Asymptotics of stochastic particle systems via Schur generating functions

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Object to study

- Discrete particle system: $l_1 > l_2 > \cdots > l_N$, $l_i \in \mathbb{Z}$.
- We consider a measure on $\mathbb{R}$:

$$m[l] := \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{l_i}{N}\right)$$

- If l_i are random, then $m[l]$ is a random probability measure on $\mathbb{R}$.
- We are interested in asymptotic behavior of $m[l]$ as $N \to \infty$. Concentration ? Gaussian fluctuations ?

- Given a finite-dimensional representation π of some group (e.g. $S(n), U(N), Sp(2N), SO(N)$) we can decompose it into irreducible components:

$$\pi = \bigoplus_{\lambda} c_\lambda \pi^\lambda,$$

where non-negative integers c_λ are multiplicities, and λ ranges over labels of irreducible representations.

- This decomposition can be identified with a probability measure ρ^π on labels

$$\rho^\pi(\lambda) := \frac{c_\lambda \dim(\pi^\lambda)}{\dim(\pi)}.$$

- Let π be an "interesting" sequence of representations. What are the probabilistic properties of ρ^π as the size of the group grows ? (Vershik-Kerov, 70's).
- Let π be a tensor product of irreducibles / a restriction of an irreducible representation. What are the probabilistic properties of ρ^π as the size of the group grows ? (Biane, 90's).
- One also needs to set a scaling of initial irreducibles.
- We are interested in the global properties of random signatures / Young diagrams.

Tensor products/restrictions

Symmetric group, balanced regime of growth: law of large numbers: Biane (1998); central limit theorem: Sniady (2005)

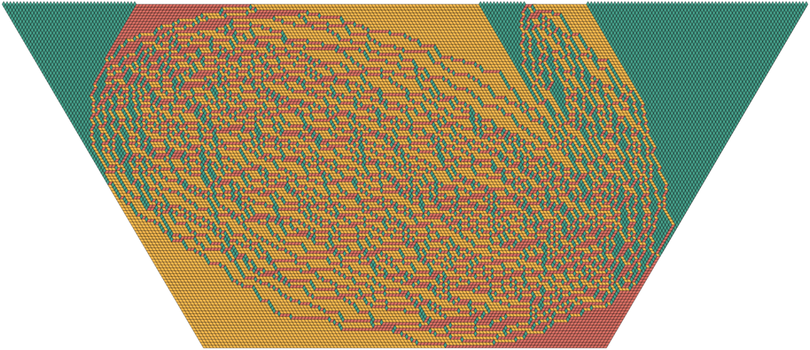
Unitary group, superlinear regime of growth: law of large numbers Biane (1995), central limit theorem Collins-Sniady (2009).

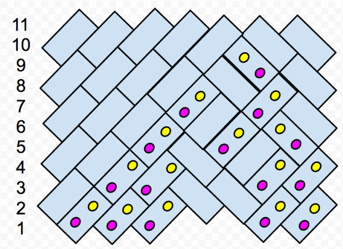
Motivation 1: LLN and CLT for a linear / balanced regime of growth for a *unitary group* ?

Analogous questions for other classical Lie groups — *symplectic group*, *orthogonal group* ?

Motivation 2: Models of statistical mechanics

- The combinatorial rule for the restrictions of representations of classical Lie groups is closely related to models of statistical mechanics, in particular, to rhombi and domino tilings of planar domains.
- The asymptotics of these models are related to random matrix asymptotics. Some tools are the same — determinantal point processes, orthogonal polynomials, variational principle. What is an analog of the moment method ?

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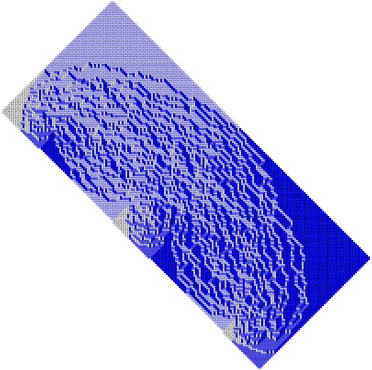


Back to particle systems

- As $N \to \infty$, we want to describe the convergence to a deterministic measure on $\mathbb{R}$

$$\frac{1}{N}\sum_{i=1}^{N}\delta\left(\frac{l_i}{N}\right) \to \mu,$$

and Gaussian fluctuations around the limit.

- Characteristic functions ??? N-tuples of integers are a dual object to the *unitary group* of dimension N. Motivation 3: Describe the convergence in terms of such characteristic functions.

Signatures and Schur functions

- A *signature* of length N is an N-tuple of integers $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$.
  For example, $\lambda = (5, 3, 3, 1, -2, -2)$ is a signature of length 6.
- $\mathrm{Sign}(N)$ — the set of all signatures of length N.
- The Schur function is defined by

$$s_\lambda(x_1, \ldots, x_N) := \frac{\det_{i,j=1,\ldots,N}\left(x_i^{\lambda_j+N-j}\right)}{\prod_{1\leq i<j\leq N}(x_i - x_j)},$$

where λ is a signature of length N. The Schur function is a Laurent polynomial in $x_1, \ldots, x_N$.

Representations of $U(N)$

- Let $U(N)$ denote the group of all $N \times N$ unitary matrices.
- It is known that all irreducible representations of $U(N)$ are parameterized by signatures ($=$ highest weights). Let π^λ be an irreducible representation of $U(N)$ corresponding to λ.
- The character of π^λ is a function on $U(N)$. Its value on all matrices with the same eigenvalues is the same.
- The values of the character of π^λ is the Schur function $s_\lambda(x_1, \dots, x_N)$, where x_i are eigenvalues of an element from $U(N)$.

Schur generating functions

$\mathrm{Sign}(N)$ — the set of all signatures of length N. Let $\rho(\lambda)$ be a probability measure on $\mathrm{Sign}(N)$.

$$\Phi(x_1, \dots, x_N) := \sum_{\lambda \in \mathrm{Sign}(N)} \rho(\lambda) \frac{s_\lambda(x_1, \dots, x_N)}{s_\lambda(1, 1, \dots, 1)}.$$

Definition: $\Phi(x_1, \dots, x_N)$ is the *Schur generating function* of ρ. Note that $\Phi(1, \dots, 1) = 1$.

Claim: A Schur generating function is a good analog of a characteristic function.

Analogs of cumulants

$\rho_N(\lambda)$ — sequence of probability measures on $\mathrm{Sign}(N)$;
$\Phi_N(x_1, \ldots, x_N)$ — sequence of its Schur generating functions.
We are interested in the asymptotic behavior of the random signature distributed according to ρ_N as $N \to \infty$.

- $\partial_{x_1}^k \log \Phi_N(x_1, \ldots, x_N)\big|_{x_i=1}$ are analogs of the first cumulant (for any k),
- $\partial_{x_1}^k \partial_{x_2}^l \log \Phi_N(x_1, \ldots, x_N)\big|_{x_i=1}$ are analogs of the second cumulant,
- $\partial_{x_1}^k \partial_{x_2}^l \partial_{x_3}^m \log \Phi_N(x_1, \ldots, x_N)\big|_{x_i=1}$ are analogs of the third cumulant, etc.

Description of results

$N \to \infty$.

- Law of Large Numbers+Central Limit Theorem: "first cumulants" are of order N, ""second cumulants" tend to constants, "order 3 and higher cumulants" tend to 0..
- These conditions about cumulants are equivalent to LLN + CLT.
- Analogous multi-dimensional results.

Bufetov-Gorin (2013, 2016, 2017+).

Measures related to signatures

It will be convenient for us to encode a signature λ by the *counting measure* $m[\lambda]$:

$$m[\lambda] := \frac{1}{N} \sum_{i=1}^{N} \delta \left(\frac{\lambda_i + N - i}{N} \right).$$

$\rho_N(\lambda)$ is a probability measure on $\mathrm{Sign}(N)$. Consider

$$m[\rho] := \frac{1}{N} \sum_{i=1}^{N} \delta \left(\frac{\lambda_i + N - i}{N} \right),$$

where λ is distributed according to $\rho_N(\lambda)$; note that this is a **random probability measure** on $\mathbb{R}$.

LLN (Bufetov-Gorin'13)

$\Phi_N(x_1, \dots, x_N)$ are Schur generating functions of $\rho_N(\lambda)$.
Assume

$$\lim_{N\to\infty} \frac{1}{N} \partial_{x_1} \log \Phi_N(x_1, \dots, x_k, 1^{N-k}) = F(x_1),$$

$F(x)$ — analytic function; convergence is uniform in a neighborhood of $(x_1, x_2, \dots, x_k) = (1^k)$; k is fixed.
Then the random measures $m[\rho_N]$ converge as $N \to \infty$ to a *deterministic* measure μ on $\mathbb{R}$, such that

$$\int_{\mathbb{R}} x^k d\mu(x) = \frac{1}{(2\pi\mathbf{i})} \oint_{|z|=\epsilon} \frac{dz}{1+z} \left(\frac{1}{z} + 1 + (1+z)F(1+z) \right)^{k+1},$$

where $\epsilon \ll 1$. Convergence in probability, in the sense of moments.

CLT (Bufetov-Gorin'16)

Let $p_k := \int_{\mathbb{R}} x^k dm[\rho_N](x)$ be the kth moment of $m[\rho_N]$.
Assume additionally

$$\lim_{N\to\infty} \partial_{x_1}\partial_{x_2} \log \Phi_N(x_1, \ldots, x_k, 1^{N-k}) = G(x_1, x_2),$$

G(x,y) — analytic function; convergence is uniform in a neighborhood of $(x_1, x_2, \ldots, x_k) = (1^k)$.
Then the collection

$$\{N(p_k - \mathbf{E}p_k)\}_{k\in\mathbb{N}}$$

converges, as $N \to \infty$, in the sense of moments, to the Gaussian vector with zero mean and covariance ....

Formula for covariance

$$\frac{1}{(2\pi\mathbf{i})^2} \oint_{|z|=\epsilon} \oint_{|w|=2\epsilon} \mathcal{F}^{k_1}(z)\mathcal{F}(w)^{k_2} \times \left(G(1+z, 1+w) + \frac{1}{(z-w)^2} \right) dzdw,$$

with

$$\mathcal{F}(z) := \frac{1}{z} + 1 + (1+z)F(1+z).$$

Further results

$\Phi_N(x_1, \ldots, x_N) := \mathbf{E}(s_\lambda(x_1, \ldots, x_N)/s_\lambda(1, \ldots, 1))$. For a probability measure on $\mathrm{Sign}(N) \times \mathrm{Sign}(M)$ a two-dimensional analog:

$$\mathbf{E} \frac{s_\lambda(x_1, \ldots, x_N)}{s_\lambda(1, \ldots, 1)} \frac{s_\mu(y_1, \ldots, y_M)}{s_\mu(1, \ldots, 1)}.$$

- Multi-dimensional analogs of CLT (Bufetov-Gorin'17+, in progress).
- LLN and CLT imply the asymptotics of the logarithm of Schur generating functions — inversion of the results formulated above (Bufetov-Gorin'17+, in progress).

Applications

Where can we compute asymptotics of a Schur generating function ???

- Tensor products of irreducible representations of classical Lie groups.
- Models of lozenge tilings without and with additional symmetries.
- domino tilings.
- more general Schur processes and their variations.
- ensembles of non-intersecting paths.
- probabilistic models related to extreme characters of infinite-dimensional groups.
- Other applications ?

Applications: Tensor products

- Given a finite-dimensional representation π of $U(N)$ we can decompose it into irreducible components:

$$\pi = \bigoplus_{\lambda} c_\lambda \pi^\lambda,$$

where non-negative integers c_λ are multiplicities.

- This decomposition can be identified with a probability measure ρ^π on signatures of length N such that

$$\rho^\pi(\lambda) := \frac{c_\lambda \dim(\pi^\lambda)}{\dim(\pi)}.$$

Tensor product

- Let λ and μ be signatures of length N. π^λ and π^μ — irreducible representations of $U(N)$.
- We consider the decomposition of the (Kronecker) tensor product $\pi^\lambda \otimes \pi^\mu$ into irreducible components

$$\pi^\lambda \otimes \pi^\mu = \bigoplus_\eta c_\eta^{\lambda,\mu} \pi^\eta,$$

where η runs over signatures of length N.

- Assume that two sequences of signatures $\lambda = \lambda(N)$ and $\mu = \mu(N)$ satisfy

$$m[\lambda] \xrightarrow[N\to\infty]{} m_1, \qquad m[\mu] \xrightarrow[N\to\infty]{} m_2, \qquad \text{weak convergence,}$$

  where m_1 and m_2 are probability measures. For example, $\lambda_1 = \cdots = \lambda_{[N/2]} = N$, $\lambda_{[N/2]+1} = \cdots = \lambda_N = 0$, or $\lambda_i = N - i$, for $i = 1, 2, \ldots, N$.
- We are interested in the **asymptotic behaviour** of the decomposition of the tensor product into irreducibles, i.e., we are interested in the asymptotic behaviour of the random probability measure $m[\rho^{\pi^\lambda \otimes \pi^\mu}]$.

Limit results for tensor products

Under assumptions above, we have (Bufetov-Gorin'13, Bufetov-Gorin'16):

- Law of Large Numbers:

$$\lim_{N \to \infty} m[\pi^{\lambda} \otimes \pi^{\mu}] = m_1 \otimes m_2,$$

  where $m_1 \otimes m_2$ is a deterministic measure on $\mathbb{R}$. We call $m_1 \otimes m_2$ the *quantized free convolution* of measures m_1 and m_2.
- Central Limit Theorem: Fluctuations around the limit measure are Gaussian and given by an explicit double contour integral formula.

Random matrices

- Let A be a $N \times N$ Hermitian matrix with eigenvalues $\{a_i\}_{i=1}^N$. Let
$$m[A] := \frac{1}{N} \sum_{i=1}^{N} \delta(a_i)$$
be the *empirical* measure of A.
- For each $N = 1, 2, \ldots$ take two sets of real numbers $a(N) = \{a_i(N)\}_{i=1}^N$ and $b(N) = \{b_i(N)\}_{i=1}^N$.
- Let $\mathcal{A}(N)$ be the uniformly (= Haar distributed) random $N \times N$ Hermitian matrix with fixed eigenvalues $a(N)$ and let $\mathcal{B}(N)$ be the uniformly (= Haar distributed) random $N \times N$ Hermitian matrix with fixed eigenvalues $b(N)$ such that $\mathcal{A}(N)$ and $\mathcal{B}(N)$ are independent.

Limit of sums of random matrices

Suppose that as $N \to \infty$ the empirical measures of $\mathcal{A}(N)$ and $\mathcal{B}(N)$ weakly converge to probability measures $\mathbf{m}^1$ and $\mathbf{m}^2$, respectively.

- (Voiculescu'91) The random empirical measure of the sum $\mathcal{A}(N) + \mathcal{B}(N)$ converges (weak convergence; in probability) to a *deterministic* measure $\mathbf{m}^1 \boxplus \mathbf{m}^2$ which is the free convolution of $\mathbf{m}^1$ and $\mathbf{m}^2$.
- Fluctuations around the limit are Gaussian and given by an explicit formula (Pastur-Vasilchuk'06); second order freeness (Mingo-Speicher'04, Mingo-Sniady-Speicher'04, Collins-Mingo-Sniady-Speicher'06)

Related results

- In the case of the symmetric group similar results were obtained by Biane (1998), Sniady (2005).
- In our situation coordinates of signatures λ and μ grow linearly in N. The situation when this growth is superlinear was considered by Biane (1995), and Collins-Sniady (2009). The resulting operation on measures is the conventional free convolution; fluctuations also coincide with random matrix case.
- LLN for *Perelomov-Popov measures* (Bufetov-Gorin'13); again, it is given by a conventional free convolution. Free independence: Collins-Novak-Sniady'16.

More examples: Extreme characters of $U(\infty)$: Edrei-Voiculescu theorem

$$\alpha^{\pm} = \alpha_1^{\pm} \geq \alpha_2^{\pm} \geq \cdots \geq 0, \qquad \beta^{\pm} = \beta_1^{\pm} \geq \beta_2^{\pm} \geq \cdots \geq 0,$$

$$\delta^{\pm} \geq 0, \quad \sum_{i=1}^{\infty} (\alpha_i^{\pm} + \beta_i^{\pm}) \leq \delta^{\pm}, \quad \beta_1^{+} + \beta_1^{-} \leq 1.$$

$$\Phi^{\omega}(x) = \exp(\gamma^{+}(x-1) + \gamma^{-}(x^{-1}-1)) \prod_{i=1}^{\infty} \frac{(1+\beta_i^{+}(x-1))}{(1-\alpha_i^{+}(x-1))} \times \frac{(1+\beta_i^{-}(x^{-1}-1))}{(1-\alpha_i^{-}(x^{-1}-1))},$$

$$\chi^{\omega}(U) := \prod_{x \in \mathrm{Spectrum}(U)} \Phi^{\omega}(x), \qquad U \in U(\infty),$$

A Schur generating function

$$\left(\Phi^{\omega}(x_1)\Phi^{\omega}(x_2)\dots\Phi^{\omega}(x_N)\right)^N$$

satisfies our conditions. Therefore, we know LLN and CLT for such restrictions of extreme characters (LLN was done in Borodin-Bufetov-Olshanski'13).

Schur-Weyl measure

More examples: Schur-Weyl measure

$$\Phi(x_1, \ldots, x_N) = \frac{(x_1 + \cdots + x_N)^{cN^2}}{N^{cN^2}}, \qquad c > 0$$

LLN: Biane'00, CLT: Meliot'10.

One can consider "multidimensional versions".

Schur processes

More examples:

$$\sum_{\lambda} s_\lambda(x_1, \ldots, x_N) s_\lambda(y_1, \ldots, y_N) = \prod_{i=1}^{N} \prod_{j=1}^{N} \frac{1}{1 - x_i y_j}.$$

If $P(\lambda) \sim s_\lambda(1, \ldots, 1) s_\lambda(\phi)$ the Schur generating function can be computed. Thus, we automatically know global LLN and CLT.

Schur measures (Okounkov'00) and processes (Okounkov-Reshetikhin'01).

More examples: One can use other facts for Schur functions:

$$\sum_{\lambda:\lambda' \text{ is even}} s_\lambda(x_1, \ldots, x_N) = \prod_{i=1}^{N} \frac{1}{1 - x_i x_j},$$

$$\sum_{\lambda} s_\lambda(x_1, \ldots, x_N) = \prod_{i=1}^{N} \frac{1}{1 - x_i} \prod_{i=1}^{N} \frac{1}{1 - x_i x_j}$$

λ is summed over partitions with all even columns in the first formula, and over all partitions in the second one.

$P(\lambda) \sim s_\lambda(a, \ldots, a)$, for $a < 1$; number of a's tends to infinity.

Pfaffian Schur processes (Sasamoto-Imamura'04, Borodin-Rains'05).

Restriction of π^λ

Let λ be a signature of length N. Let us restrict π^λ to $U(N-1)$:

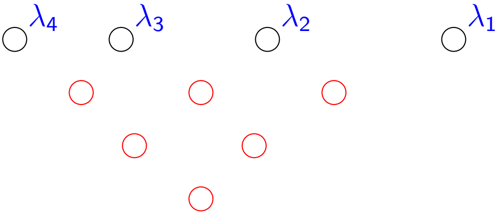
$$\pi^\lambda\Big|_{U(N-1)} = \bigoplus_{\mu \prec \lambda} \pi^\mu,$$

where $\mu \prec \lambda$ means that they *interlace*:

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{N-1} \geq \mu_{N-1} \geq \lambda_N.$$

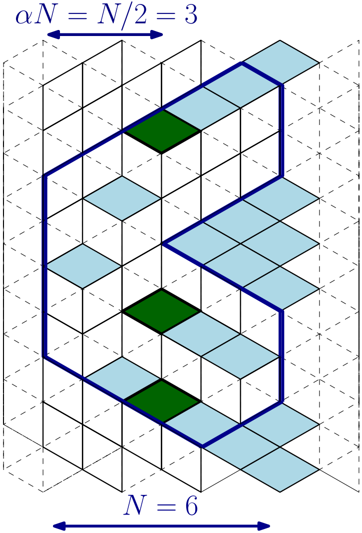
Gelfand-Tsetlin arrays

Restricting π^λ to $U(M)$, for $M < N$, we obtain the following picture:

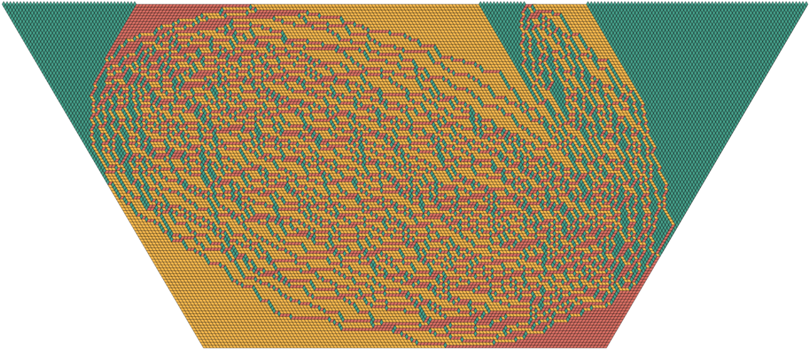


For large N we consider uniformly random Gelfand-Tsetlin arrays with fixed upper row λ. What is the behavior of the signature on level $[\alpha N]$, $0 < \alpha < 1$. ?

Projections and random tilings



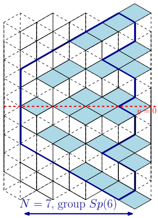
Projections and random tilings

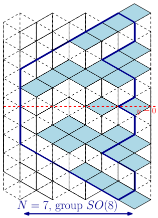


Application of results

- LLN implies the limit shape phenomenon for uniform random lozenge tilings of certain polygons. The existence of the limit shape was known (Cohn-Kenyon-Propp (2001), Kenyon-Okounkov-Sheffield (2006), Kenyon-Okounkov (2007)).
- CLT (Bufetov-Gorin'16) extends the result of Petrov'12. The covariance can be written in terms of a *Gaussian Free Field* (an idea of such a description first push forward by Kenyon).
- Connection of tilings with free probability.

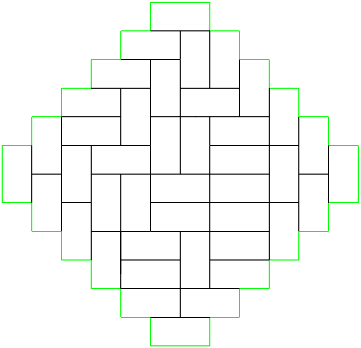
Tilings with symmetries

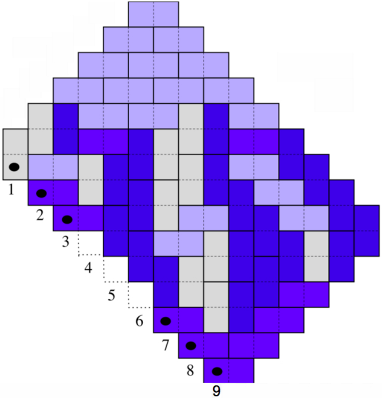


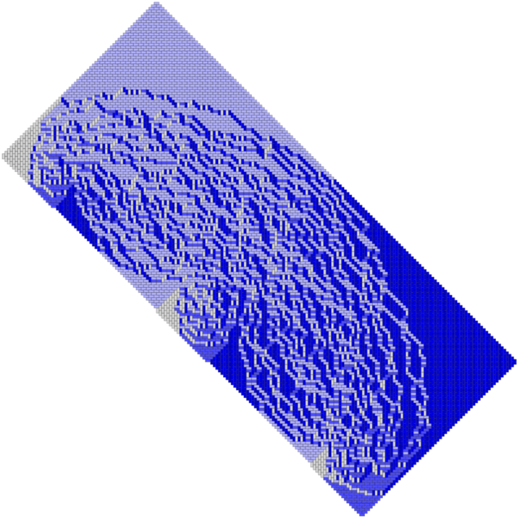


- In a similar way, branching rules for symplectic and orthogonal groups give results for certain symmetric tilings.
- Another model of symmetric tilings was studied by Panova (2014).

Domino tilings

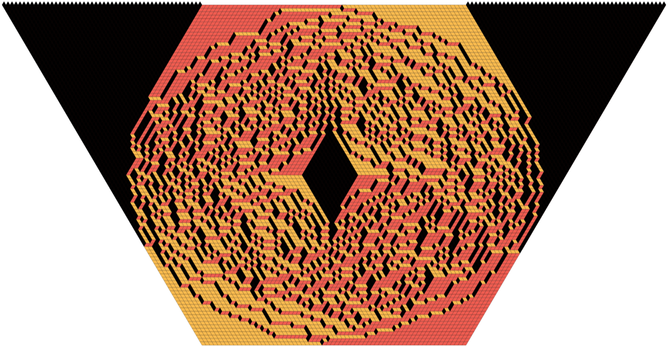






Domino tilings

- Concentration phenomenon for general domains: Cohn-Kenyon-Propp'01.
- The most studied example: Aztec diamond; Jochush-Propp-Shor (1995), Johansson (2003), Chhita-Johansson-Young (2012).
- Bufetov-Knizel'16: rectangular Aztec diamonds: arbitrary boundary conditions on one of the sides. Limit shapes, explicit formulae for frozen boundary curves, central limit theorem.



(Figure by L. Petrov)

Bufetov-Gorin'17+ combined with Borodin-Gorin-Guionnet'15: LLN+CLT for lozenge tilings of domains with holes.

- An analog of a characteristic function for random discrete measures — a Schur generating function. General results.
- Applications: Tensor products of irreducible representations of classical Lie groups and other measures related to Schur functions. Connection with random matrices and free probability.
- Applications: random lozenge and domino tilings.

A. Bufetov, V. Gorin, *Representations of classical Lie groups and quantized free convolution*, `arXiv:1311.5780`.
A. Bufetov, V. Gorin, *Fluctuations of particle systems determined by Schur generating functions*, `arXiv:1604.01110`.
A. Bufetov, A. Knizel, *Asymptotics of random domino tilings of rectangular Aztec diamonds*, `arXiv:1604.01491`