Characters of infinite-dimensional unitary group and particle systems

Leonid Petrov

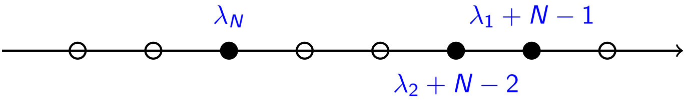
University of Virginia and IITP

Extreme characters of the infinite-dimensional unitary group

Irreducible characters of unitary groups $U(N)$ [Clebsch 1872, Schur, Weyl]

Irreducible representations of $U(N)$ are indexed by highest weights (signatures)

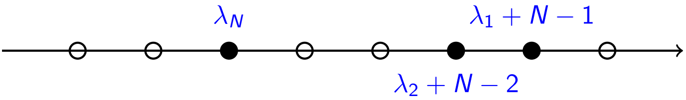
$$\widehat{U(N)} = \{\lambda = (\lambda_1 \ge \ldots \ge \lambda_N), \quad \lambda_i \in \mathbb{Z}\}.$$



Irreducible characters of unitary groups $U(N)$ [Clebsch 1872, Schur, Weyl]

Irreducible representations of $U(N)$ are indexed by highest weights (signatures)

$$\widehat{U(N)} = \{\lambda = (\lambda_1 \geq \ldots \geq \lambda_N), \quad \lambda_i \in \mathbb{Z}\}.$$



$$U(N) \ni U = D \begin{pmatrix} e^{\mathbf{i}\theta_1} & 0 & \ldots & 0 \\ 0 & e^{\mathbf{i}\theta_2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & e^{\mathbf{i}\theta_N} \end{pmatrix} D^{-1}.$$

Let $x_j = e^{\mathbf{i}\theta_j}$. The values of the irreducible character corresponding to λ are given by the Schur polynomial:

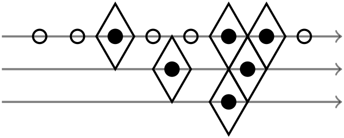
$$\chi_\lambda(x_1, \ldots, x_N) = s_\lambda(x_1, \ldots, x_N) = \frac{\det_{i,j=1}^{N} \left[x_i^{\lambda_j + N - j} \right]}{\det_{i,j=1}^{N} \left[x_i^{N-j} \right]}.$$

Branching and lozenge tilings

Restriction to $U(N-1) \subset U(N)$:

$$s_\lambda(x_1, \ldots, x_{N-1}, 1) = \sum_{\lambda_N \le \mu_{N-1} \le \lambda_{N-1} \le \ldots \le \lambda_2 \le \mu_1 \le \lambda_1} s_\mu(x_1, \ldots, x_{N-1})$$

which implies that there is a distinguished Gelfand–Tsetlin basis in every irreducible V_λ indexed by lozenge tilings of a sawtooth domain with *fixed top row* $\lambda + \rho = (\lambda_1 + N - 1, \lambda_2 + N - 2, \ldots, \lambda_N)$



$(\lambda_1 + 2, \lambda_2 + 1, \lambda_3) \in \widehat{U(3)}$

$(\mu_1 + 1, \mu_2) \in \widehat{U(2)}$

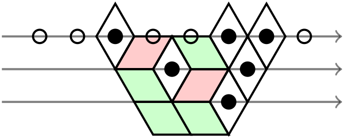
$(\nu_1) \in \widehat{U(1)}$

Branching and lozenge tilings

Restriction to $U(N-1) \subset U(N)$:

$$s_\lambda(x_1, \ldots, x_{N-1}, 1) = \sum_{\lambda_N \le \mu_{N-1} \le \lambda_{N-1} \le \ldots \le \lambda_2 \le \mu_1 \le \lambda_1} s_\mu(x_1, \ldots, x_{N-1})$$

which implies that there is a distinguished Gelfand–Tsetlin basis in every irreducible V_λ indexed by lozenge tilings of a sawtooth domain with *fixed top row* $\lambda + \rho = (\lambda_1 + N - 1, \lambda_2 + N - 2, \ldots, \lambda_N)$

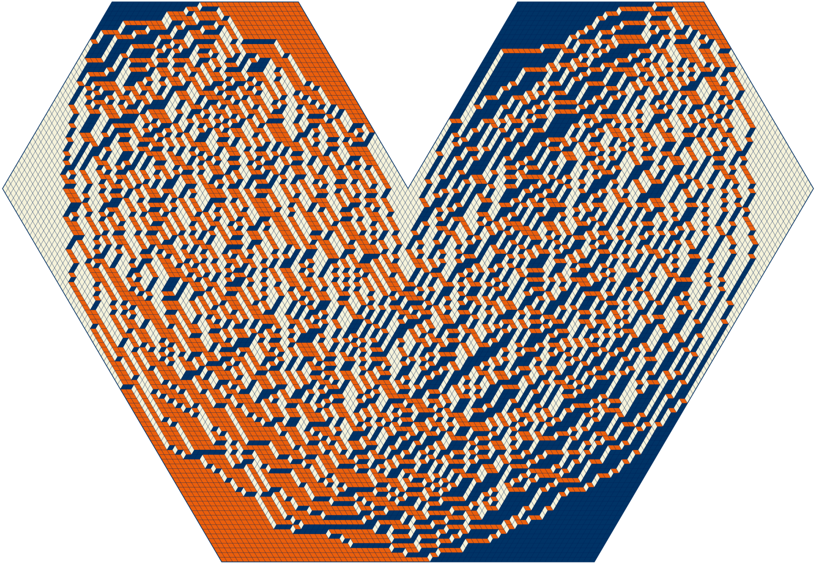


$(\lambda_1 + 2, \lambda_2 + 1, \lambda_3) \in \widehat{U(3)}$

$(\mu_1 + 1, \mu_2) \in \widehat{U(2)}$

$(\nu_1) \in \widehat{U(1)}$

Remark: uniformly random lozenge tilings with fixed top row



(asymptotics: a somewhat different story; e.g. **[Bufetov's talk next]**)

Asymptotic representation theory of unitary groups

Let

$$U(\infty) = \bigcup_{N=1}^{\infty} U(N), \qquad \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1N} & 0 & 0 & \dots \\ u_{21} & u_{22} & \dots & u_{2N} & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ u_{N1} & u_{N2} & \dots & u_{NN} & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \in U(\infty)$$

A (normalized) character of $U(\infty)$ is a function χ on $U(\infty)$ which is

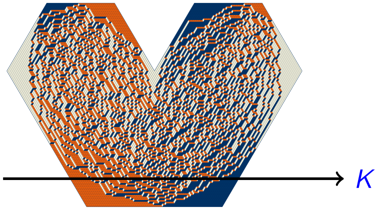
- Continuous
- $\chi(AB) = \chi(BA)$ (constant on conjugacy classes)
- $\chi(1) = 1$ (normalized)
- χ is positive definite

Normalized characters of $U(\infty)$ form a convex set, and its extreme points are analogues of irreducible characters.

Asymptotic representation theory of unitary groups

Each extreme character of $U(\infty)$ is a limit of normalized irreducible characters
$\dfrac{s_{\lambda(N)}(\cdots)}{s_{\lambda(N)}(1,\ldots,1)}$, $N \to +\infty$ **[Vershik '74]**, **[Vershik–Kerov '80s]**
Combinatorially, the problem of asymptotic character theory of $U(\infty)$ is equivalent to the following:

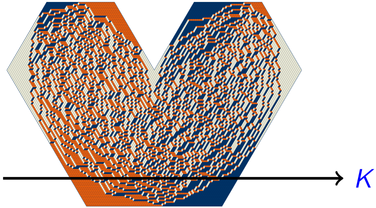
Find all sequences $\lambda(N)$ such that the uniformly random tilings of domains of height N with fixed top rows $\lambda(N)$ have weak limits when restricted to any fixed height K.

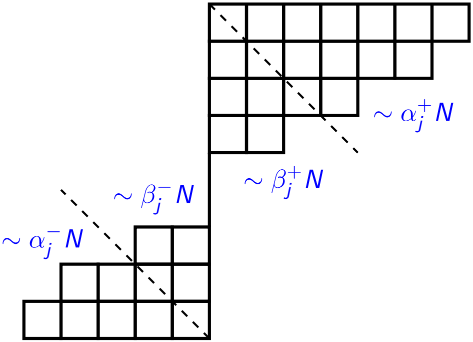


Asymptotic representation theory of unitary groups

Each extreme character of $U(\infty)$ is a limit of normalized irreducible characters
$\dfrac{s_{\lambda(N)}(\cdots)}{s_{\lambda(N)}(1,\ldots,1)}$, $N \to +\infty$ **[Vershik '74]**, **[Vershik–Kerov '80s]**
Combinatorially, the problem of asymptotic character theory of $U(\infty)$ is equivalent to the following:

Find all sequences $\lambda(N)$ such that the uniformly random tilings of domains of height N with fixed top rows $\lambda(N)$ have weak limits when restricted to any fixed height K.





Asymptotic representation theory of unitary groups

Theorem **[Edrei '53]**, **[Voiculescu '76]**, also **[Vershik–Kerov '80s]**, **[Okounkov–Olshanski '90s]**, yet other proofs **[Borodin–Olshanski '11]**, **[P. '12]**, **[Gorin–Panova '13]**

Extreme characters of $U(\infty)$ are parametrized* by tuples in

$$\widehat{U(\infty)} := \Big\{\omega = (\alpha^\pm, \beta^\pm, \gamma^\pm) \in \mathbb{R}^{4\infty+2} \colon \alpha^\pm = (1 > \alpha_1^\pm \ge \alpha_2^\pm \ge \ldots \ge 0), \quad \beta^\pm = (\beta_1^\pm \ge \beta_2^\pm \ge \ldots \ge 0),\ \gamma^\pm \ge 0\Big\}$$

and are given by

$$\chi(x_1, x_2, \ldots) = \prod_{i=1}^{\infty} \frac{\Phi(x_i)}{\Phi(1)}, \qquad |x_i| = 1,$$

where

$$\Phi(x) := e^{\gamma^+ x + \gamma^- x^{-1}} \prod_{j=1}^{\infty} \frac{1 + \beta_j^+ x}{1 - \alpha_j^+ x} \prod_{j=1}^{\infty} \frac{1 + \beta_j^- x^{-1}}{1 - \alpha_j^- x^{-1}}$$

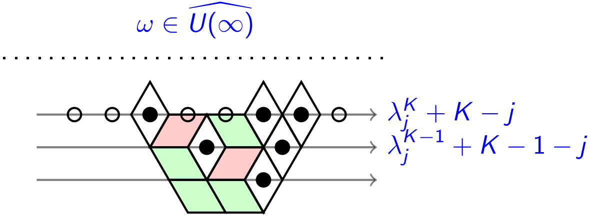
Particle systems from flows on $\widehat{U(\infty)}$

Random lozenge tilings / interlacing arrays

For any $\omega \in \widehat{U(\infty)}$ the restriction to the first K levels is a random tiling with *random top row*.

Gibbs property

Conditioned on fixed configuration λ^K at any level K, the distribution of the tiling at levels $1, 2, \dots, K-1$ is uniformly random (provided the interlacing).



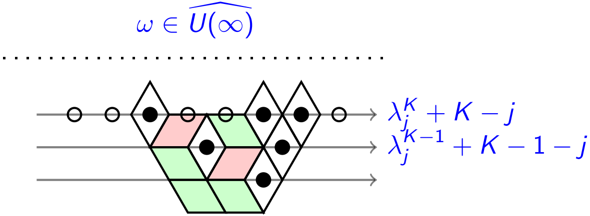
If $\omega = 0$, then $\lambda^K = (0, 0, \dots, 0)$, and there is only one densely packed tiling.

Random lozenge tilings / interlacing arrays

For any $\omega \in \widehat{U(\infty)}$ the restriction to the first K levels is a random tiling with *random top row*.

Gibbs property

Conditioned on fixed configuration λ^K at any level K, the distribution of the tiling at levels $1, 2, \dots, K-1$ is uniformly random (provided the interlacing).



If $\omega = 0$, then $\lambda^K = (0, 0, \dots, 0)$, and there is only one densely packed tiling.

The distribution of each λ^K is a *Schur measure* (or its analogue with additional α^-, β^-, γ^- parameters), and the whole tiling is distributed as a *Schur process* **[Okounkov '01]**, **[Okounkov-Reshetikhin '03]**.

How to see the structure of Schur measures

Restrict the extreme character χ of $U(\infty)$ to any $U(K) \subset U(\infty)$. This is still a normalized character (but not necessarily irreducible), and we can decompose:

$$\chi(x_1, \ldots, x_K, 1, 1, \ldots) = \prod_{i=1}^{K} \frac{\Phi(x_i)}{\Phi(1)} = \sum_{\lambda_1 \geq \ldots \geq \lambda_K} \mathrm{Prob}_K(\lambda) \frac{s_\lambda(x_1, \ldots, x_K)}{s_\lambda(1, \ldots, 1)},$$

where $\mathrm{Prob}_K(\lambda)$ is a probability distribution on K-signatures λ.

The expansion into Schur polynomials can be computed explicitly. Let $\dfrac{\Phi(x)}{\Phi(1)} = \sum_{n \in \mathbb{Z}} \varphi_n$, then

$$\mathrm{Prob}_K(\lambda) = s_\lambda(1, \ldots, 1) \det_{i,j=1}^{K} \left[\varphi_{\lambda_j + i - j}\right].$$

Markov kernel Λ_K^∞

$\mathrm{Prob}_k(\lambda)$ defines a **Markov kernel** $\Lambda_K^\infty : \widehat{U(\infty)} \dashrightarrow \widehat{U(K)}$, $\Lambda_K^\infty(\omega, \lambda) = \mathrm{Prob}_K(\lambda)$.

If $\alpha^- = \beta^- = \gamma^- = 0$ then $\det_{i,j=1}^{K} \left[\varphi_{\lambda_j + i - j}\right] = s_\lambda(\alpha; \beta; \gamma)$ is a specialization of the Schur function, and that is how we get Schur measures.

Deterministic flows on $\widehat{U(\infty)}$

Let $\mathbf{T}_{\alpha^+}, \mathbf{T}_{\beta^+}, \mathbf{T}_{\Delta\gamma^+} : \widehat{U(\infty)} \to \widehat{U(\infty)}$ be operations of adding a new parameter α^+, β^+, or changing $\gamma^+ \to \gamma^+ + \Delta\gamma^+$, respectively. Similarly for $\mathbf{T}_{\alpha^-}, \mathbf{T}_{\beta^-}, \mathbf{T}_{\Delta\gamma^-}$.

Remark

$\lim_{M\to+\infty}(\mathbf{T}_{\alpha^+_M})^M = \mathbf{T}_{\gamma^+}$ with $\alpha^+_M = \gamma^+/M$;
$\lim_{M\to+\infty}(\mathbf{T}_{\beta^+_M})^M = \mathbf{T}_{\gamma^+}$ with $\beta^+_M = \gamma^+/M$; and similarly for $\alpha^-, \beta^-, \gamma^-$.

Deterministic flows on $\widehat{U(\infty)}$

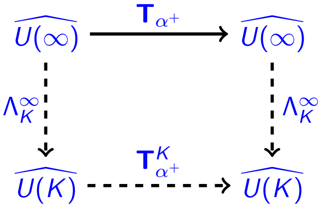
Let $\mathbf{T}_{\alpha^+}, \mathbf{T}_{\beta^+}, \mathbf{T}_{\Delta\gamma^+} : \widehat{U(\infty)} \to \widehat{U(\infty)}$ be operations of adding a new parameter α^+, β^+, or changing $\gamma^+ \to \gamma^+ + \Delta\gamma^+$, respectively. Similarly for $\mathbf{T}_{\alpha^-}, \mathbf{T}_{\beta^-}, \mathbf{T}_{\Delta\gamma^-}$.

Remark

$\lim_{M\to+\infty}(\mathbf{T}_{\alpha^+_M})^M = \mathbf{T}_{\gamma^+}$ with $\alpha^+_M = \gamma^+/M$;
$\lim_{M\to+\infty}(\mathbf{T}_{\beta^+_M})^M = \mathbf{T}_{\gamma^+}$ with $\beta^+_M = \gamma^+/M$; and similarly for $\alpha^-, \beta^-, \gamma^-$.

Stochastic particle dynamics [Borodin–Ferrari '08], [Borodin '10]

Each of these flows defines a *unique* discrete time or a continuous time Markov evolution of each level λ^K, $K = 1, 2, \ldots$; denote by $\mathbf{T}^K_{\alpha^+}$, etc.



[Borodin–Gorin '10]: q-variant with random flows

Particle dynamics $\mathbf{T}^K_{\alpha^+}$ etc. on $\widehat{U(K)}$

$K=1$ — single-particle dynamics

- (α^+) At each discrete time step, the particle jumps to the right by $j\in\mathbb{Z}_{\ge 0}$ with probability $(\alpha^+)^j(1-\alpha^+)$;
- (β^+) At each discrete time step, the particle jumps to the right by 1 with probability $\frac{\beta^+}{1+\beta^+}$, and otherwise stays put;
- (γ^+) Continuous time Poisson process: in continuous time, the particle jumps to the right by 1 after exponential random time intervals

For $\alpha^-,\beta^-,\gamma^-$, reverse the direction and jump to the left.

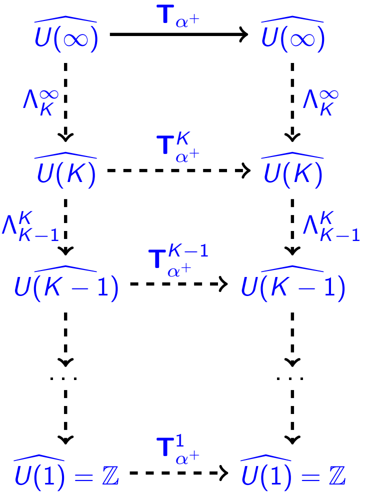
General K: discrete-space analogues of Dyson Brownian motion

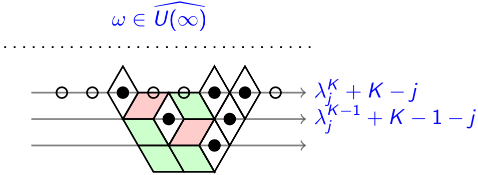
Dynamics of K *independent* particles λ^K_j+K-j evolving as $\mathbf{T}^1_{\alpha^+}$ etc., but conditioned to *never collide*.

From the point of view of characters: $\chi_\lambda\left(\chi_{\alpha^+}\big|_{U(K)}\right)$ as character of $U(K)$ decomposes into irreducible characters with weights corresponding to dynamics $\mathbf{T}^K_{\alpha^+}$.

More particle systems, now in 2 dimensions

2-dimensional particle systems



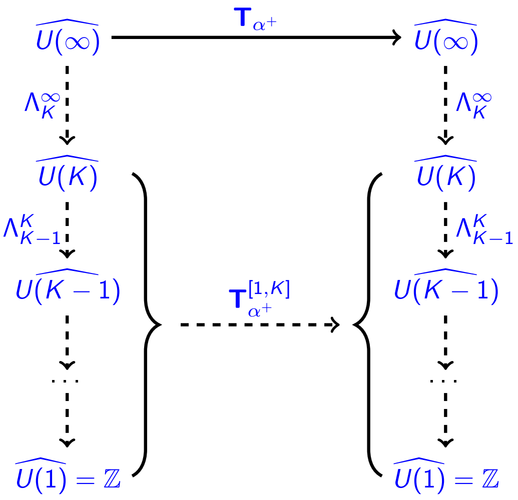


$$\Lambda^K_{K-1}(\lambda, \mu) = \frac{s_\mu(1^{K-1})}{s_\lambda(1^K)} \mathbf{1}_{\lambda \text{ and } \mu \text{ interlace}}$$

(Gibbs property)

If $\mathbf{T}^K_{\alpha^+}, \mathbf{T}^{K-1}_{\lambda^+}, \ldots, \mathbf{T}^1_{\alpha^+}$ evolve independently, this *does not preserve interlacing* (i.e., there is no consistent tiling picture)

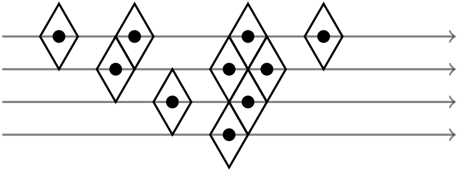
2-dimensional particle systems [Borodin–Ferrari '08], [Borodin–P. '13], [Matveev–P. '15]



- $\mathbf{T}^{[1,K]}_{\alpha^+}$ preserves Gibbs measures;
- on Gibbs measures, restriction of $\mathbf{T}^{[1,K]}_{\alpha^+}$ to any level m is $\mathbf{T}^{m}_{\alpha^+}$;
- The construction of $\mathbf{T}^{[1,K]}_{\alpha^+}$ is highly non-unique (as opposed to single-level dynamics $\mathbf{T}^{K}_{\alpha^+}$);
- Some possible constructions of $\mathbf{T}^{[1,K]}_{\alpha^+}$ involve Robinson–Schensted–Knuth correspondences **[O'Connell '03]**;
- Let us give an example of $\mathbf{T}^{[1,K]}_{\Delta\gamma^+}$ not directly related to Robinson–Schensted–Knuth **[Borodin–Ferrari '08]**

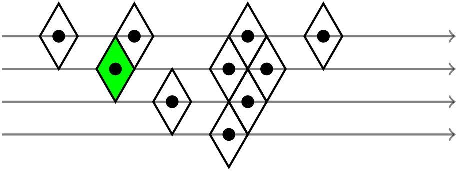
Example: $\mathbf{T}^{[1,K]}_{\Delta\gamma^+}$ [Borodin–Ferrari '08]

Each particle waits for a random exponential time and then jumps to the right by 1, unless it is blocked from below. If needed, it pushes the particles above.



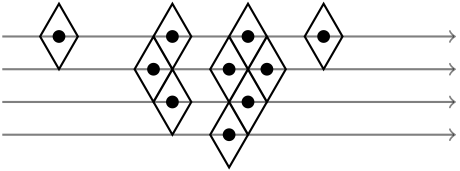
Example: $\mathbf{T}^{[1,K]}_{\Delta\gamma^+}$ [Borodin–Ferrari '08]

Each particle waits for a random exponential time and then jumps to the right by 1, unless it is blocked from below. If needed, it pushes the particles above.



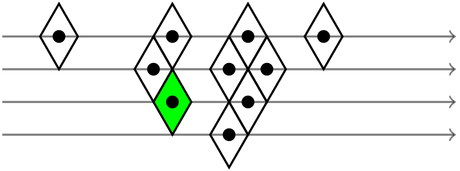
Example: $\mathbf{T}^{[1,K]}_{\Delta\gamma^+}$ [Borodin–Ferrari '08]

Each particle waits for a random exponential time and then jumps to the right by 1, unless it is blocked from below. If needed, it pushes the particles above.



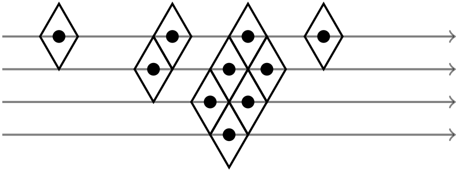
Example: $\mathbf{T}^{[1,K]}_{\Delta\gamma^+}$ [Borodin–Ferrari '08]

Each particle waits for a random exponential time and then jumps to the right by 1, unless it is blocked from below. If needed, it pushes the particles above.



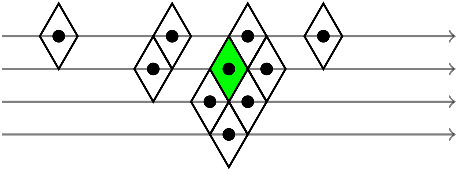
Example: $\mathbf{T}^{[1,K]}_{\Delta\gamma^+}$ [Borodin–Ferrari '08]

Each particle waits for a random exponential time and then jumps to the right by 1, unless it is blocked from below. If needed, it pushes the particles above.



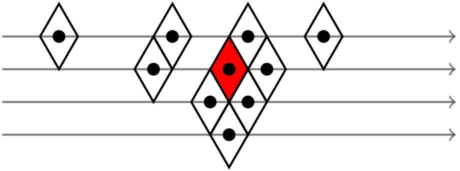
Example: $\mathbf{T}_{\Delta\gamma^{+}}^{[1,K]}$ [Borodin–Ferrari '08]

Each particle waits for a random exponential time and then jumps to the right by 1, unless it is blocked from below. If needed, it pushes the particles above.



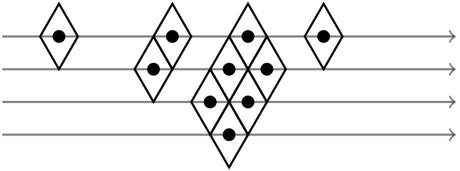
Example: $\mathbf{T}^{[1,K]}_{\Delta\gamma^+}$ [Borodin–Ferrari '08]

Each particle waits for a random exponential time and then jumps to the right by 1, unless it is blocked from below. If needed, it pushes the particles above.



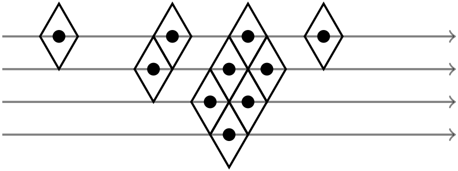
Example: $\mathbf{T}_{\Delta\gamma^+}^{[1,K]}$ [Borodin–Ferrari '08]

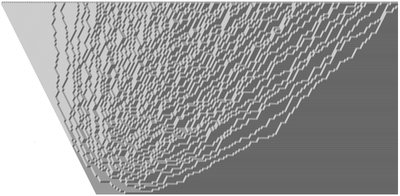
Each particle waits for a random exponential time and then jumps to the right by 1, unless it is blocked from below. If needed, it pushes the particles above.



Example: $\mathbf{T}_{\Delta\gamma^+}^{[1,K]}$ [Borodin–Ferrari '08]

Each particle waits for a random exponential time and then jumps to the right by 1, unless it is blocked from below. If needed, it pushes the particles above.



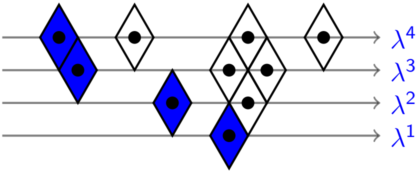


Long-time ($\gamma^+ \to \infty$) asymptotics [Borodin–Ferrari '08]: Anisotropic KPZ

- Frozen boundary is a parabola;
- Local fluctuations along the boundary are of order $(\gamma^+)^{1/3}$ in normal direction and have the GUE Tracy–Widom F_2 distribution (and multipoint ones are governed by the Airy line ensemble);
- Global fluctuations inside the parabola are governed by the Gaussian Free Field

TASEPs

TASEPs



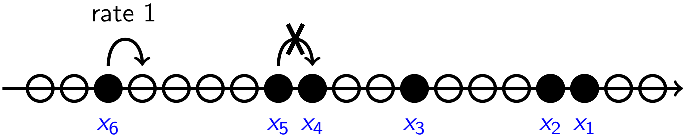
The evolution of the leftmost particles is *marginally Markovian*, i.e., it does not depend on the rest of the tiling

Let $x_j := \lambda^j_j - j$, so $x_1 > x_2 > x_3 > \dots$ is a particle configuration in $\mathbb{Z}$.

Continuous time TASEP **[MacDonald–Gibbs–Pipkin '68]**, **[Spitzer '70]**,

In continuous time, each particle in the configuration jumps to the right by 1 after an exponential waiting time with mean 1, unless the destination is occuppied

GUE Tracy–Widom asymptotics of fluctuations for $x_j(0) = -j$: **[Johansson '00]**

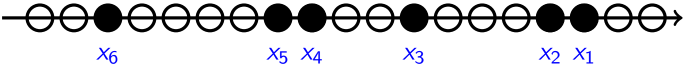


Discrete time TASEPs for $\mathbf{T}^{[1,K]}_{\alpha^+}$ and $\mathbf{T}^{[1,K]}_{\beta^+}$

Geometric case α^+. Notation $\mathbf{G}_{\alpha^+}$

At each dicrete time step $t \to t+1$, each particle x_j, independently of others, jumps to the right by a geometric random distance with parameter α^+, and the jump is stopped if x_j would overjump $x_{j-1}(t)$ (since $\alpha < 1$ this is well-defined):

$$\mathrm{Prob}(x_j(t+1) = x_j(t) + m) = \begin{cases} (\alpha^+)^m(1-\alpha^+), & 0 \le m < x_{j-1}(t) - x_j(t) - 1; \\ (\alpha^+)^m, & m = x_{j-1}(t) - x_j(t) - 1. \end{cases}$$



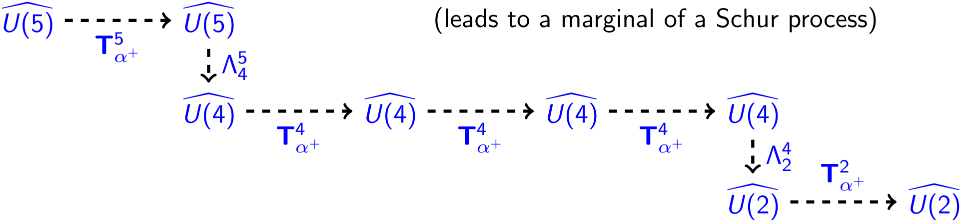
Bernoulli case β^+. Notation $\mathbf{B}_{\beta^+}$

At each dicrete time step $t \to t+1$, each particle x_j jumps to the right by 1 with probability $\frac{\beta^+}{1+\beta^+}$, unless blocked. We update from right to left (from x_1 to x_K), so blockage is defined as $x_j(t) = x_{j-1}(t+1) - 1$.

Joint distributions

The constructions explained so far allow to obtain determinantal structure of joint distributions in any of the TASEPs along **space-like paths**, for example:

$$x_5(t), x_5(t+1), x_4(t+3), x_4(t+4), x_2(t+5)$$

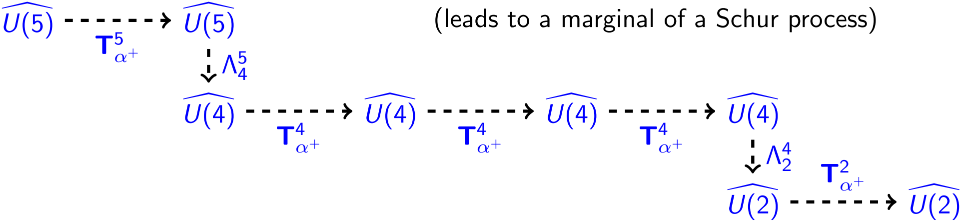


(leads to a marginal of a Schur process)

Joint distributions

The constructions explained so far allow to obtain determinantal structure of joint distributions in any of the TASEPs along **space-like paths**, for example:

$$x_5(t), x_5(t+1), x_4(t+3), x_4(t+4), x_2(t+5)$$



(leads to a marginal of a Schur process)

What about **time-like paths** such as

$$x_5(t), x_{10}(t+8), x_{10}(t+9), x_{12}(t+10) \qquad ?$$

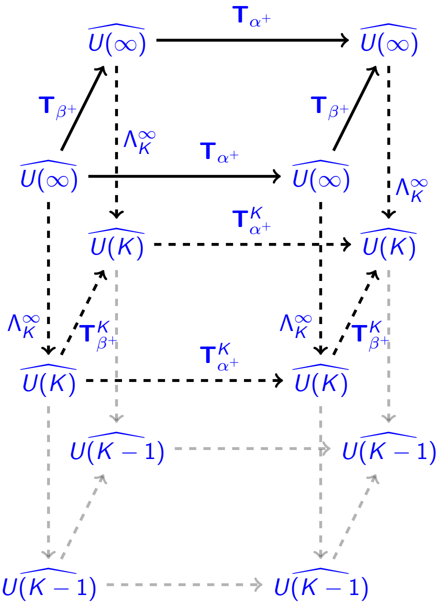
[Johansson '15]: asymptotics of time-like two-point distribution in last passage percolation (in bijection with TASEP started from $x_j(0) = -j$). Different asymptotic regime, fluctuation exponent 1 (scale $O(1)$). Slow decorrelation **[Ferrari '08]**, **[Corwin–Ferrari-Péché '10]**

Next

An attempt to understand some of the algebraic structure of time-like joint distributions

Commuting flows on $\widehat{U(\infty)}$ and particle systems of a new sort

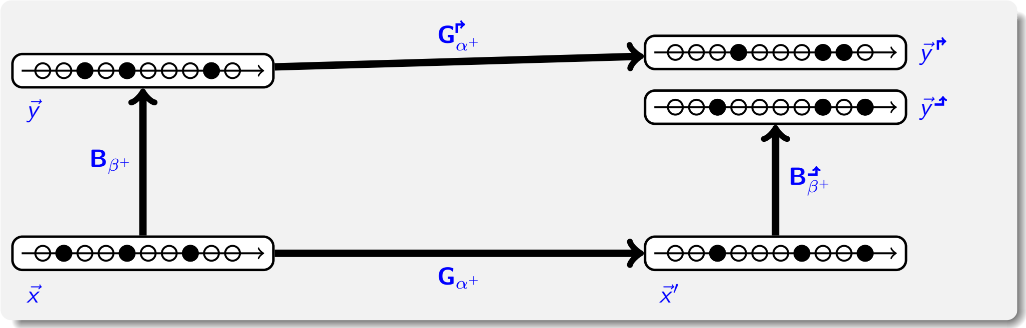
Commuting flows and Markov kernels



- Flows $\mathbf{T}_{\alpha^+}$ and $\mathbf{T}_{\beta^+}$ on $\widehat{U(\infty)}$ commute for trivial reasons
- Markov dynamics $\mathbf{T}^K_{\alpha^+}$ and $\mathbf{T}^K_{\beta^+}$ also commute*, for simple Schur reasons
- What about 2-dimensional dynamics $\mathbf{T}^{[1;K]}_{\alpha^+}$ and $\mathbf{T}^{[1;K]}_{\beta^+}$?
- What about TASEPs $\mathbf{G}_{\alpha^+}$ and $\mathbf{B}_{\beta^+}$?

In the following we focus on TASEPs, which **indeed commute***. Their commutation can be traced to the **Yang–Baxter equation** for (a degenerate case of) the higher spin six-vertex model

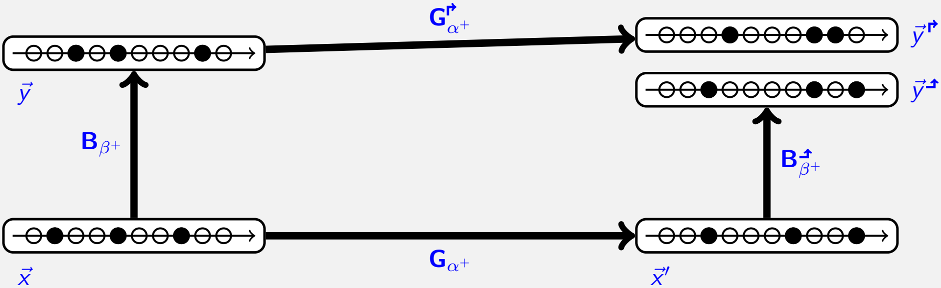
Commutation of TASEPs $\mathbf{G}_{\alpha^+}$ and $\mathbf{B}_{\beta^+}$



If all four geometric and Bernoulli steps are independent (given $\vec{x}$), then commutation means that for fixed $\vec{x}$ the distributions of $\vec{y}^{\,⬏}$ and $\vec{y}^{\,↱}$ are the same.

However, unknown how to couple so that the steps $\mathbf{G}_{\alpha^+}$ and $\mathbf{B}_{\beta^+}$ are still independent but $\vec{y}^{\,⬏} = \vec{y}^{\,↱}$.

Commutation of TASEPs $\mathbf{G}_{\alpha^+}$ and $\mathbf{B}_{\beta^+}$



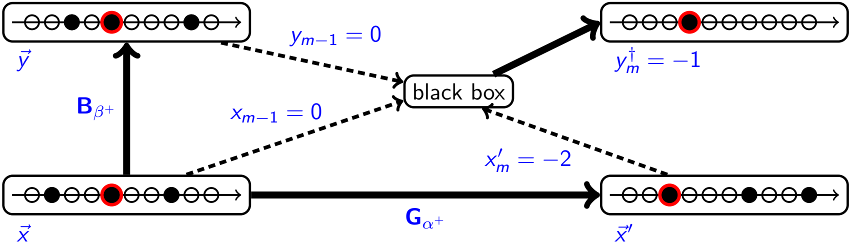
If all four geometric and Bernoulli steps are independent (given $\vec{x}$), then commutation means that for fixed $\vec{x}$ the distributions of $\vec{y}^{\,\uparrow}$ and $\vec{y}^{\,\Rsh}$ are the same.

However, unknown how to couple so that the steps $\mathbf{G}_{\alpha^+}$ and $\mathbf{B}_{\beta^+}$ are still independent but $\vec{y}^{\,\uparrow} = \vec{y}^{\,\Rsh}$.

Idea

Having some partial information about $\vec{x}, \vec{x}, \vec{y}$, we are able to say something about the top right configuration.

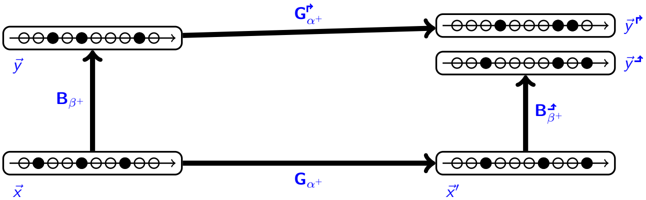
Coupling lemmas



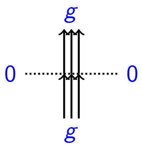
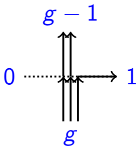
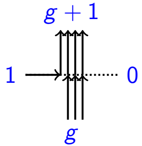
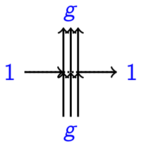
Lemmas [Orr–P. '16]

There is a rule for sampling $y_m^{\dagger}$ given x_{m-1}, y_{m-1}, x'_m, such that given $\vec{x}$:

$$(y_{m-1}, y_m^{\dagger}) \stackrel{d}{=} (y_{m-1}, y_m^{\Rsh}), \qquad (x'_m, y_m^{\dagger}) \stackrel{d}{=} (x'_m, y_m^{\Lsh}).$$



Q: What is in the black box? A: stochastic vertex weights

	$\dfrac{1+\beta^+\mathbf{1}_{g>0}}{1+\beta^+}$		$\dfrac{\beta^+\mathbf{1}_{g>0}}{1+\beta^+}$	$\sum = 1$
	$\dfrac{1-\alpha^+\mathbf{1}_{g=0}}{1+\beta^+}$		$\dfrac{\beta^+ + \alpha^+\mathbf{1}_{g=0}}{1+\beta^+}$	$\sum = 1$

At a vertex: fixing incoming arrows (from left and bottom), we have probability distribution on outgoing arrow configuration (to the right and upwards).

Parameters: $\alpha^+ \in (0,1)$, $\beta^+ \in (0,+\infty)$. Inputs: $x_{m-1} - x'_m - 1$, $y_{m-1} - x_{m-1}$.

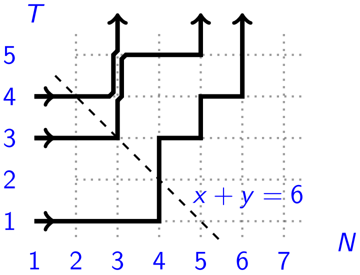
Remark

This is $q = 0$ case of a more general stochastic higher spin six vertex model **[Reshetikhin et al. '80s]**, **[Borodin '14]**, **[Corwin–P. '15]**. There is also *space-time inhomogeneous* version ($\beta^+ \to a_x \beta^+_y$, $\alpha^+ \to \alpha^+_x$) for which we can compute observables for $q > 0$ **[Borodin–P. '16]** and $q = 0$ **[Knizel–P. '17, in prep.]**. We mostly stick to the homogeneous case.

Coupling of TASEPs to a stochastic vertex model

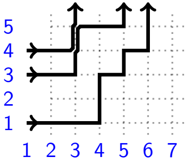
- **Step Bernoulli boundary conditions** in vertex model: paths are started independently on the right with probabilities $\frac{\beta^+}{1+\beta^+}$; nothing enters from below.
- Having **stochastic vertex weights**, define probability distribution on path ensembles by induction on $x + y$.

  (At most one horizontal arrow per edge.)



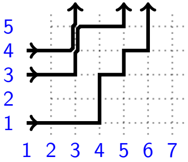
- Define the **height function** $h(N, T)$ to be the number of vertical arrows crossing the T-th horizontal (weakly) to the right of N. In short, paths are level lines for the height function.

Coupling of TASEPs to a stochastic vertex model



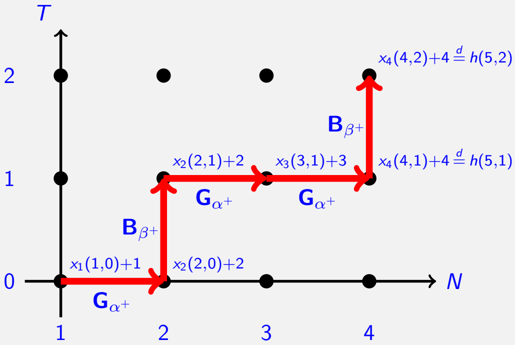
Let $h(N, T)$ be the height function of the vertex model with the step Bernoulli boundary condition, and $x(N, T)$ be a TASEP started from the usual initial configuration $x_j(0, 0) = 0$. To get to configuration $x(N, T)$, TASEP performs $N - 1$ geometric steps $\mathbf{G}_{\alpha^+}$ and T Bernoulli steps $\mathbf{B}_{\beta^+}$ **in some order**.

Coupling of TASEPs to a stochastic vertex model



Let $h(N, T)$ be the height function of the vertex model with the step Bernoulli boundary condition, and $x(N, T)$ be a TASEP started from the usual initial configuration $x_j(0, 0) = 0$. To get to configuration $x(N, T)$, TASEP performs $N - 1$ geometric steps $\mathbf{G}_{\alpha^+}$ and T Bernoulli steps $\mathbf{B}_{\beta^+}$ **in some order**.

Theorem **[Orr-P. '16]**



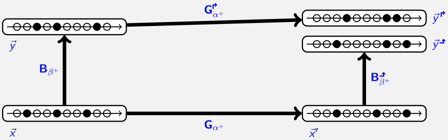
Joint distribution of $\{h(N_t + 1, T_t)\}$ along any up-right path is the same as for $\{x_{N_t}(N_t, T_t) + N_t\}$ in a mixed geometric/Bernoulli TASEP as above, with

- Horizontal step $N \to N + 1$ $\leftrightarrow$ geometric $\mathbf{G}_{\alpha^+}$
- Vertical step $T \to T + 1$ $\leftrightarrow$ Bernoulli $\mathbf{B}_{\beta^+}$

Remarks

Schur structure

Joint distributions of $(x_{m-1}, y_{m-1}, y^{\Rsh}_{m-1})$ and $(x_{m-1}, x'_{m-1}, y^{\Lsh}_{m-1})$ are **marginals of Schur processes**. The passage $m-1 \to m$ in the last coordinates seems to bring us **outside the Schur processes formalism**.



Marginal distributions are Schur-type; access to single-point asymptotics

We have $h(N+1, T) = x_N(N, T) + N$, which does not depend on the path.
At the same time, $x_N(N, T) + N$ is equal in distribution to λ_N
under the Schur measure $\propto s_\lambda(\underbrace{\alpha^+, \ldots, \alpha^+}_{N-1}; \underbrace{\beta^+, \ldots, \beta^+}_{T}) s_\lambda(\underbrace{1, \ldots, 1}_{N})$.

Shifting

For fixed k, $\{x_{N_t+k}(N_t, T_t) + N_t + k\} \stackrel{d}{=} \{h^{(k)}(N_t + 1, T_t)\}$, where $h^{(k)}$ is the height function of the vertex model with the so-called **generalized step Bernoulli boundary condition** (depending on k).

Time-like joint distributions

Vertex model structure of joint distribution of, e.g., $x_4(4,3), x_4(4,4), x_5(5,4), x_6(6,4)$

- an informal picture
- joint distribution does not seem accessible (and not via Schur processes)

$$\widehat{U(4)} \xrightarrow{\text{Bernoulli step}} \widehat{U(4)} \xrightarrow{\text{Geometric; next particle}} \widehat{U(5)} \xrightarrow{\text{Geometric; next particle}} \widehat{U(6)}$$

Accessible time-like joint distributions

In the higher spin model we can compute (at least for $q > 0$ **[Borodin–P. '16]**) joint observables of

$$x_N(N,T), x_{N+1}(N+1,T), x_{N+2}(N+2,T), \ldots$$

which are time-like (though very specific)

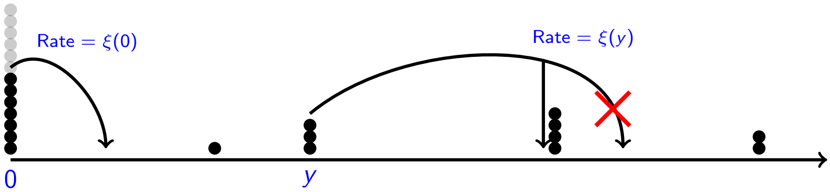
Bonus: asymptotics of TASEP in continuous inhomogeneous space

Continuous space TASEP

A (mostly straightforward) limit of the inhomogeneous stochastic vertex model (powered by TASEPs with particle-dependent jumps) brings the following continuous time particle system on ordered particles $y_1 \ge y_2 \ge \dots$ in $\mathbb{R}_{\ge 0}$:

- initially there are infinitely many particles at 0;
- one particle can leave a stack at location y at rate $\xi(y)$, where ξ is an arbitrary (positive, nice) function;
- the jumping particle wants to jump an exponential distance with mean $1/L$;
- particles preserve order (so a flying particle joins the existing stack)

Define the height function $H(t, y) := \#\{\text{number of particles} \ge y \text{ at time } t\}$.



Asymptotics via Schur measures **[Knizel–P. '17, in prep.]**

Let $L \to \infty$ (jumps are small), $t = \tau L$ (many particles). There exists a limit shape $\mathbf{H}(\tau, y)$ such that

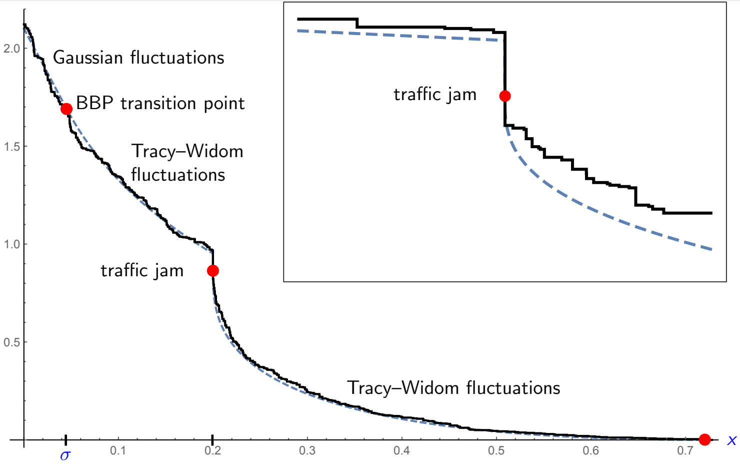
$$\lim_{L\to\infty} \mathbb{P}\left(\frac{H(\tau L, y) - L\mathbf{H}(\tau, y)}{\sigma(y) L^{\frac{1}{3}}} \geq -r\right) = F_2(r).$$

- If $\xi(0) \geq \lim_{\varepsilon\to 0} \xi(\varepsilon)$ then the Tracy–Widom fluctuations are for all $y > 0$;
- If $\xi(0) < \lim_{\varepsilon\to 0} \xi(\varepsilon)$ then there are Tracy–Widom fluctuations for $y > \sigma > 0$, Gaussian fluctuations on the scale $L^{\frac{1}{2}}$ on $(0, \sigma)$, and a Baik–Ben Arous–Péché phase transition at σ.
- The limit shape $\mathbf{H}$ can be explicitly described. If ξ is piecewise constant then $\mathbf{H}$ is piecewise algebraic.
- There is another type of phase transitions in this model — traffic jams caused by slowdown in ξ

Asymptotics via Schur measures [Knizel–P. '17, in prep.]

Let $L \to \infty$ (jumps are small), $t = \tau L$ (many particles). There exists a limit shape $\mathbf{H}(\tau, y)$ such that

$$\lim_{L\to\infty} \mathbb{P}\left(\frac{H(\tau L, y) - L\mathbf{H}(\tau, y)}{\sigma(y) L^{\frac{1}{3}}} \geq -r\right) = F_2(r).$$



Overview and perspectives:

- Commuting Markov operators from Schur processes / asymptotic character theory of $U(\infty)$ bring new particle systems and tools to study them
- Particle systems in continuous inhomogeneous space display traffic jam phase transitions with Tracy–Widom fluctuations on both sides
- How to better describe the structure of time-like joint distributions not falling under Schur processes formalism?
- Anything new if we include the other group of parameters $\alpha^-, \beta^-, \gamma^-$?
- How to extend this to commuting Markov operators on 2-dimensional particle configurations?

Overview and perspectives:

- Commuting Markov operators from Schur processes / asymptotic character theory of $U(\infty)$ bring new particle systems and tools to study them
- Particle systems in continuous inhomogeneous space display traffic jam phase transitions with Tracy–Widom fluctuations on both sides
- How to better describe the structure of time-like joint distributions not falling under Schur processes formalism?
- Anything new if we include the other group of parameters $\alpha^-, \beta^-, \gamma^-$?
- How to extend this to commuting Markov operators on 2-dimensional particle configurations?

Things left out:

- Along the way of studying the continuous space TASEP we discovered new continuous-parameter limits of Schur measures and processes. They live on half-infinite or infinite configurations
- Everything discussed in the talk except connections to irreducible characters has a q-deformation related to q-Whittaker polynomials ($=$ Macdonald polynomials with $t = 0$). Asymptotic behavior of the continuous space inhomogeneous model for fixed q is the same (with q-modified $\mathbf{H}$) **[Borodin–P., arXiv:1702.????]**