Edge-colored graphs as higher-dimensional maps

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Combinatorial maps

- Discrete surfaces made of gluings of polygons
- Have vertices, edges and faces
- Topological classification

$$F(M) - E(M) + V(M) = 2 - 2g(M)$$

Universality

- Families differing microscopically have same macroscopic behavior
- Triangulations, p-angulations, generic maps
- Asymptotically

$$\# \text{ planar } p\text{-angulations with } n \text{ faces} \sim K_p \rho_p^{-n} n^{-5/2}$$

Exponent $-5/2$ is universal: independent of p

What do we know?
Maps: from Tutte to today

Enumeration

- Count maps with possible decorations (Ising, Potts, loops)
- Exact generating functions or their properties
- Random matrix model techniques
- Bijections [Cori-Vauquelin-Schaeffer, Bouttier-Di Francesco-Guitter]
- Tutte's equations and topological recursion [Borot, Eynard, Orantin]

Physics motivations and applications

- Two-dimensional quantum gravity coupled to matter
- Liouville theory coupled to conformal field theories
- Statistical mechanics
- Celebrated KPZ relations

What do we know? II

Geometric applications of bijections

- Two-point, three-point functions
- Local limit
- Continuum limit [Brownian sphere]

Enumerative geometry

- Intersection numbers [Kontsevich-Witten]
- Hurwitz numbers
- Integrable hierarchies, etc.

What we want

Incorporate combinatorial maps into theory of higher-dimensional spaces

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Incorporate combinatorial maps into theory of higher-dimensional spaces

Let's be realistic... What we can do:

- Natural families generalizing p-angulations
- Polygons become building blocks known as bubbles
- Schwinger-Dyson eqs generalizing loop eqs on generating functions
- Combinatorial extensions of Euler's formula
- For some choice of building blocks, find equivalent of planar maps
- Universality classes
  - Trees for large class of building blocks in 3d
  - Trees, planar maps and trees of maps in even d
- Explicit enumeration
- Topological recursion applies on some cases

Challenges

Difficult interplay between topology and combinatorics

- Numerous families of cellular complexes in topology
- Not designed for enumeration (too wild or too restrictive)
- Attempts at fixed topology
  - locally constructible
  - numerical simulations
- Maps are built regardless of topology

Random tensors

- Generalize famous relation between maps and random matrices to random tensors
- This is our inspiration, but no techniques specific to random tensors

Two key properties

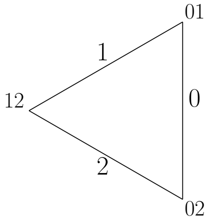
- Natural families to generalize p-angulations and study universality
- Combinatorial extension of genus instead of topological

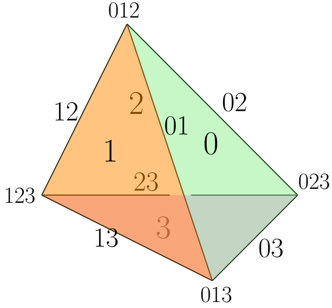
Colored triangulations

- Introduced in topology (crystallization, graph-encoded manifold) in 80s because provide graphical representation of PL-manifolds!
- Closely related to Stanley's balanced complexes
  Difference here: vertices do not define a unique simplex
- Never considered for enumeration purposes

Colored simplex

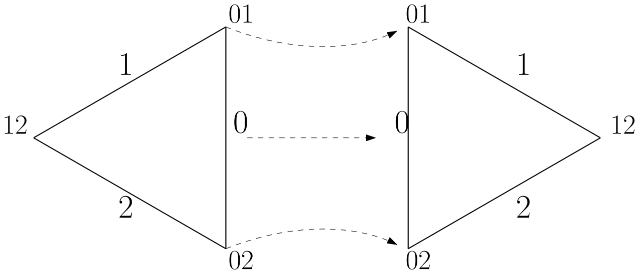
- Faces have a color from $\{0, 1, \ldots, d\}$
- $(d-2)$-simplices have pairs of colors, and so on





Attaching map

- Unique gluing which respects all subcolorings



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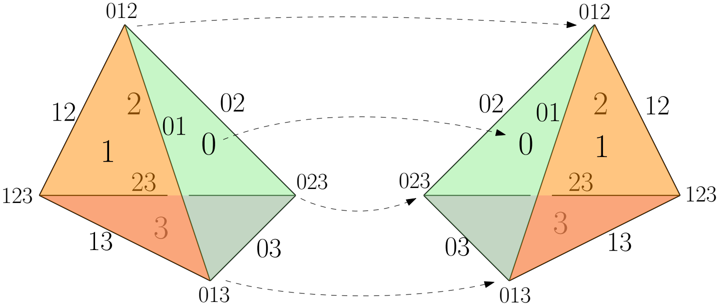
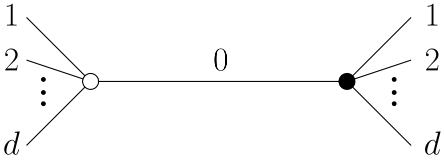
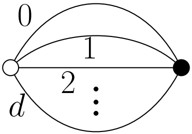


Colored graphs

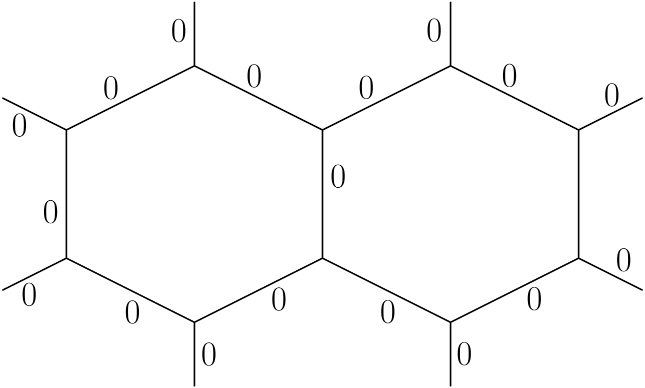
- Gluing determined by color of face
- Graphical representation



d-simplex	vertex
$(d-1)$-simplex of color c	edge of color c
$(d-2)$-simplex with two colors	conn. comp. with two colors



The 2D case



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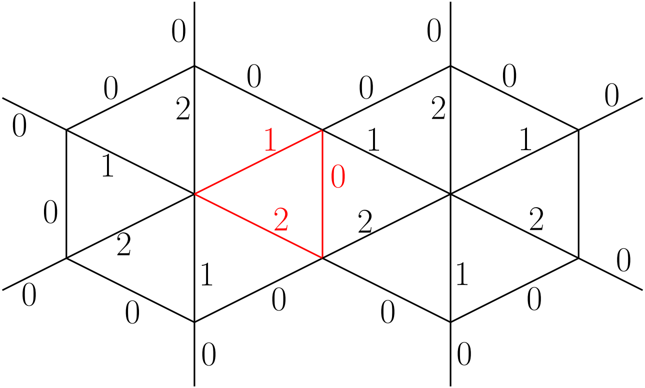

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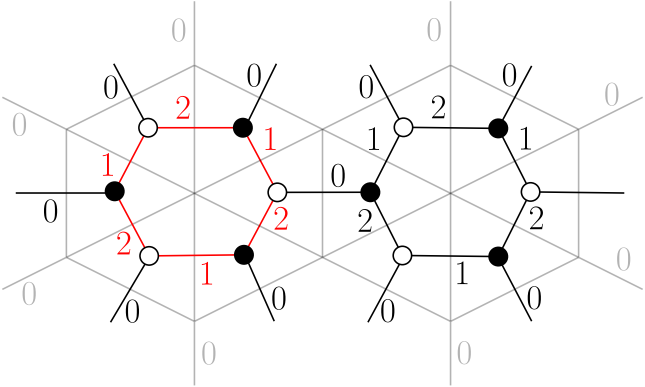

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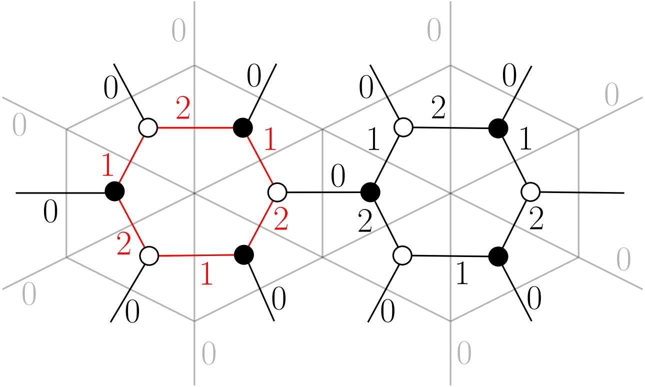
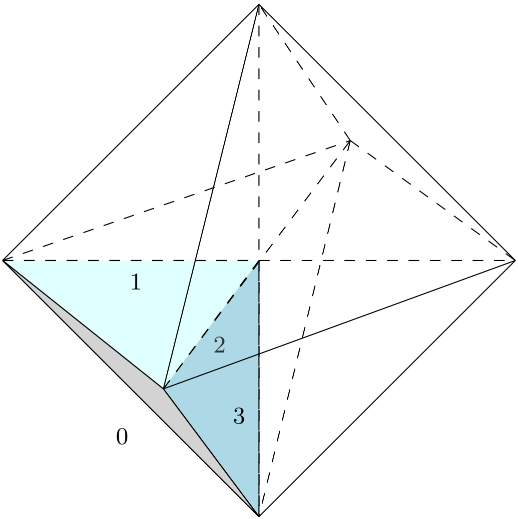


2p–angle

- Gluing of $2p$ triangles with boundary of color 0
- Dually: Components with all colors but 0

The 3D case



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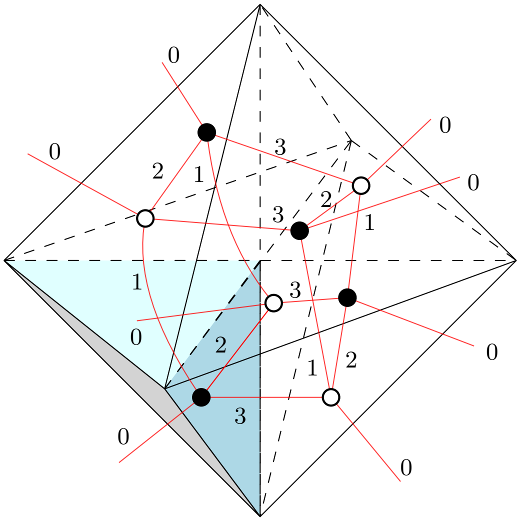

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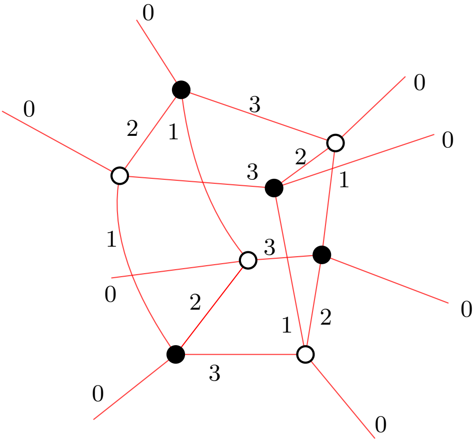

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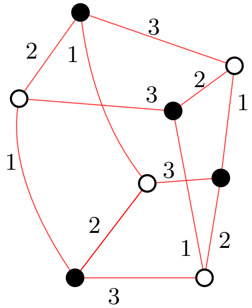

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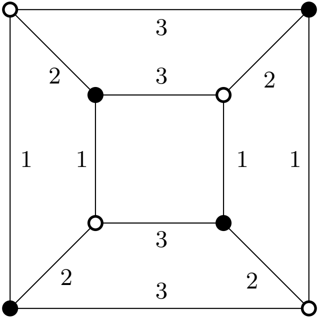
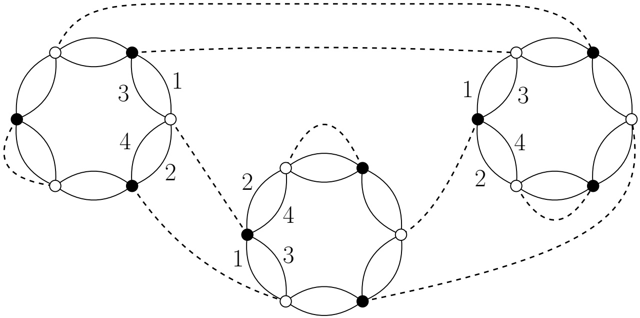


Bubbles as building blocks

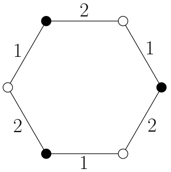
- Triang. dim. d = colored graphs with colors $0, 1, \ldots, d$
- Bubble = building block with all colors except 0
- All graphs obtained by gluing bubbles along edges of color 0



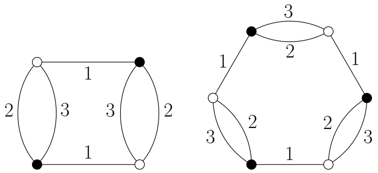
- Bubble is determined by boundary triangulation

Bubbles

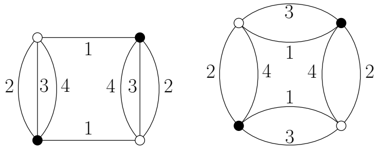
2d: determined by length

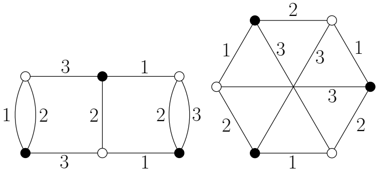


3d: labeled by surfaces



4d: new one with 4 vertices





Gurau's degree theorem

- Finite set of bubbles $B_1, B_2, \ldots,$ graph G
- 2d: genus classification

Bound on $(d-2)$-simplices

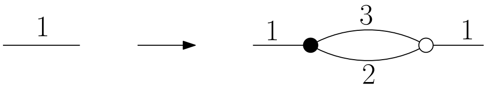
There exists $\omega(G) \geq 0$

$$\Delta_{d-2}(G) - \frac{d(d-1)}{4}\Delta_d(G) = d - \omega(G) \leq d$$

- $d = 2 \quad \Rightarrow \quad \omega(G) = 2g(G)$
- For $d \geq 3$, bound can be saturated only for melonic bubbles
- Maximizing graphs (melonic) are series–parallel
- Gurau–Schaeffer classification according to the degree
- Genuine combinatorial extensions of genus exist!

Towards other behaviors

- Melonic insertion



- Colored triangulations built from non-melonic bubbles grow fewer $(d-2)$-simplices
- Need a bubble-dependent degree

$$\Delta_{d-2}(G) - \alpha(B_1, B_2, \dots)\Delta_d(G) = d - \omega_B(G) \leq d$$

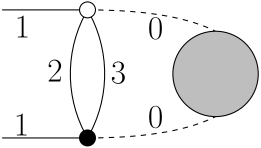
with

$$\alpha(B_1, B_2, \dots) > d(d-1)/4$$

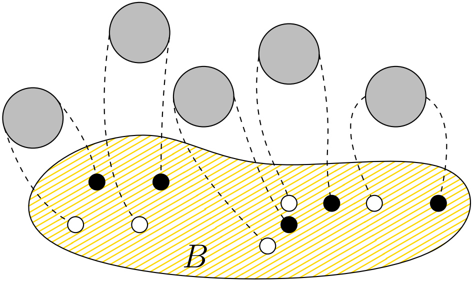
- Finding α is challenging!
- Provides notion of higher-dimensional "planar" maps
- Identify graphs which maximize the bound

3D: 3 colors + 0

Melonic bubbles

- 2 parallel edges: 

- Partition into 2-edge-cuts

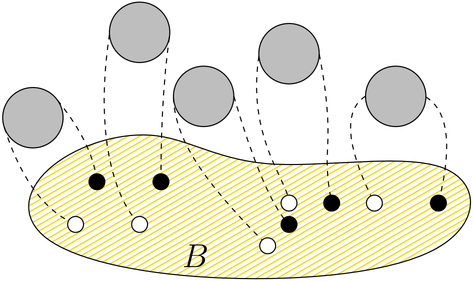


- Tree structure

3D with spherical bubbles

Conjecture/Work in progress

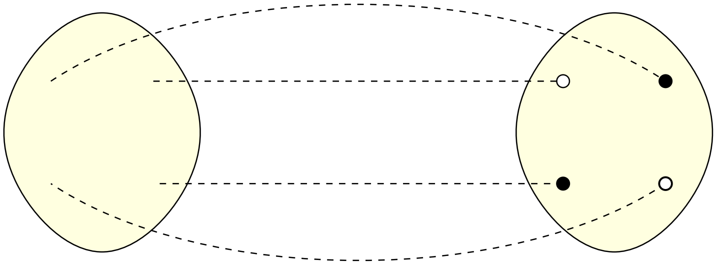
- Bubbles homeomorphic to 3-balls
- Same combinatorial result
- Maximizing edges



- 3-spheres
- Already proved for octahedra/bipyramids

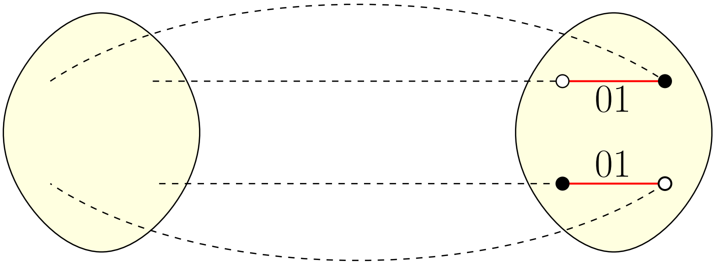
4-edge-cut

- Bicolored cycles with colors 01, 02, 03 along each edge of color 0



4-edge-cut

- What path do they follow on the right?



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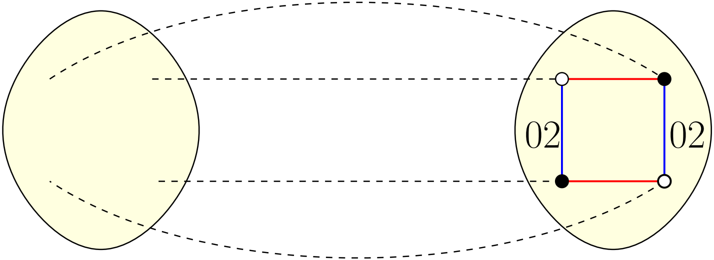
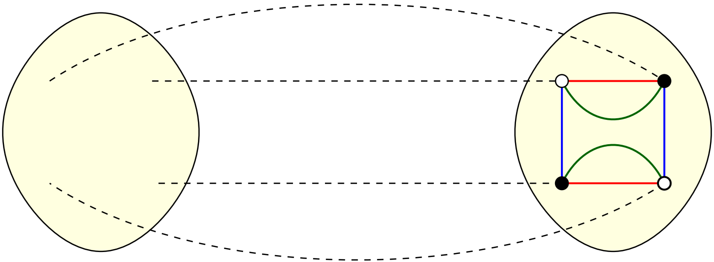


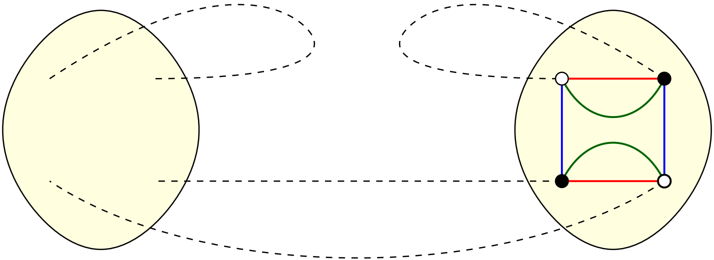
4-edge-cut

- At least 1 pair connected by two paths



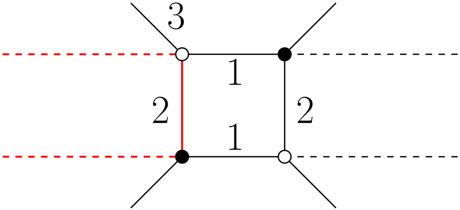
4-edge-cut

- Perform a flip, increase number of bicolored cycles
- 4-edge-cut cannot happen



Spherical bubbles

- Spherical bubble have planar boundary
- Colored graph is 3-regular, planar, bipartite
- Has either face of degree 2, or (at least six) faces of degree 4
- Perform two flips
- Number of bicolored cycles does not decrease



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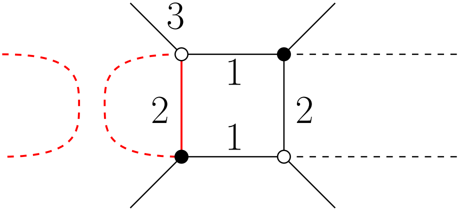

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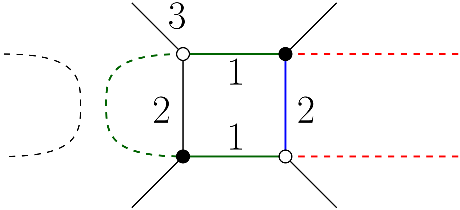

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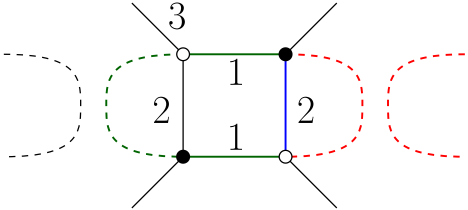
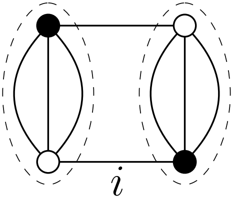


Quartic case, $d = 4$

Melonic bubbles



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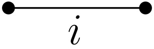
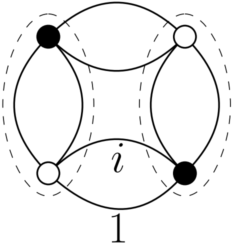


Quadrangular bubbles



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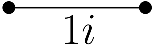
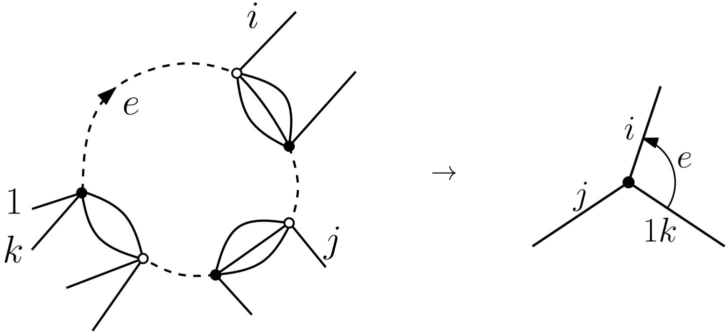


Bijection

- Cycles of color 0 and pairs of vertices $\to$ counter–clockwise star–map



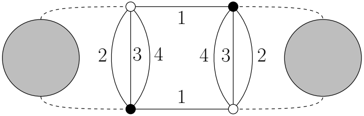
- In 2D, bubbles are quadrangles $\to$ Tutte's bijection

Quartic case, $d = 4$

- Maps of arbitrary degree
- Monocolored edges, colors $1, 2, 3, 4$
- Bicolored edges, colors $12, 13, 14$
- Bicolored cycles $(0c)$ are faces of color c

Maximizing triangles = maximizing faces

- Monocolored edges are bridges



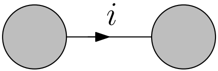
- Bicolored form planar components
- Bicolored types $1c$ and $1c'$ touch on cut–vertices (similar to $O(n)$ model on planar maps)

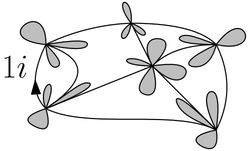
The quartic case

- Generating function of (rooted) maps for k types of bicolored edges

$$f_k(t,\lambda) = \sum_M t^{\#\text{edges}}\ \lambda^{\#\text{monocol. edges}}$$

- $P(t)$ the generating function of planar non-separable maps





$$f_k(t,\lambda) = 1 + \qquad t\lambda f_k(t,\lambda)^2 \qquad + \quad k\Big(P\big(tf_k(t,\lambda)^2\big) - 1\Big)$$

implies algebraicity

$$\begin{cases} tf^2 = u(1-u)^2 \\ f = k(1-u)(1+3u) - k + 1 + \lambda u(1-u)^2 \end{cases}$$

- Generic planar maps for $\lambda = 0$ and $k = 1$

$$27t^2A(t)^2 + (1-18t)A(t) + 16t - 1 = 0$$

Explicit singularity analysis for $k = 1$

- Quartic eq on $f(t, \lambda)$
- For $\lambda < 3$, singularity at $t_1(\lambda) = \frac{27}{4(\lambda+9)^2}$

$$f(t,\lambda) = \frac{4}{27}(\lambda+9) + \frac{16(\lambda+3)(\lambda+9)^3}{729(\lambda-3)}(t_1(\lambda)-t) + \frac{64(\lambda+9)^{11/2}}{6561(3-\lambda)^{5/2}}(t_1(\lambda)-t)^{3/2} + o((t_1(\lambda)-t)^{3/2})$$

- For $\lambda > 3$, singularity at $t_2(\lambda) = \frac{\lambda}{4(1+\lambda)^2}$

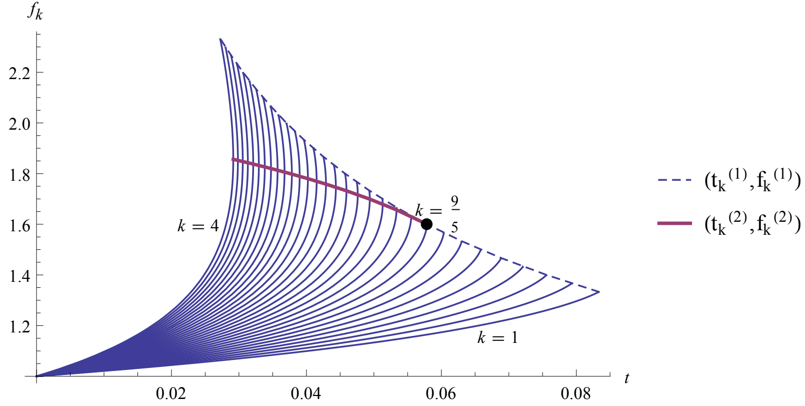
$$f(t,\lambda) = 2\frac{\lambda^2-1}{\lambda^2} - \frac{4(1+\lambda)^2}{\lambda^{5/2}}\sqrt{\lambda^2-2\lambda-3}\,(t_2(\lambda)-t)^{1/2} + o((t_2(\lambda)-t)^{1/2})$$

- $\lambda = 3$, proliferation of baby universes

$$f(t,\lambda=3) = \frac{16}{9} - \frac{128}{3^{5/3}}\left(\frac{3}{64}-t\right)^{2/3} + o\left(\left(\frac{3}{64}-t\right)^{2/3}\right)$$

Same results with respect to k

- $\lambda = 0$, no monocolored edges
- k small enough: universality class of maps
- k large enough: branching process and square–root singularity
- k critical: singularity exponent $2/3$



Summary of maximizing number of $(d-2)$-simplices

3D

- Universality class of trees
- Conjectured for all spherical bubbles

4D

- Universality class depends on bubbles!
- Transitions between planar maps and trees
- Proliferation of baby universes

Colored triangulations offer a frame for exact solutions to higher dim. spaces

1-bubble triangulations

- 2D unicellular maps have a single polygon

$$\langle 2n\text{-gon}\rangle = \sum_{\text{perfect matchings of edges}} N^{\#\text{ vertices}}$$

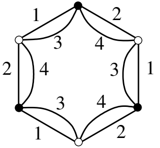
$$= \text{Harer-Zagier polynomial}(N) = N^{n+1}\big(\text{Catalan}(n) + \mathcal{O}(1/N)\big)$$

- Generic case using bubble B

$$\langle B\rangle = \sum_{\text{Add edges color 0}} N^{\#\text{ bicolored cycles}}$$

- Various behaviors at large N

$$\langle\text{melon}\rangle = 1 \qquad \Big\langle \quad \Big\rangle = \text{Catalan}(n)$$



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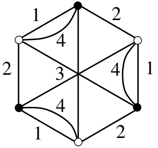
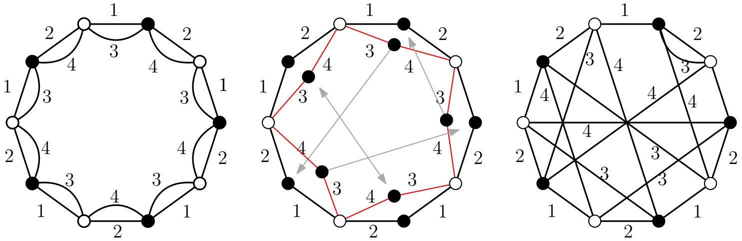
$\langle\text{melon}\rangle = 1 \qquad \Big\langle$ 

$\Big\rangle = \text{Motzkin}(n)$

4D double-gon

- 1 cycle (12), 1 cycle (34)
- cycle (34) obtained by permuting black vertices with σ

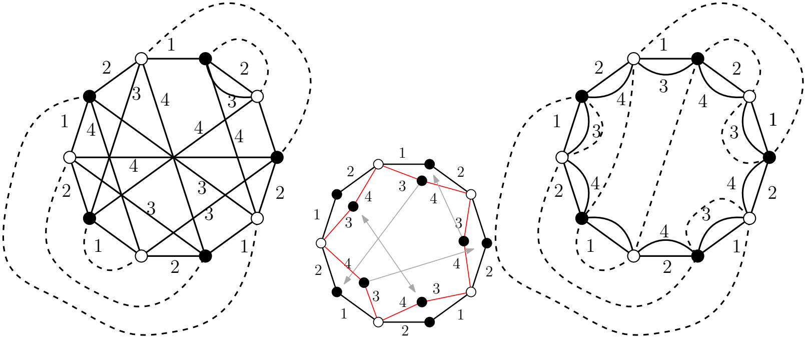


- Add color 0 and maximize number of bicolored cycles

$$\langle B_\sigma \rangle = |\text{subset of meanders with } |\sigma| \text{ roads}|$$

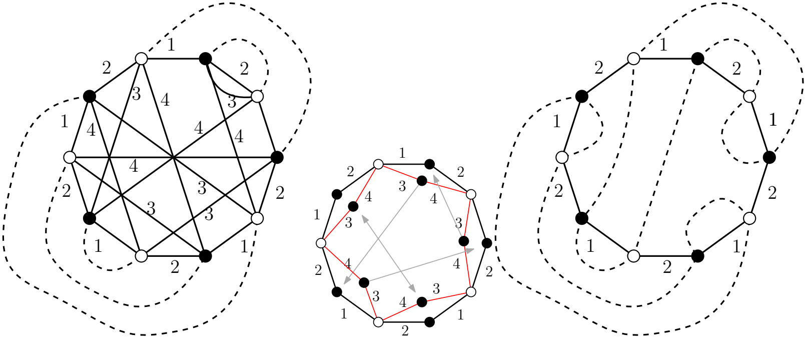
4D double-gon and meanders

- Add color 0
- Straighten cycle (34) and deform color 0 accordingly
- Remove colors 3, 4



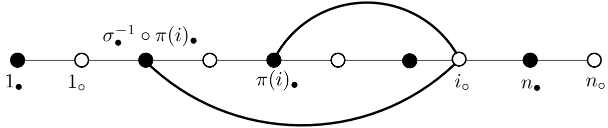
4D double-gon and meanders

- Add color 0
- Straighten cycle (34) and deform color 0 accordingly
- Remove colors 3, 4



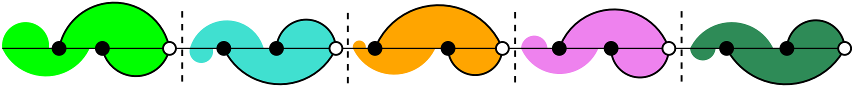
4D double-gon and meanders

- Let $\mathcal{M}_{\sigma_\bullet}$ the set of meandric systems



- **Block decomposition** σ has a block decomposition $(\sigma_1, \cdots, \sigma_p)$ such that σ_1 stabilizes $[1, i_1 - 1]$, σ_2 stabilizes $[i_1, i_2 - 1]$ and so on
- σ is **connected** if it only stabilizes $[1, n]$
- **Connected block decomposition** σ has a unique maximal block decomposition
  $\sigma = (\sigma_1, \ldots, \sigma_p)$ with all σ_j connected

Factorization



$$\mathcal{M}_{\sigma_1} \times \cdots \times \mathcal{M}_{\sigma_p} \subset \mathcal{M}_\sigma$$

1-reducible meandric system: 1 cut creates 2 disconnected pieces

- A **Planar permutation** π corresponds to a planar arch configuration
  $\mathrm{Pl}\,\mathfrak{S}_n$ set of planar permutations
  (Not a group for the composition, but for TL composition)
- There is a bijective map between

$$\mathcal{M}_{\sigma_1} \times \cdots \times \mathcal{M}_{\sigma_p} \times \mathrm{Pl}\,\mathfrak{S}_p \;\to\; \mathcal{M}_\sigma$$

  which implies

$$\mathcal{M}_\sigma = \mathrm{Cat}_p \prod_{j=1}^{p} \mathcal{M}_{\sigma_j}$$

Conclusion

- Colored triangulations are genuine generalization of maps
- Admit generalization of genus, but bubble-dependent
- Conjecture large class of tree-like behaviors in odd dim.
- Universality classes depend on bubbles in even dim., unlike 2D
- At least some enumeration is feasible in dim $d > 2$!
- Use of bijections with maps [generic case: VB & L. Lionni]
- Beyond maximizing number of $(d-2)$-simplices in quartic case
  $\rightarrow$ Topological recursion! [VB & S. Dartois]
- More to be studied