Hook formulas for skew shapes and asymtptotics of skew SYTs

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Symmetric group S_n
Irreps $\mathbb{S}_\lambda$, $\lambda \vdash n$

$$Tr_{\mathbb{S}_\lambda}[\pi] = \chi^\lambda(\pi)$$

General linear group GL_N
V_λ, $\ell(\lambda) \leq N$

$$Tr_{V_\lambda}(diag(x_1, \ldots)) = s_\lambda(x_1, x_2, \ldots)$$

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2	6	10		
5	8			

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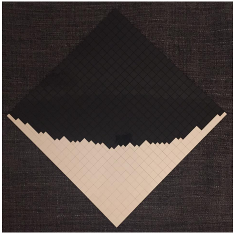

(Lego art by Dan Betea)

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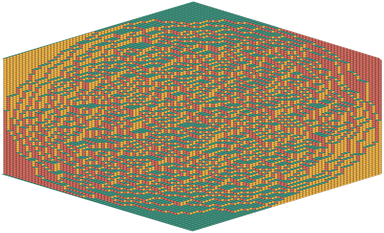
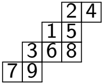


(Computer art by Leonid Petrov)

Counting skew SYTs

Outer shape λ, inner shape μ, e.g. for $\lambda = (5,4,4,2), \mu = (3,2,1)$



Jacobi-Trudi[Feit 1953]:

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{\ell(\lambda)}.$$

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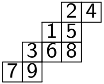

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Littlewood-Richardson:

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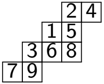

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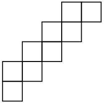
No product formula, e.g. $\lambda/\mu = \delta_{n+2}/\delta_n$:



$f^{\delta_{n+2}/\delta_n} = E_{2n+1}$:

$$1 + E_1 x + E_2 \frac{x^2}{2!} + E_3 \frac{x^3}{3!} + E_4 \frac{x^4}{4!} + \ldots = \sec(x) + \tan(x).$$

Euler numbers: $2, 5, 16, 61 \ldots.$

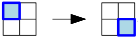
Hook-Length formula for skew shapes

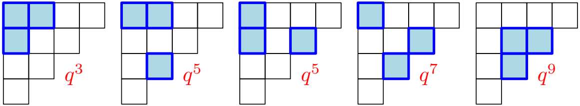
Theorem (Naruse, 2014)

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)},$$

where $\mathcal{E}(\lambda/\mu)$ *is the set of* excited diagrams *of* λ/μ.

Excited diagrams:

$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$



$$f^{(4321/21)} = 7! \left(\frac{1}{1^4 \cdot 3^3} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^2 \cdot 3^3 \cdot 5^2} + \frac{1}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7} \right) = 61$$

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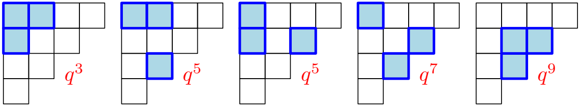


$$s_{\lambda/\mu}(1,q,q^2,\ldots) = \sum_{T \in SSYT(4321/21)} q^{|T|} = \frac{q^3}{(1-q)^4(1-q^3)^3} + 2 \times \frac{q^5}{(1-q)^3(1-q^3)^3(1-q^5)}$$
$$+ \frac{q^7}{(1-q)^2(1-q^3)^3(1-q^5)^2} + \frac{q^9}{(1-q)^2(1-q^3)^2(1-q^5)^2(1-q^7)}$$

Theorem (Morales-Pak-P)

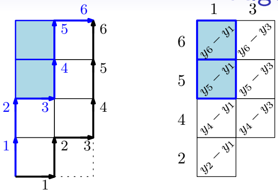
$$\sum_{T \in SSYT(\lambda/\mu)} q^{|T|} = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in [\lambda] \setminus D} \left[\frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}} \right].$$

Theorem (Morales-Pak-P)

$$\sum_{\pi \in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S \in PD(\lambda/\mu)} \prod_{u \in S} \left[\frac{q^{h(u)}}{1 - q^{h(u)}} \right].$$

where $PD(\lambda/\mu)$ is the set of pleasant diagrams.

Algebraic proof for SSYTs:



$v = 245613$, $w = 361245$

[Ikeda-Naruse, Kreiman]: Let $w \preceq v$ be Grassmannian permutations whose unique descent is at position d with corresponding partitions $\mu \subseteq \lambda \subseteq d \times (n-d)$. Then

$$[X_w]\big|_v = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} \left(y_{v(d+j)} - y_{v(d-i+1)}\right).$$

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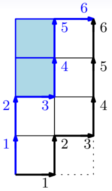

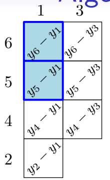

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Factorial Schur functions:

$$s_\mu^{(d)}(\mathbf{x}|\mathbf{a}) := \frac{\det\big[(x_j - a_1)\cdots(x_j - a_{\mu_i+d-i})\big]_{i,j=1}^d}{\prod_{1\le i<j\le d}(x_i - x_j)},$$

[Knutson-Tao, Lakshmibai–Raghavan–Sankaran] Schubert class at a point:

$$[X_w]\big|_v = (-1)^{\ell(w)} s_\mu^{(d)}\big(y_{v(1)},\ldots,y_{v(d)}|y_1,\ldots,y_{n-1}\big).$$

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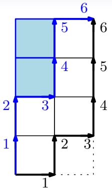

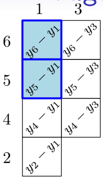

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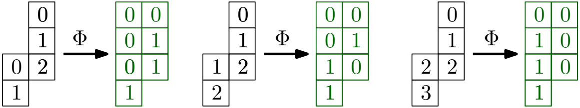
Evaluation at $y = 1, q, q^2, \ldots$, $v(d+1-i) = \lambda_i + d + 1 - i$, $\rightarrow$ Jacobi-Trudi

$$s_\mu^{(d)}(q^{v(1)},\ldots|1,q,\ldots) = \frac{\det[\prod_{r=1}^{\mu_j+d-j}(q^{v(i)} - q^r)]}{\Delta(y_v)} = \ldots.$$

$$= \det[h_{\lambda_i - i - \mu_j + j}(1,q,\ldots)] = s_{\lambda/\mu}(1,q,\ldots)$$

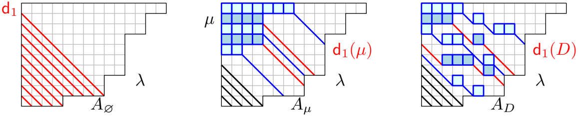
Combinatorial proofs:

Hillman-Grassl Reverse Plane Partitions of shape λ to Arrays of shape λ.



Theorem (Morales-Pak-P)

The Hillman-Grassl map is a bijection to the SSYTs of shape λ/μ to the excited arrays (diagrams in $\mathcal{E}(\lambda/\mu)$ with nonzero entries on the broken diagonals) .



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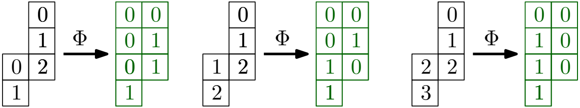

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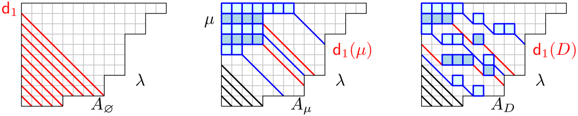


Without the restriction of strictly increasing columns, we have skew reverse plane partitions and a wider class of arrays/diagrams, called *pleasant diagrams*: $PD(\lambda/\mu)$.

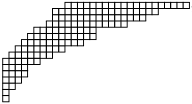
Theorem (MPP)

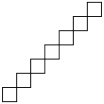
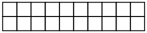
The HG map is a bijection between skew RPPs of shape λ/μ and arrays with certain nonzero entries (at the "high peaks"):

$$\sum_{\pi \in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S \in PD(\lambda/\mu)} \prod_{u \in S} \left[\frac{q^{h(u)}}{1-q^{h(u)}} \right].$$

With P-partitions/limit: combinatorial proof of original Naruse Hook-Length Formula for $f^{\lambda/\mu}$..

Asymptotics of the number of skew SYTs

$\lambda/\mu =$ , $|\lambda/\mu| = n \to \infty$

 $f^{\lambda/\mu} = n!$  $f^{\lambda/\emptyset} = C_{n/2} \approx 2^n \frac{2\sqrt{2}}{n^{3/2}\sqrt{\pi}}$

Question: What is the asymptotic value of $f^{\lambda/\mu}$, $|\lambda/\mu| = n$ as $n \to \infty$ and λ, μ change under various regimes:

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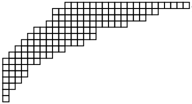
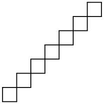
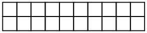
$\lambda/\mu =$  , $|\lambda/\mu| = n \to \infty$



$f^{\lambda/\mu} = n!$  $\qquad f^{\lambda/\emptyset} = C_{n/2} \approx 2^n \frac{2\sqrt{2}}{n^{3/2}\sqrt{\pi}}$

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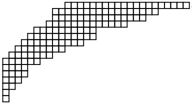
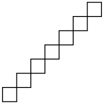
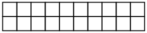
0. If $\mu = \emptyset$, then $f^{\lambda} \sim \sqrt{n!}(1 + O(1/n))$ for $\lambda \sim$ Plancherel.

Asymptotics of the number of skew SYTs

$\lambda/\mu =$  , $|\lambda/\mu| = n \to \infty$

 $f^{\lambda/\mu} = n!$  $\qquad f^{\lambda/\emptyset} = C_{n/2} \approx 2^n \frac{2\sqrt{2}}{n^{3/2}\sqrt{\pi}}$

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1. [Stanley, 2001]: when μ is fixed, $\lambda^n \to (a; b)$ (Frobenius limit):

$$f^{\lambda^n/\mu} \sim f^{\lambda^n} s_\mu(\rho_a^+; \rho_b^-)(1 + O(1/n)),$$

where ρ_a^+, ρ_b^- are the corresponding specializations.
Similar results in [Corteel-Goupil-Schaeffer]
[Okounkov-Olshanski]

Tool

Naruse Hook-Length formula:

$$f^{\lambda/\mu} = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in D} \frac{1}{h_u}.$$

Define the "naive" hook-length formula:

$$F(\lambda/\mu) := \prod_{u \in \lambda/\mu} \frac{1}{h_u}.$$

			5	4	1
		5	3	2	
	7	4	2	1	
	4	1			
4	2				
3	1				
1					

$$F((6,5,5,3,2,2,1)/(3,2,1,1)) = \frac{1}{5\cdot4\cdot1\cdot5\cdot3\cdot2\cdot7\cdot4\cdot2\cdot1\cdot4\cdot1\cdot4\cdot2\cdot3\cdot1\cdot1}$$

Corollary

$$F(\lambda/\mu) \le f^{\lambda/\mu} \le |\mathcal{E}(\lambda/\mu)| F(\lambda/\mu)$$

General bounds for posets (folklore)

P – poset, $e(P)$ – number of linear extensions,
$P = A_1 \sqcup \ldots \sqcup A_\ell$ – antichains, $P = C_1 \sqcup \ldots \sqcup C_p$ – chains.

$$|A_1|!|A_2|!\cdots|A_\ell|! \leq e(P) \leq \frac{n!}{|C_1|!|C_2|!\cdots|C_p|!}$$

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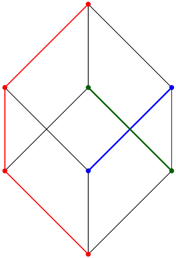

$36 = 1!3!3!1! \leq 48 \leq \frac{8!}{4!2!2!} = 420$

[Brightwell-Tetali]: Boolean lattice
$\frac{\log_2 e(B_n)}{2^n} = \binom{n}{n/2} - \frac{3}{2}\log_2(e) + o(1)$

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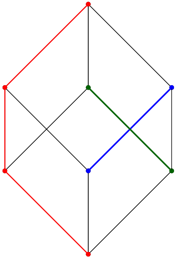

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In our case:
Theorem[MPP]: When $P = \lambda/\mu$ and A_i – ith antidiagonal, then

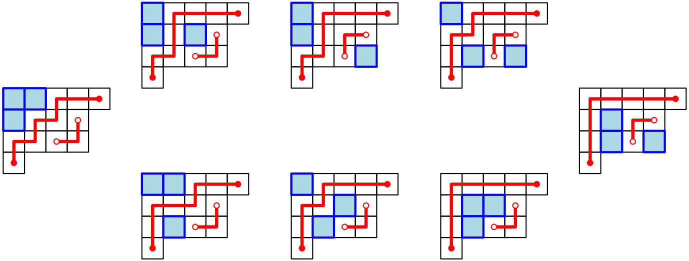
$$|A_1|!\cdots|A_\ell|! \leq F(\lambda/\mu)$$

if $|A_i| \leq |A_{i+1}|$.

General bounds: size of $\mathcal{E}(\lambda/\mu)$

$$F(\lambda/\mu) \leq f^{\lambda/\mu} \leq |\mathcal{E}(\lambda/\mu)| F(\lambda/\mu)$$

$\mathcal{E}(\lambda/\mu) = \{$ Non-intersecting Lattice Paths in λ/μ $\}$



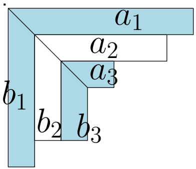
Lemma (MPP)

If $|\lambda/\mu| = n$ then $\mathcal{E}(\lambda/\mu) \leq 2^n$.

Lemma (MPP)

If d is the Durfee square size of λ, then $\mathcal{E}(\lambda/\mu) \leq n^{2d^2}$.

The "linear" regime



$a(\lambda) = (a_1, a_2, \ldots)$, $b(\lambda) = (b_1, b_2, \ldots)$ – Frobenius coordinates of λ. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta := (\beta_1, \ldots, \beta_k)$ be fixed sequences in $\mathbb{R}_+^k$.

Thoma–Vershik–Kerov (TVK) *limit* if $a_i/n \to \alpha_i$ and $b_i/n \to \beta_i$ as $n \to \infty$, for all $1 \le i \le k$.

Theorem (MPP)

Let $\{\lambda^{(n)}/\mu^{(n)}\}$ be a sequence of skew shapes with a TVK limit, i.e. suppose $\lambda^{(n)} \to (\alpha, \beta)$, where $\alpha_1, \beta_1 > 0$, and $\mu^{(n)} \to (\pi, \tau)$ for some $\alpha, \beta, \pi, \tau \in \mathbb{R}_+^k$. Then

$$\log f^{\lambda^{(n)}/\mu^{(n)}} = cn + o(n) \quad \textit{as} \quad n \to \infty,$$

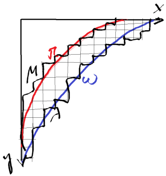
where

$$c = \gamma \log \gamma - \sum_{i=1}^{k} (\alpha_i - \pi_i) \log(\alpha_i - \pi_i) - \sum_{i=1}^{k} (\beta_i - \tau_i) \log(\beta_i - \tau_i)$$

and

$$\gamma = \sum_{i=1}^{k} (\alpha_i + \beta_i - \pi_i - \tau_i).$$

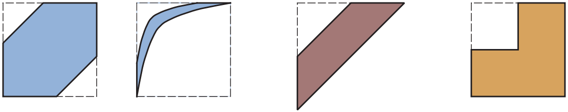
The stable shape: $\sqrt{n}$ scale



Theorem (MPP)

Let $\omega, \pi : [0, a] \to [0, b]$ be continuous non-increasing functions, and suppose that area$(\omega/\pi) = 1$. Let $\{\lambda^{(n)}/\mu^{(n)}\}$ be a sequence of skew shapes with the stable shape ω/π, i.e. $[\lambda^{(n)}]/\sqrt{n} \to \omega$, $[\mu^{(n)}]/\sqrt{n} \to \pi$. Then

$$\log f^{\lambda^{(n)}/\mu^{(n)}} \sim \frac{1}{2} n \log n \quad \text{as} \quad n \to \infty.$$



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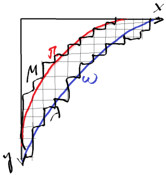


Theorem (MPP)

Suppose $(\sqrt{N}-L)\omega \subset [\lambda^{(n)}](\sqrt{N}+L)\omega$ *for some* $L>0$*, and similarly for* $\mu^{(n)}$ *wrt* π*, then*

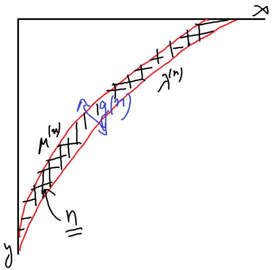
$$-\big(1+c(\omega/\pi)\big)n+o(n) \le \log f^{\lambda^{(n)}/\mu^{(n)}} - \frac{1}{2}n\log n \le -\big(1+c(\omega/\pi)\big)n+\log \mathcal{E}(\lambda^{(n)}/\mu^{(n)})+o(n),$$

as $n\to\infty$*, where*

$$c(\omega/\pi) = \iint_{\omega/\pi} \log h(x,y)\,dxdy,$$

where $h(x,y)$ *is the hook length from* (x,y) *to* ω*.*

Subpolynomial depth, “thin” shapes



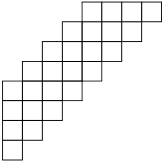
Suppose
depth:= $\max_{u\in\lambda/\mu} h_u =: g(n) = n^{o(1)}$
(subpolynomial growth).

Theorem (MPP)

Let $\{\nu_n = \lambda^{(n)}/\mu^{(n)}\}$ *be a sequence of skew partitions with a subpolynomial depth shape associated with the function* $g(n)$*. Then*

$$\log f^{\nu_n} = n\log n - \Theta\big(n\log g(n)\big) \quad \text{as} \quad n\to\infty.$$

Thick ribbons



Theorem (MPP)

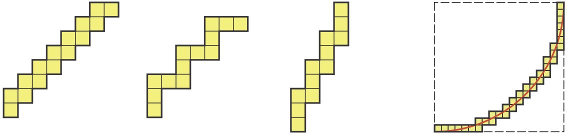
Let $\gamma_k := (\delta_{2k}/\delta_k)$, *where* $\delta_k = (k-1, k-2, \ldots, 2, 1)$. *Then*

$$\frac{1}{6} - \frac{3}{2}\log 2 + \frac{1}{2}\log 3 + o(1) \leq \frac{1}{n}\left(\log f^{\gamma_k} - \frac{1}{2} n \log n\right) \leq \frac{1}{6} - \frac{7}{2}\log 2 + 2\log 3 + o(1),$$

where $n = |\gamma_k| = k(3k-1)/2$.

Question: What (if it is exists) is $c =$?: $c = \lim_{n\to\infty} \frac{1}{n}\left(\log f^{\gamma_k} - \frac{1}{2} n \log n\right)$.
Jay Pantone's implementation (method of differential approximants) on 150+ terms of the sequence $\{\log f^{\gamma_k}\}$ to approximate the constant to $c \approx -0.1842$.

Thin ribbons



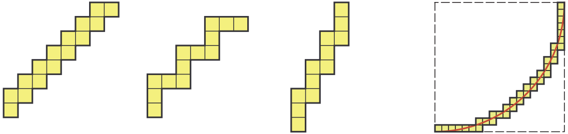
Zigzag: $\rho_k := \delta_{k+2}/\delta_k$, $E_n = |\{\sigma \in S_n : \sigma(1) < \sigma(2) > \sigma(3) < \cdots\}|$ – Euler numbers, alternating permutations.

$$f^{\rho_n} = E_{2n+1}; \qquad E_m \sim m!(2/\pi)^m 4/\pi(1 + o(1))$$

From theorem: $F(\rho_k) = n!/3^k$, $\mathcal{E}(\rho_k) = C_k$, so

$$\frac{(2k+1)!}{3^k} \leq E_{2k+1} \leq \frac{(2k+1)!\,C_k}{3^k}$$

Thin ribbons



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Problem: If $\gamma_n := \lambda/\mu$ is a border strip (ribbon of thickness 1, n boxes) approaching a given curve γ under rescaling by n, what is $\log f^{\gamma_n} - n\log n$ in terms of γ? Is it true that $\frac{\log f^{\gamma_n} - n\log n}{n} \to c(\gamma)$ for some constant $c(\gamma)$? (Permutations with certain descent sequences)

Lozenge tilings

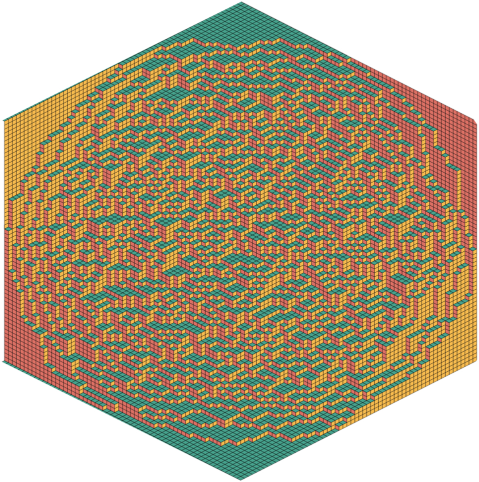


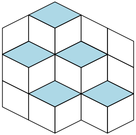
Image: Leonid Petrov

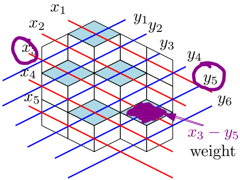
Lozenge tilings with multivariate weights

Plane partitions with base μ, height d

weights of horizontal lozenges $= x_i - y_j$

3	2	1
2	1	





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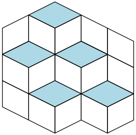
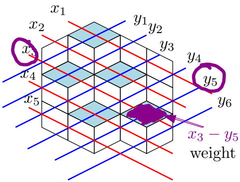




Theorem (Morales-Pak-P)

Introduce the x and y coordinates as above. Let $d' = d + \ell(\mu)$ and $n := d' + d + \mu_1$. Then

$$\sum_T \prod_{(i,j)\text{—— horizontal lozenge of } T} (x_i - y_j) = \frac{\det[(x_i - z_1)\cdots(x_i - z_{\mu_j+d'-j})]_{i,j=1}^n}{\Delta(x)}$$

$$=: s_\mu^{(d')}(x_1, \ldots, x_{d'} | z_1, \ldots, z_{n-1}) \text{ (Factorial Schur function)}$$

where the sum is over lozenge tilings T of support μ and height d, and $z_{\lambda_i+(d'+1-i)} = x_i$ and $z_{d'+j-\lambda'_j} = y_j$

Theorem (Morales-Pak-P)

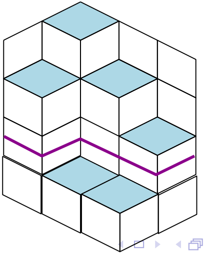
Consider tilings of the $a \times b \times c \times a \times b \times c$ (base $a \times b$, height c) hexagon with horizontal lozenges having weights $x_i - y_j$. The partition function is given by

$$Z(a,b,c) := \sum_T \prod_{(i,j)\in HT(T)} (x_i - y_j) = \frac{\det \left[\begin{cases} (x_i - y_1)\cdots(x_i - y_{a+b+c-j}) \\ (x_i - y_1)\cdots(x_i - y_{a+c})(x_i - x_{b+c})\ldots(x_i - x_{b-j}) \end{cases} \right]}{\Delta(x)}$$

The probability that a tiling contains a vertical line passing through the points on vertical lines $1, 2, \ldots$ at heights $d_1, d_2, \ldots, d_{a+b}$ (necessarily $|d_i - d_{i+1}| \leq 1$, $d_i \leq d_{i+1}$ if $i \leq b$ and $d_i \geq d_{i+1}$ if $i > b$, and $d_1 = d_{a+b}$) is given by

$$\frac{s_\mu^{b+d_1}(x_1, \ldots, x_{d+b}|z_1, z_2, \ldots) s_{\bar\mu}^{b+c-d_1}(x_{b+c}, \ldots, x_{d_1+1}|z')}{Z}$$

where $d := d_1$, $\ell(\mu) = b$, $\mu_1 = a$ and μ has diagonals given by $d_i - d_1$ and z is determined by μ as in Theorem 12. Here $\bar\mu$ is the complementary shape of μ in a $(a+c-d_1)\times(b+c-d_1)$ rectangle and z' is determined accordingly, with variables $x_{b+c}, x_{b+c-1}, \ldots$ and $y_{a+c}, y_{a+c-1}, \ldots$.



$\mu = (3,1)$

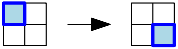
$\bar\mu = (2,0)$

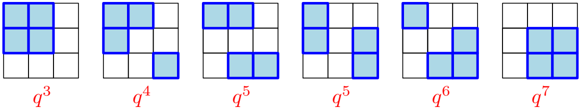
Excited diagrams and factorial Schur functions

Factorial Schur functions.

$$s_\mu^{(d)}(x|a) := \frac{\det\left[(x_j - a_1)\cdots(x_j - a_{\mu_i+d-i})\right]_{i,j=1}^{d}}{\prod_{1\le i<j\le d}(x_i - x_j)},$$

where $x = (x_1, x_2, \ldots, x_d)$ and $a = (a_1, a_2, \ldots)$ is a sequence of parameters.
Excited diagrams $\mathcal{E}(\lambda/\mu)$: Start with λ/μ. Move cells of μ inside λ via:





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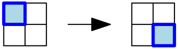

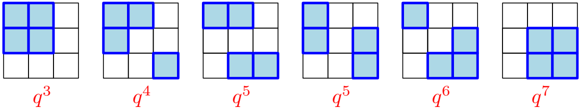

Theorem (Ikeda-Naruse Multivariate "Hook-Length Formula")

Let $\mu \subset \lambda \subset d \times (n-d)$. Let v be the Grassmannian permutation with unique descent at position d corresponding to λ, i.e. $v(d'+1-i) = \lambda_i + (d'+1-i)$ and $v(j) = d' + j - \lambda'_j$. Then

$$s_\mu^{(d)}(y_{v(1)}, \ldots, y_{v(d)} | y_1, \ldots, y_{n-1}) = \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} (y_{v(d-i+1)} - y_{v(d+j)})$$

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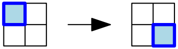

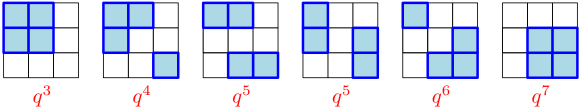

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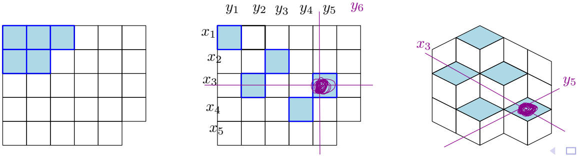

Other results

- Proof of multivariate Naruse formula using Lascoux-Pragacz determinant, and manipulatorial proofs for border strips.

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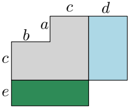
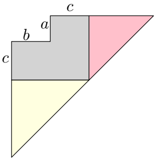
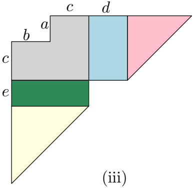


(i)



(ii)



(iii)

Other results

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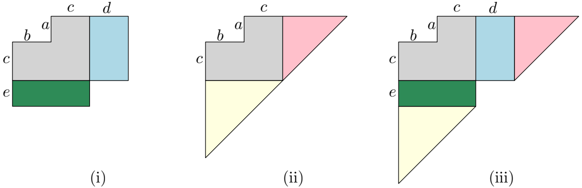
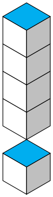


- Generalizations to Grothendieck polynomials.

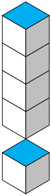
More problems?

- More precise asymptotics of $f^{\lambda/\mu}$ in various regimes.
- Asymptotics of lozenge tilings using the multivariate weights, new regimes?
- Asymptotics of $\frac{s_{\lambda/\mu}(x_1,\ldots,x_k,1^{n-k})}{s_{\lambda/\mu}(1^n)}$ (Schur generating functions of tilings of arbitrary domains)
- Asymptotics of Littlewood-Richardson coefficients, $c^{\lambda}_{\mu,\nu}\ldots$ (e.g. if $\lambda \vdash 2n$, $\mu,\nu \vdash n$, when is it maximal)
- Maximal $f^{\lambda/\mu}$ under constraints...

Thank you



Thank you



Thank you for organizing, Valentin Féray, Pierre-Loïc Méliot!