Functional equations in number theory and André's reflection principle

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Sommaire

Introduction

Rank 1 example: Let us explain the relationship between

- André's reflection principle:

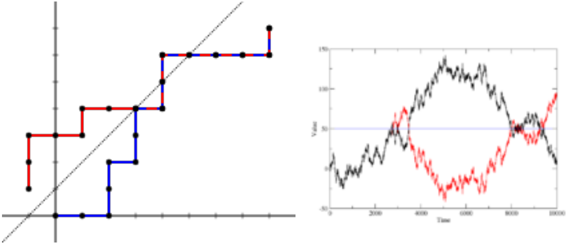

Figure: André's reflection principle (src:Wikipedia)

- The functional equation in number theory

$$\forall s \in \mathbb{C}, \Lambda(s) = \Lambda(1-s)$$

where

$$\begin{cases} \Lambda(s) & = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \\ \zeta(s) & = \sum_{n=1}^{\infty} \frac{1}{n^s} \end{cases}$$

André's reflection principle

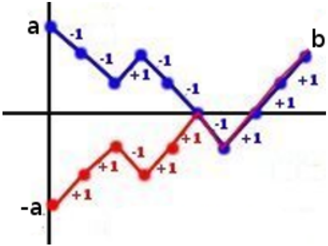

Figure: André's reflection principle

Let $p(n, a, b)$ be the number of simple walks from a to b, constrained on remaining positive. $q(n, a, b)$ the number of *unconstrained* walks. The reflection principle states:

$$p(n, a, b) = q(n, a, b) - q(n, -a, b) = \binom{n}{\frac{n+b-a}{2}} - \binom{n}{\frac{n+b+a}{2}}$$

(Asymptotic variant of) André's reflection principle

Let $z > 1$ and $W^{(z)}$ be SRW such that:

$$\forall t,\ \mathbb{P}\left(W^{(z)}_{t+1} - W^{(z)}_t = 1\right) = 1 - \mathbb{P}\left(W^{(z)}_{t+1} - W^{(z)}_t = -1\right) = \frac{z}{z + z^{-1}} > \frac{1}{2}$$

and killed upon exiting $\mathbb{N}$.

Lemma

$$\mathbb{P}_x\left(W^{(z)} \ \textit{survives}\ \right) = 1 - z^{-2(x+1)}$$

Corollary (Probabilistic representation of SL_2-characters)

$$s_{(x)}\left(z, z^{-1}\right) := \frac{z^{x+1} - z^{-(x+1)}}{z - z^{-1}} = \frac{z^{x+1}}{z - z^{-1}} \mathbb{P}_x\left(W^{(z)} \ \textit{survives}\ \right)$$

where s_λ is the Schur function associated to the shape λ.

(Variant of) André's reflection principle - Geometric case

Let $\mu > 0$ and $W^{(\mu)}$ be the sub-Markovian process on $\mathbb{R}$ with generator:

$$\mathcal{L}^{(\mu)} = \frac{1}{2}\partial_x^2 + \mu\partial_x - 2e^{-2x}$$

i.e Brownian motion with drift μ and killing measure by a potential $V(x) = 2e^{-2x}$.

Proposition (Probabilistic representation of the K-Bessel/Whittaker function)

If

$$K_\mu(y) = \int_0^\infty \frac{dt}{t} t^\mu e^{-y\left(t+t^{-1}\right)}$$

then

$$K_\mu\left(e^{-2x}\right) = \Gamma(\mu)e^{\mu x}\mathbb{P}_x\left(W^{(\mu)} \text{ survives }\right)$$

Remark: Previous work shows that $K_\mu\left(e^{-2x}\right)$ plays the role of characters associated to a SL_2-geometric crystals. In all cases, we have symmetric and analytic extension of survival probabilities.

Functional equation I: The classical Eisenstein series

$\mathbb{H} = \{\Im z > 0\}$; $g \cdot z = \frac{az+b}{cz+d}$ usual action of SL_2 on $\mathbb{H}$.

Non-holomorphic Eisentein serie defined for $\Re(s) > 2$ and $z = x + iy \in \mathbb{H}$:

$$E(z;s) = \frac{1}{2} \sum_{(m,n)\ \text{coprime}} \frac{y^s}{(mz+n)^s}$$

$$E^*(z;s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z;s) \qquad \text{(Normalization)}$$

Properties:

- Automorphic: $\forall g \in SL_2(\mathbb{Z})$, $E^*(g \cdot z; s) = E^*(z;s)$.
- 1-periodicity in $x \rightsquigarrow$ Expansion in Fourier-Whittaker coefficients:

$$E^*(z = x+iy; s) = \sum_{n\in\mathbb{Z}} A_n(y;s) e^{i2\pi xn} \ .$$

Functional equation II: Computing coefficients

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$$A_n(y; s) = \begin{cases} * & \text{if } n = 0 \\ y^{\frac{1}{2}} \sigma_{s-\frac{1}{2}}(n) K_{s-\frac{1}{2}}(\pi |n| y) & \text{if } n \neq 0 \end{cases}$$

where

$$\begin{aligned} K_s(y) = & \quad \int_0^\infty \frac{dt}{t} t^s e^{-y\left(t+t^{-1}\right)} && \text{(Bessel K-function)} \\ \sigma_s(n) = & \quad \sum_{d|n} \left(\frac{n}{d^2}\right)^s && \text{(Normalized divisor function)} \end{aligned}$$

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Symmetries of A_1-type:

$$\forall s \in \mathbb{C}, \;\; K_s(y) = K_{-s}(y) \; , \sigma_s(n) = \sigma_{-s}(n)$$

$$\rightsquigarrow \text{(Functional equation)} \quad E^*(s; z) - E^*(1-s; z) = 0$$

Relationship

An important remark: σ_s is weakly multiplicative

$$\sigma_s(n) = \prod_{p|n} \sigma_s\left(p^{\nu_p(n)}\right)$$

$$\sigma_s\left(p^k\right) = s_k\left(p^s, p^{-s}\right)$$

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As such, the functional equations:

$$E^*(s;z) = E^*(1-s;z); \qquad \Lambda(s) = \Lambda(1-s) \quad \text{(Constant term *)}$$

are implied by the A_1-symmetry in:

$K_s(\cdot)$	The Archimedean Whittaker function
$s_k\left(p^s, p^{-s}\right)$	The non-Archimedean Whittaker function at the prime p

Taking the problem backwards: What if these functions were expressed as survival probabilities from the start? Then
Functional equations + Analytic continuation $\Leftrightarrow$ André's reflection principle

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Notations

For concreteness only: GL_n instead of any (split) Chevalley-Steinberg group. Also $\mathbb{F}_q((T))$ instead of a non-Archimedean local field.

- $\mathcal{K} = \mathbb{F}_q((T))$: field of Laurent series in the formal variable T with coefficients in the finite field $\mathbb{F}_q$. If $x \in \mathcal{K}$, then $\mathrm{val}(x)$ is the index of the first non-zero monomial. If $k = \mathrm{val}(x)$:
$$x = a_k T^k + a_{k+1} T^{k+1} + \dots$$
with $a_k \neq 0$. The ring of integers is made of elements of non-negative valuation and denoted $\mathcal{O} = \mathbb{F}_q[[T]]$.
- $G = GL_n(\mathcal{K})$: group of $\mathcal{K}$-points.
- $K = GL_n(\mathcal{O})$: maximal compact (open) subgroup.
- $A = \left\{ T^{-\mu} = diag(T^{\mu_1}, T^{\mu_2}, \dots, T^{\mu_n}) \mid \mu \in \mathbb{Z}^n \right\} \approx \mathbb{Z}^n$
- $A_+ = \left\{ T^{-\lambda} \mid \lambda_1 \geq \lambda_2 \dots \geq \lambda_n \right\} \approx \mathbb{N}^n$
- $N(\mathcal{K})$ is the unipotent subgroup of lower triangular matrices.

Facts:
$$G = NAK \quad \text{(Iwasawa or "Gram-Schmidt")}$$

Number theory facts

- (Langlands, Shahidi) Eisenstein series generalize from $\mathbb{H} = SL_2(\mathbb{R})/SO_2(\mathbb{R})$ to other Lie groups
  $\rightsquigarrow$ Functions on G/K , general symmetric spaces.

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$$\Phi_z(g) := e^{\langle z, \mu \rangle} \text{ when } g \in NT^\mu K \text{ (“Spherical vector”)}$$

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- (Jacquet's thesis, 1967) Fourier-Whittaker coefficients of such generalized Eisenstein series are given by the Whittaker function $\mathcal{W}_z$. Let $\varphi_N : N \to (\mathbb{C}^*, \times)$ be a character trivial on integer points and w_0 is the permutation matrix $(1n)\,(2(n-1))\,(3(n-2))\ldots$.

$$\mathcal{W}_z(g) := \int_N \Phi_z\left(w_0 ng\right) \varphi_N(n)^{-1} dn$$

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Important message: As in rank 1, the functional equations for Eisenstein series are implied by the $W = S_n$-symmetry after a normalization $\widetilde{\mathcal{W}_z}$

$$\forall \sigma \in S_n, \forall g \in G, \forall z \in \mathbb{R}^n, \widetilde{\mathcal{W}_{\sigma z}}(g) = \widetilde{\mathcal{W}_z}(g)$$

The Shintani-Casselman-Shalika formula

Recall:

$$\mathcal{W}_z(g) = \int_N \Phi_z\left(w_0 ng\right)\varphi_N(n)^{-1}dn$$

The Whittaker function is essentially a function on $A \approx \mathbb{Z}^n$:

$$\forall (n, T^{-\lambda}, k) \in N \times A \times K,\ \mathcal{W}_z\left(nT^{-\lambda}k\right) = \varphi_N(n)\mathcal{W}_z(T^{-\lambda})$$

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Theorem (Shintani 76 for GL_n, Casselman-Shalika 81 for G general)

$$\mathcal{W}_z\left(T^{-\lambda}\right) = \begin{cases} e^{\frac{1}{2}\sum i\lambda_i} \prod_{i<j}\left(1 - q^{-1}e^{z_i - z_j}\right) s_\lambda(z) & \text{if } \lambda \text{ dominant,} \\ 0 & \text{otherwise.} \end{cases}$$

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Step 1: A crucial remark

There is a seemingly innocent remark:

Remark (Gelfand-Graev-Piatetski-Shapiro 1969)

If $\mathcal{H}$ is a K bi-invariant measure on G, then:

$$\mathcal{W}_z \star \mathcal{H} = \mathcal{S}(\mathcal{H})(z)\mathcal{W}_z \ , \quad \textit{for a certain } \mathcal{S}(\mathcal{H})(z) \in \mathbb{C}$$

.

"The Whittaker function is a convolution eigenfunction" $\rightsquigarrow$ Harmonicity for a probabilist!

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Proposition

There exists a left-invariant random walk $\left(B_t\left(W^{(z)}\right); t \geq 0\right)$ on NA such that

- *(Harmonicity) A simple transform of $\mathcal{W}_z$ is harmonic for $\left(B_t\left(W^{(z)}\right); t \geq 0\right)$.*
- *$B_t\left(W^{(z)}\right) = N_t\left(W^{(z)}\right) T^{-W_t^{(z)}}$ and $W^{(z)}$ is the lattice walk on $\mathbb{N}^n$ with:*

$$\mathbb{P}\left(W_{t+1}^{(z)} - W_t^{(z)} = e_i\right) = \frac{z_i}{\sum_j z_j}$$

- *The N-part of the increments are Haar distributed on $N(\mathcal{O})$.*
- *(Exit law) $N_t\left(W^{(z)}\right) \stackrel{t\to\infty}{\longrightarrow} N_\infty\left(W^{(z)}\right)$*

Step 1: GL_2 example

Consider $\left(W^1, W^2\right)$ a simple random walk on $\mathbb{N}^2$ by steps $(1,0)$ and $(0,1)$.

$$W_t^{(z)} = W_t^1 - W_t^2, \quad \mathrm{SRW}$$

Form the random walk on $GL_2(\mathcal{K})$:

$$B_{t+1}(W) = B_t(W) \begin{pmatrix} T^{-(W_{t+1}^1 - W_t^1)} & 0 \\ U_t & T^{-(W_{t+1}^2 - W_t^2)} \end{pmatrix}$$

where $(U_t; t \geq 0)$ are i.i.d Haar distributed on $\mathcal{O}$. A simple computation gives that:

$$B_t(W) = \begin{pmatrix} T^{-W_t^1} & 0 \\ T^{-W_t^1} \sum_{s=0}^t T^{-(W_s^2 - W_s^1)} U_s & T^{-W_t^2} \end{pmatrix},$$

hence

$$N_\infty\left(W^{(z)}\right) = \begin{pmatrix} 1 & 0 \\ \sum_{s=0}^\infty T^{W_s^{(z)}} U_s & 1 \end{pmatrix}.$$

Step 2: Trivial key lemma

Imperfect relation to tropical calculus:

$$\text{val}\left(x+y\right) \geq \min\left(\text{val}(x), \text{val}(y)\right)$$

The perfect marriage of probability and non-Archimedean fields appears in:

Lemma ("The trivial key lemma")

If U and U' are independent and Haar distributed on $\mathcal{O}$, then for any two integers a and b in $\mathbb{Z}$:

$$T^a U + T^b U' \stackrel{\mathcal{L}}{=} T^{\min(a,b)} U$$

In a sense, non-Archimedean Haar random variables are "tropical calculus friendly". By this lemma:

$$N_\infty\left(W^{(z)}\right) = \begin{pmatrix} 1 & 0 \\ \sum_{s=0}^{\infty} T^{W_s^{(z)}} U_s & 1 \end{pmatrix} \stackrel{\mathcal{L}}{=} \begin{pmatrix} 1 & 0 \\ T^{\min_{0\leq s} W_s^{(z)}} U & 1 \end{pmatrix}$$

and U Haar distributed on $\mathcal{O}$.

Step 3: Poisson formula

$$\begin{aligned}\mathcal{W}_z(T^{-\lambda}) = &\quad e^{\frac{1}{2}\sum i\lambda_i}\prod_{i<j}\left(1-q^{-1}e^{z_i-z_j}\right)e^{\langle z,\lambda\rangle}\mathbb{E}\left(\varphi_N\left(T^{-\lambda}N_\infty(W^{(z)})T^{\lambda}\right)\right)\\ \sim &\quad e^{\langle z,\lambda\rangle}\mathbb{E}\left(\varphi_N\left(T^{-\lambda}N_\infty(W^{(z)})T^{\lambda}\right)\right)\end{aligned}$$

Factor $\varphi_N = \psi\circ\varphi$ as the composition of two characters. Here φ satisfies $\varphi\left(\mathrm{id}+tE_{i+1,i}\right)=t$ and $\psi:(\mathcal{K},+)\to(\mathbb{C}^*,\times)$.

$$\begin{aligned}\mathcal{W}_z(T^{-\lambda}) &\overset{\text{Def of }\varphi_N}{=} e^{\langle z,\lambda\rangle}\mathbb{E}\left[\prod_{i=1}^{n-1}\psi\left(T^{\lambda_i-\lambda_{i+1}+\min_{0\leq s}W_s^i-W_s^{i+1}}U_i\right)\right]\\ &\overset{\text{Average }U_i}{=} e^{\langle z,\lambda\rangle}\mathbb{E}\left[\prod_{i=1}^{n-1}\mathbb{1}_{\{\lambda_i-\lambda_{i+1}+\min_{0\leq s}W_s^i-W_s^{i+1}\geq 0\}}\right]\\ &= \quad e^{\langle z,\lambda\rangle}\mathbb{P}\left(\lambda+W^{(z)}\text{ remains in }C\right)\end{aligned}$$

Step 4: Reflection principle (1)

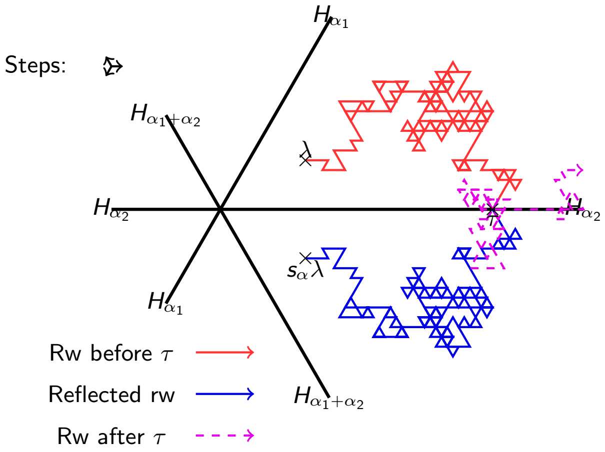
Theorem (Reflection principle)

We have:

$$\mathbb{P}\left(\lambda + W^{(z)} \textit{ remains in } C\right) = \sum_{w \in S_n} (-1)^{\ell(w)} e^{\langle w(\lambda+\rho)-(\lambda+\rho), z\rangle}$$

Step 4: Reflection principle (2)

Figure: Illustration of the reflection principle for the A_2-type weight lattice



Shintani-Casselman-Shalika formula in short

Theorem

Non-Archimedean Whittaker functions are proportional to Schur functions.

Sketch of probabilistic proof.

- Non-Archimedean Whittaker functions are harmonic for certain random walks $\left(B_t\left(W^{(z)}\right), t \geq 0\right)$ on $GL_n(\mathcal{K})$, driven by a lattice walk $W^{(z)}$ on $\mathbb{Z}^n$ (with drift).
- Poisson formula (not the summation one!): The Whittaker function is an expectation of the random walks' exit law.
- The stationary distribution is made of many sums which can be simplified thanks to the "Trivial key lemma", giving infinimas.
- Integrate out the Haar random variables. Get indicator functions. Whittaker function is proportional to the probability of a lattice walk W staying inside the Weyl chamber.
- The reflection principle gives a determinant. This determinant is the Schur function.

□

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Acknowledgments

Thank you for your attention!