Lecture 3: Integrable probabilities

Alexey Bufetov and Vadim Gorin

January 31, 2017

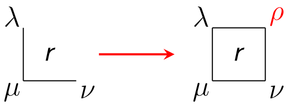
RSK-algorithm (Robinson, Schensted, Knuth, Fomin, Fulton, Viennot).

Stanley, "Enumerative combinatorics", Chapter 7; Fomin growth diagrams.

Exposition for Schur processes with alpha- and beta-specializations: Betea-Boutillier-Bouttier-Chapuy-Corteel-Vuletic'14, Matveev-Petrov'15.

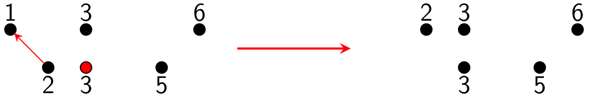
INPUT: three Young diagrams $\mu \prec \lambda$, $\mu \prec \nu$, $r \in \mathbb{Z}_{\geq 0}$.

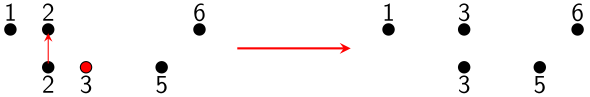
OUTPUT: Young diagram ρ such that $\lambda \prec \rho$, $\nu \prec \rho$, also $|\rho| - |\lambda| = |\nu| - |\mu| + r$.



MICROUPDATE; INPUT: $\tilde{\mu}, \tilde{\lambda}, j$ such that $\tilde{\mu} \prec \tilde{\lambda}$, $\tilde{\mu}$ has length N, $\tilde{\lambda}$ has length $N+1$; $1 \le j \le N$.
OUTPUT: updated $\tilde{\mu} \prec \tilde{\lambda}$.

IF $\tilde{\lambda}_j > \tilde{\mu}_j$ THEN $\tilde{\lambda}_{j+1} \mapsto \tilde{\lambda}_{j+1} + 1$.
ELSE $\tilde{\lambda}_j \mapsto \tilde{\lambda}_j + 1$. (we have $\tilde{\lambda}_j = \tilde{\mu}_j$ in this case)
DO: $\tilde{\mu}_j \mapsto \tilde{\mu}_j + 1$
Examples: $MU\,[(2,5) \prec (1,3,6), 2] = (3,5) \prec (2,3,6)$,
$MU\,[(2,5) \prec (1,2,6), 2] = (3,5) \prec (1,3,6)$





RSK algorithm

INPUT: $\mu \prec \lambda$, $\mu \prec \nu$, $r \in \mathbb{Z}_{\geq 0}$. OUTPUT: ρ such that $\lambda \prec \rho$, $\nu \prec \rho$, also $|\rho| - |\lambda| = |\nu| - |\mu| + r$.

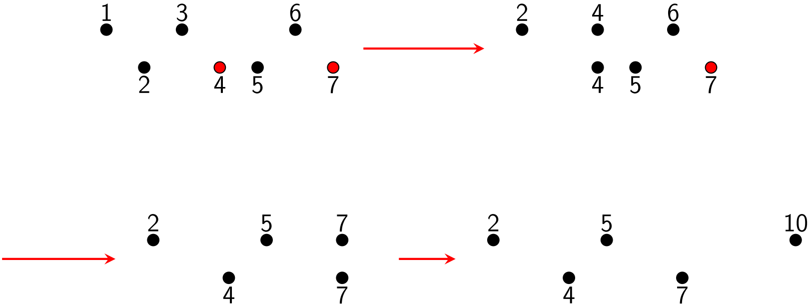
We think about μ and ν as about signatures of length N, and about λ and ρ as signatures of length $N+1$.

ALGORITHM:

ASSIGN: $\tilde{\mu} := \mu$, $\tilde{\lambda} := \lambda$.
DO $MU(\tilde{\mu}, \tilde{\lambda}, N)$ exactly $\nu_N - \mu_N$ times.
DO $MU(\tilde{\mu}, \tilde{\lambda}, N-1)$ exactly $\nu_{N-1} - \mu_{N-1}$ times.
$\dots$
DO $MU(\tilde{\mu}, \tilde{\lambda}, 1)$ exactly $\nu_1 - \mu_1$ times.
DO $\tilde{\lambda}_1 \mapsto \tilde{\lambda}_1 + r$.
OUTPUT: $\rho := \tilde{\lambda}$.

Example

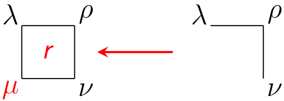
$\mu = (2, 5)$, $\lambda = (1, 3, 6)$, $\nu = (4, 7)$, $r = 3$.



OUTPUT: $\rho = (2, 5, 10)$.
Notation: $RSK(\mu, \lambda, \nu, r) = \rho$.

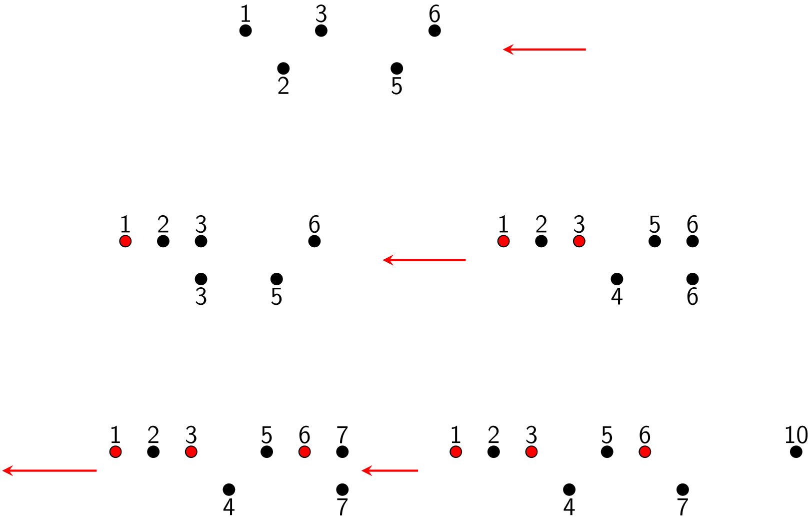
Inversion

Proposition 1: Given ρ, ν, λ such that $\nu \prec \rho$, $\lambda \prec \rho$ there exists a unique Young diagram μ and a unique $r \in \mathbb{Z}_{\geq 0}$ such that $RSK(\mu, \lambda, \nu, r) = \rho$.



PROOF: The move $\lambda_1 \to \rho_1$ is affected only by the move $\mu_1 \to \nu_1$, so rules of the jumps imply $r = \rho_1 - \max(\lambda_1, \nu_1)$
Each micro update can be inverted.
Basically, we just need to rotate a picture by 180 degrees and use the same rules.

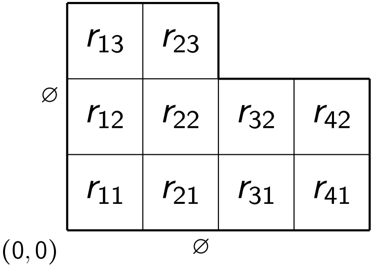
$\lambda = (1, 3, 6)$, $\nu = (4, 7)$, $\rho = (2, 5, 10)$.



COROLLARY: For any λ, ν we construct a bijection between pairs (μ, r) which satisfy $\mu \prec \lambda$, $\mu \prec \nu$, $r \in \mathbb{Z}_{\geq 0}$ and ρ such that $\lambda \prec \rho$, $\nu \prec \rho$.

This bijection has a property $|\rho| - |\lambda| = |\nu| - |\mu| + r$.

Our next goal is to construct the RSK field $\lambda(k, l)$, $k, l \in \mathbb{Z}_{\geq 0}$ given the a field of inputs r_{kl}, $k, l \in \mathbb{Z}_{\geq 0}$.

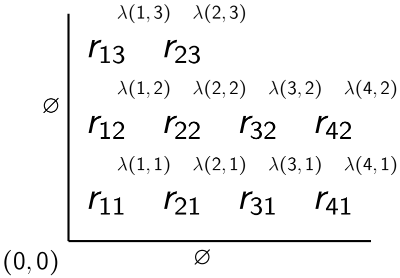


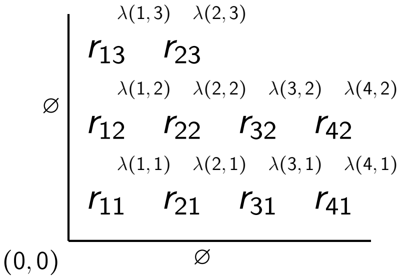
Set $\lambda(k,0) = \varnothing$, $\lambda(0,k) = \varnothing$, for any $k \in \mathbb{Z}_{\geq 0}$. Then, define inductively

$$\lambda(k+1, l+1) = RSK(\lambda(k,l), \lambda(k,l+1), \lambda(k+1,l), r_{kl}).$$

That is, we add boxes one by one using elementary steps described before.

Note that by construction for any (k,l) we have $\lambda(k,l) \prec \lambda(k+1,l)$, $\lambda(k,l) \prec \lambda(k,l+1)$.





For this example we have

$$\varnothing < \lambda(1,3) < \lambda(2,3) > \lambda(2,2) < \lambda(3,2) < \lambda(4,2) > \lambda(4,1) > \varnothing$$

We have already seen such interlacing arrays...

Let us fix a down-right path C on the grid.
PROPOSITION 2: We have a bijection between
$\{r_{ij} : (i,j) \text{ below } C, r_{ij} \in \mathbb{Z}_{\geq 0}\}$ and
$\{\lambda(k,l) : (k,l) \in C, \text{ interlacing constraints}\}$ such that for any $k \in \mathbb{Z}_{\geq 1}$ we have

$$\sum_{j:(kj) \text{ below } C} r_{kj} = |\lambda(k,L)| - |\lambda(k,L-1)|,$$

for a unique L such that both $(k-1,L) \in C$, $(k,L) \in C$, and for any $l \in \mathbb{Z}_{\geq 1}$ we have:

$$\sum_{i:(il) \text{ below } C} r_{il} = |\lambda(K,l)| - |\lambda(K-1,l)|,$$

for a unique K such that both $(K,l) \in C$, $(K-1,l) \in C$.

Proof of Proposition 2: Sequential use of Proposition 1 and the property $|\rho| - |\lambda| = |\nu| - |\mu| + r$ for each box (ij) below C.

Corollary: The same statement for C which first makes all steps to the right, and then makes all steps down (this is a typical form of RSK).

Applications to Schur measures and Schur processes.

Recall that $s_{\lambda/\mu}(a) = a^{|\lambda|-|\mu|}$.

Schur measure with alpha-parameters $a_1, \dots, a_M > 0$ and $b_1, \dots, b_N > 0$:

$$Prob(\lambda) \sim s_\lambda(a_1, \dots, a_M) s_\lambda(b_1, \dots, b_N)$$
$$= \sum_{\lambda(1,N),\dots,\lambda(M-1,N),\lambda(M,N-1),\dots} a_1^{|\lambda(1,N)|-|\lambda(0,N)|} a_2^{|\lambda(2,N)|-|\lambda(1,N)|}$$
$$\times \dots a_M^{|\lambda(M,N)|-|\lambda(M-1,N)|} b_N^{|\lambda(M,N)|-|\lambda(M,N-1)|} \dots b_1^{|\lambda(M,1)|-|\lambda(M,0)|}.$$

such that $\lambda = \lambda(M, N)$ and

$$\varnothing = \lambda(0, N) \prec \lambda(1, N) \prec \dots \prec \lambda(M, N)$$
$$\succ \lambda(M, N-1) \succ \dots \succ \lambda(M, 0) = \varnothing$$

Note that expressions like $|\lambda(k+1,N)| - |\lambda(k,N)|$ are exactly the sums of r_{ij}.

Consider a down-right path C consisting of M steps to the right, then N steps down. If each integer r_{ij} has weight $(a_i b_j)^{r_{ij}}$, then the product of these weights over (i,j) equals one term in the sum for the product of Schur functions (by Proposition 2) !

Also, by Proposition 2 we have bijections with integer arrays.

Proposition 3 a: Let r_{ij} be independent random variables with geometric distribution $Prob(r_{ij} = x) = (1 - a_i b_j)(a_i b_j)^x$, $x = 0, 1, 2, \dots$. Then $\lambda(M,N)$ is distributed according to the Schur measure with parameters $a_1, \dots, a_M, b_1, \dots, b_N$.

Proposition 3 b: Let r_{ij} be independent random variables with geometric distribution $Prob(r_{ij} = x) = (1 - a_i b_j)(a_i b_j)^x$. For any $M_1 \geq M_2 \geq M_k$ and $N_1 \leq N_2 \leq \cdots \leq N_k$ the random Young diagrams are distributed according to the Schur process with specializations $\rho_0^+ = \{a_1, \ldots, a_{M_k}\}$, $\rho_1^- = \{b_{N_k}, \ldots, b_{N_{k-1}+1}\}$, $\ldots$, $\rho_{k-1}^+ = \{a_{M_2+1}, \ldots, a_{M_1}\}$, $\rho_k^- = \{b_{N_1}, \ldots, b_1\}$.

Idea of proof: First, we check the statements for $\{(M_i, N_i)\}$ such that $(M_{i+1}, N_{i+1}) - (M_i, N_i) = (1, 0)$ or $(0, -1)$. For such collections this is a corollary of Proposition 2 (also note that many specializations are "empty" for such paths).
For arbitrary $\{(M_i, N_i)\}$ we first consider them as a subset of a larger collection with property
$(M_{i+1}, N_{i+1}) - (M_i, N_i) = (1, 0)$ or $(0, -1)$, and then use (combinatorial) definition of skew Schur functions.

Corrolary: The partition function for Schur measure with parameters $a_1, \ldots, a_M$, $b_1, \ldots, b_N$ is $\prod_{i \le M, j \le N} (1 - a_i b_j)^{-1}$. Thus, we proved Cauchy identity.

Corrolary: The partition function for Schur process "associated" with down-right path C is

$$\prod_{(i,j) \text{ below } C} (1 - a_i b_j)^{-1}$$

This gives a method how to generate Schur process with alpha-specialisations. The same scheme can be applied to arbitrary specializations (one needs an another basic step to add β-parameter; and one needs a degeneration to obtain γ-parameter).

Consider a matrix $\{r_{ij}\}_{1\le i\le M, 1\le j\le N}$ and define the quantity

$$G(M,N) = \max_{\text{P: up-right path } (1,1)\to(M,N)} \sum_{(i,j)\in P} r_{ij}.$$

Example: $G(2,2) = 4$, $G(3,2) = 8$, $G(2,3) = 6$, $G(3,3) = 9$.

3	1	1
0	1	4
2	1	0

Last Passage Percolation.

Proposition 4: For a matrix $\{r_{ij}\}_{1\le i\le M, 1\le j\le N}$ let $\lambda(M,N)$ be a Young diagram constructed by RSK algorithm. Then

$$G(M,N) = \lambda_1(M,N)$$

Proof: We have

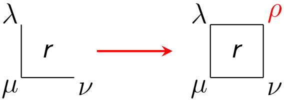
$$G(k+1,l+1) = max(G(k+1,l), G(k,l+1)) + r_{kl}$$

$$\lambda_1(k+1,l+1) = max(\lambda_1(k+1,l), \lambda_1(k,l+1)) + r_{kl}.$$

Thus, the statement follows by induction.

If r_{ij} are geometrically distributed, then the first row of a random Young diagram distributed according to the Schur measure describes the last passage percolation.

All previous discussion was based on a specific choice of RSK operation



We discussed row insertion RSK. There are also several other versions.

Properties of column insertion RSK

1) it is described by the same picture (all interlacing conditions are the same).

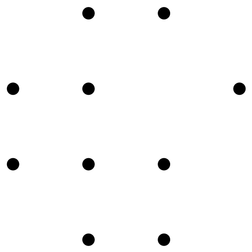
2) Propositions 1,2,3 are the same (inversion, bijection, sampling of Schur processes with alpha-specializations).

3) Proposition 4 takes a different form.

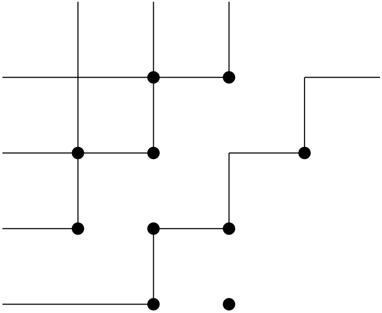
As before, let $\{r_{ij}\}_{1 \le i \le M, 1 \le j \le N}$ be a matrix with non-negative values, and define

$$\hat{r}_{ij} = \begin{cases} 1, & \text{if } r_{ij} \ge 1 \\ 0, & \text{if } r_{ij} = 0 \end{cases}$$

Let us consider the following picture:



Dots represent the places of 4×4 matrix where $\hat{r}_{ij} = 1$.



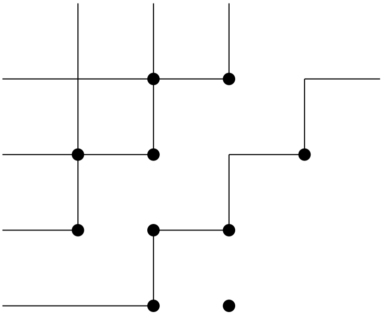
Rules:

— up-right paths coming from the left; no more that one line on each level;

— all horizontal levels are occupied on the left;

— if a horizontal line meets a dot and there is no vertical line coming from below, then this horizontal line goes up to the first not occupied level, where it turns right.

Let $H(M, N)$ be the number of vertical lines coming from the segment $[(0, N); (M, N)]$.



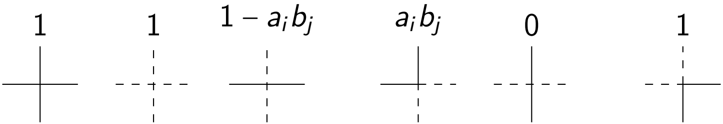
In this example $H(1,1) = 0$, $H(2,1) = 1$, $H(3,2) = 2$, $H(4,2) = 2$, $H(4,4) = 3$.

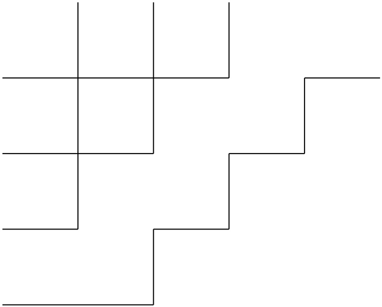
Proposition 4b: For a matrix $\{r_{ij}\}_{1 \le i \le M, 1 \le j \le N}$ let $\lambda(M,N)$ be a Young diagram constructed by (column insertion) RSK algorithm, and let $H(M,N)$ be defined through $\{\hat{r}_{ij}\}$ as discussed. Then

$$H(M,N) = \lambda'_1(M,N)$$

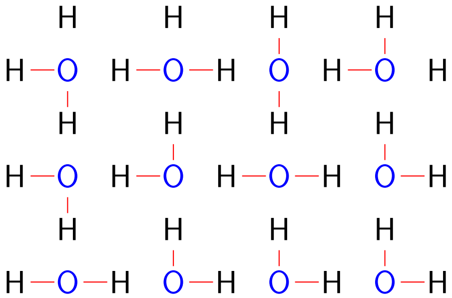
Once the (column) RSK is defined, the proof goes by exactly the same scheme...

Let us interpret this result probabilistically. Again, let r_{ij} be sampled by independent geometric distributions. We know that we have no dot at some point with probability $Prob(r_{ij} = 0) = (1 - a_i b_j)$ and we have a dot with probability $(a_i b_j)$. Therefore, this model can be interpreted as a result about *stochastic five-vertex model*...

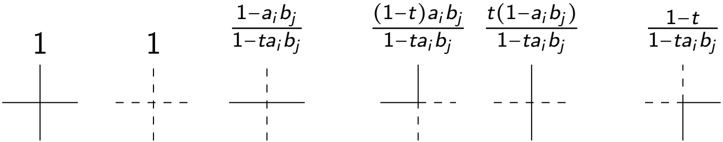

1
1
$1 - a_i b_j$
$a_i b_j$
0
1



Corollary of Proposition 4b: A height function of a stochastic five-vertex model $H(M,N)$ with weights has the same distribution as $\lambda_1'(M,N)$, where λ is distributed according to the Schur measure with parameters $a_1,\dots,a_M$, $b_1,\dots,b_N$.
One of weights of vertices was zero. What if all weights are non-zero ?
Six vertex models are of interest as models of statistical mechanics ("square ice").



We will consider one particular model: for $0 < t < 1$ let the weights have the form



This is a *stochastic six-vertex model* introduced by Gwa-Spohn'92, and recently studied in Borodin-Corwin-Gorin'14.

This is not a Schur process (we do not have determinants...). However, it can be analyzed via quite similar tools coming from the algebra of symmetric functions.

We need to use not Schur, but Hall-Littlewood functions.

Hall-Littlewood functions have a form

$$P_\lambda(x_1,\dots,x_N) := c(\lambda) \sum_{\sigma \in S_n} \sigma \left(\prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} x_1^{\lambda_1} \dots x_N^{\lambda_N} \right),$$

where σ permutes indices.

For $t = 0$ we have $P_\lambda = s_\lambda$, for $t = 1$ we have $P_\lambda = m_\lambda$.

$\{P_\lambda\}_{\lambda \in \mathbb{Y}}$ form a basis in symmetric functions.

One has formulas for linear expansions of $P_\mu P_{(r)}$ and $P_\mu P_{(1,\dots,1)}$ in linear combinations of P_λ's.

Combinatorial formula for P_λ — the sum is over interlacing arrays, but each interlacing array has a weight which depends on t.

$Q_\lambda := c_2(\lambda) P_\lambda$ — another version of Hall-Littlewood functions.

$P_{\lambda/\mu}$, $Q_{\lambda/\mu}$ — skew Hall-Littlewood functions.

Cauchy identity:

$$\sum_\lambda P_\lambda(x_1, \ldots, x_N) Q_\lambda(y_1, \ldots, y_N) = \prod_{i,j} \frac{1 - t x_i y_j}{1 - x_i y_j}$$

Identities for skew Hall-Littlewood functions.

Specializations

Hall-Littlewood positive specializations — classification is not known. Only conjecture:

For $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$, $\beta_1 \geq \beta_2 \geq \cdots \geq 0$, $\gamma > 0$ the specializations

$$p_1 \mapsto \sum_i \alpha_i + \frac{1}{1-t} \sum_i \beta_i + \gamma$$

$$p_k \mapsto \sum_i \alpha_i^k + \frac{(-1)^{k-1}}{1-t^k} \sum_i \beta_i^k.$$

are HL-positive. Kerov's conjecture: this list is exhaustive.

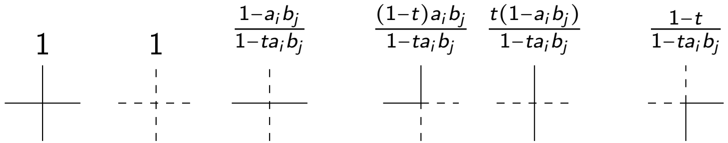
Hall-Littlewood measure: for two HL-positive specializations s_1 and s_2 we have

$$Prob(\lambda) \sim s_1(P_\lambda)s_2(P_\lambda), \qquad \lambda \in \mathbb{Y}.$$

In the case of alpha-specializations with parameters $a_1, \dots, a_M$, and $b_1, \dots, b_N$ it takes the form

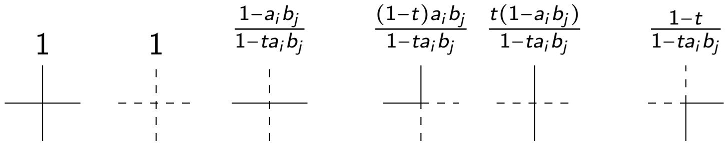
$$Prob(\lambda) = \prod_{i,j} \frac{1 - a_i b_j}{1 - t a_i b_j} P_\lambda(a_1, \dots, a_M) P_\lambda(b_1, \dots, b_N).$$

One can also similarly define Hall-Littlewood processes (subclass of Macdonald processes studied in Borodin-Corwin'11).



Fact: the height function $H(M,N)$ for a stochastic six vertex model with weights above is distributed as $\lambda_1'(M,N)$, where λ is distributed as Hall-Littlewood measure with parameters $a_1,\dots,a_M$, $b_1,\dots,b_N$.

Fact: More generally, for $M_1 \geq \dots \geq M_k$ and $N_1 \leq \dots \leq N_k$ the height functions $\{H(M_i,N_i)\}$ is distributed as first columns of diagrams from Hall-Littlewood process (proved in Borodin-Bufetov-Wheeler'16).



Fact: HL-RSK algorithm which proves these facts (Bufetov-Matveev'17+, in progress).

Recent RSK-algorithms for generalizations of Schur functions: O'Connell-Pei'12, Borodin-Petrov'13, Bufetov-Petrov'14, Matveev-Petrov'15.

The idea is to consider a random basic step, not a deterministic one.