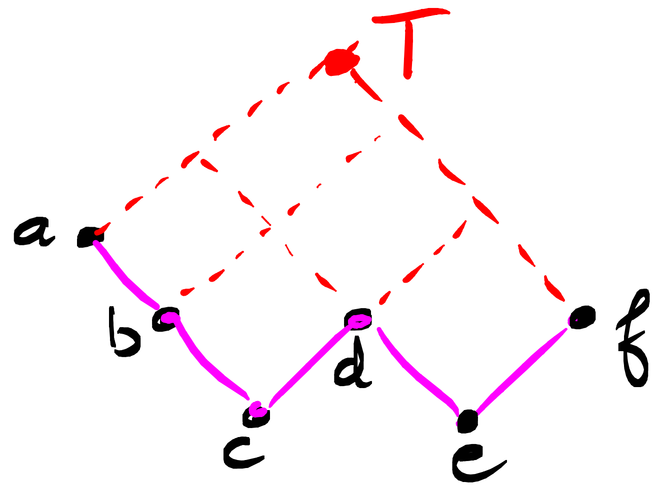
$\longleftrightarrow$

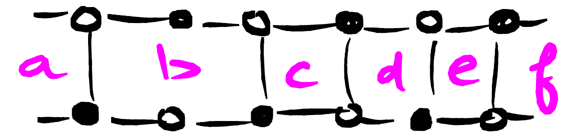


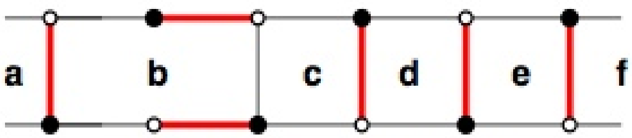
THM The solution $T_{jk} \cdot t_{jl}^{-1}$ for arbitrary p is the partition function for dimers on the square/hexagon ladder graph of the projection of (jk) onto p, with hexagonal face weights $= 1$ unless

a^{-1} $\qquad$ $(\dot{a})^{-1}$ $\qquad$ $(\dot{a})^{-1}$ $\qquad$ a^{-1}

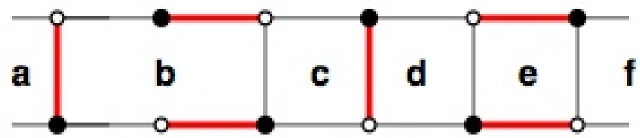
EXAMPLE



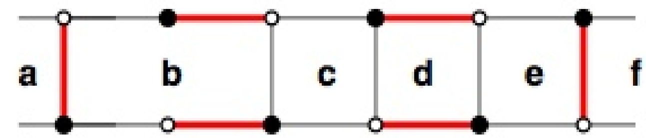




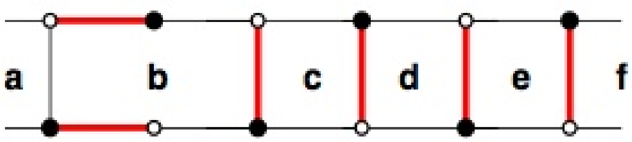
$(b^\bullet)^{-1}d^{-1}(e^\bullet)^{-1}$



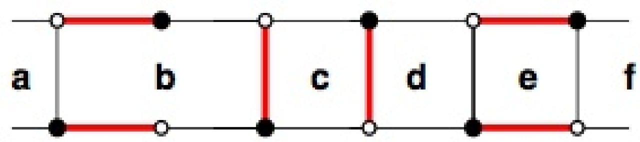
$+ \quad (b^\bullet)^{-1}e^{-1}f$



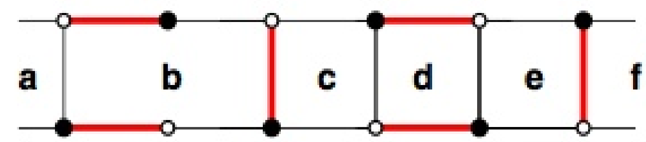
$+ \quad (b^\bullet)^{-1}c^\bullet(d^\bullet)^{-1}$



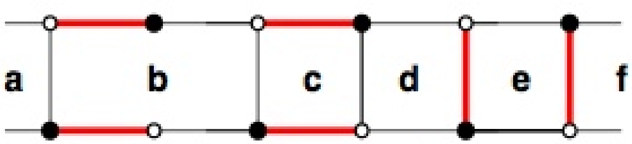
$+ \quad ab^{-1}(c^\bullet)^{-1}d^{-1}(e^\bullet)^{-1}$



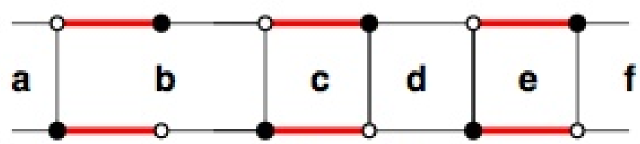
$+ \quad ab^{-1}(c^\bullet)^{-1}e^{-1}f$



$+ \quad ab^{-1}(d^\bullet)^{-1}$



$+ \quad ac^{-1}(e^\bullet)^{-1}$



$+ \quad ac^{-1}de^{-1}f$

$= T$

CONCLUSION

- NC world = very different / quasideterminants
  paths are natural NILP? GV?
  bijective weights
- NC + integrable → exact solutions
  paths / dimers ≡ "fermions".

Generalizations?

- other boundary conditions [PDF in progress] [Berenstein Retakh]
- NC cluster algebras / from surfaces?
- NC T-systems?

[arXiv:0909.0615] [arXiv:1402.2851]