Markov chains for promotion and nonabelian sandpile models or The power of $\mathscr{R}$-trivial monoids

Anne Schilling

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based on

- A. Ayyer, S. Klee, A. Schilling, J. Alg. Comb. **39** (2014)
- A. Ayyer, A. Schilling, B. Steinberg, N. Thiéry, Comm. Math. Phys. **335** (2015)
- A. Ayyer, A. Schilling, B. Steinberg, N. Thiéry, Int. J. Alg. & Comp. **25** (2015)
- A. Ayyer, A. Schilling, N. Thiéry, Exp. Math. **26** (2017)

Institut Henri Poincaré, Paris, February 20, 2017

Outline

- **Promotion Markov Chains:**
  Markov chains on linear extensions of finite posets via promotion operators:
  - Nice stationary distributions!
  - Integer eigenvalues and nice multiplicities for rooted forests (generalizations of derangements)

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- **Representation Theory of Monoids:**
  - Use the representation theory of $\mathscr{R}$-trivial monoids.

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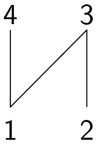
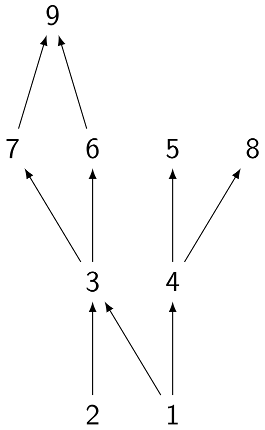
- Eg: $P =$ 

, $\mathcal{L}(P) = \{1234, 1243, 1423, 2134, 2143\}$.

Promotion (aka jeu de taquin)

[Schützenberger '72]



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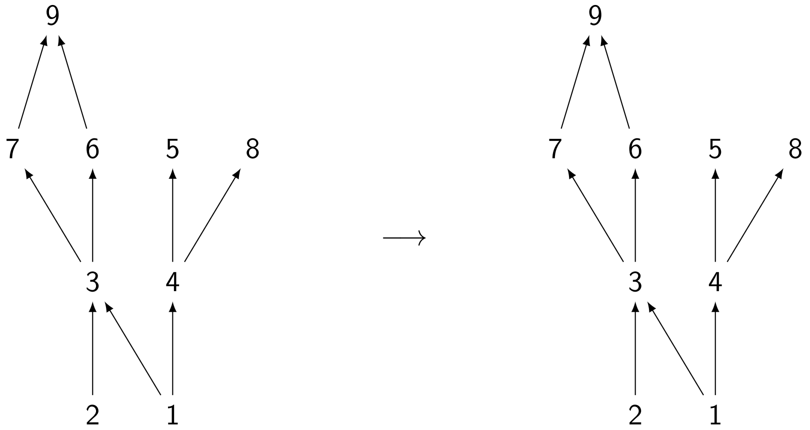

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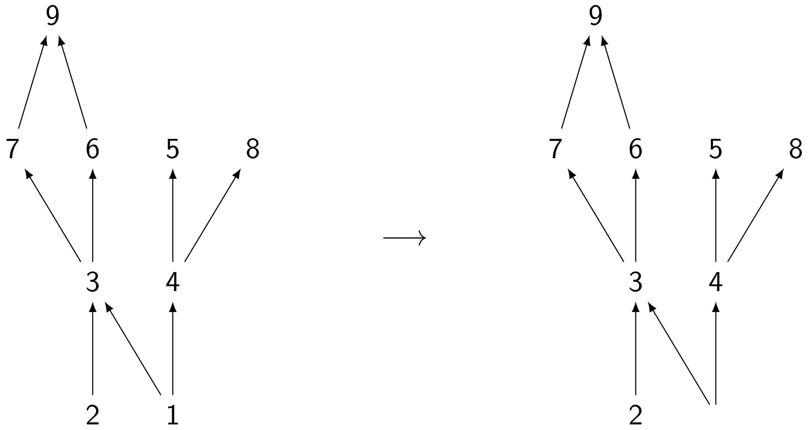

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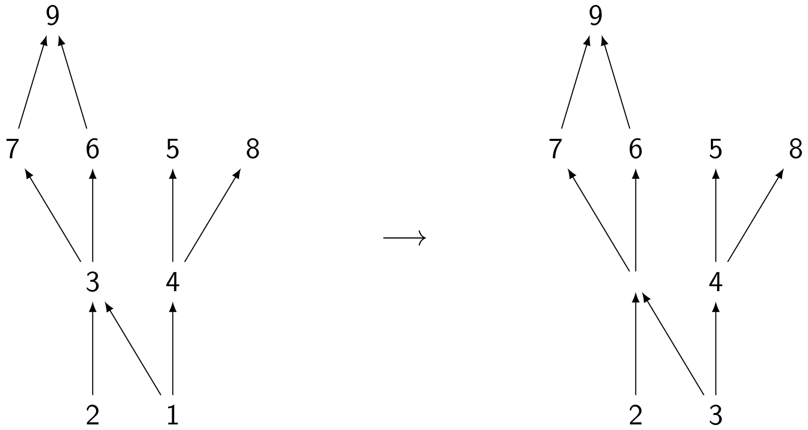

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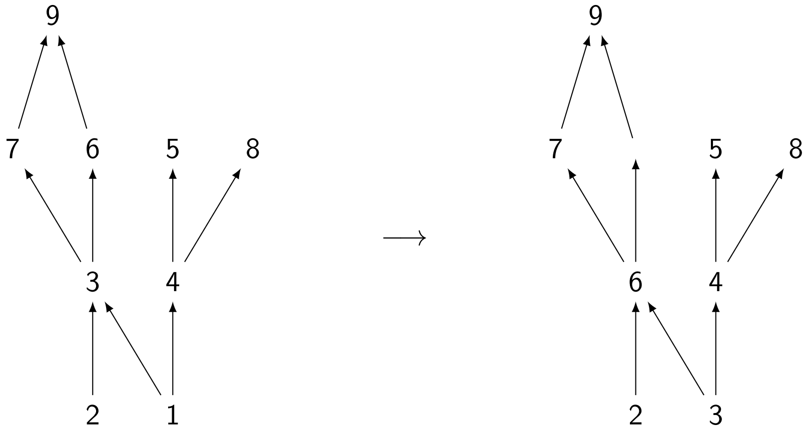

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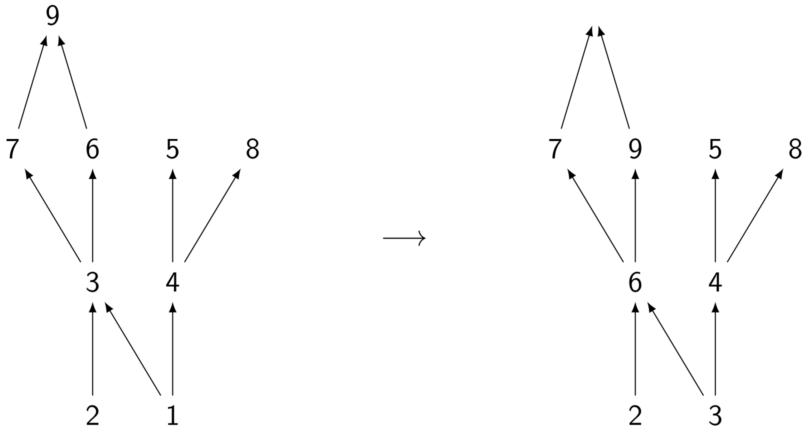

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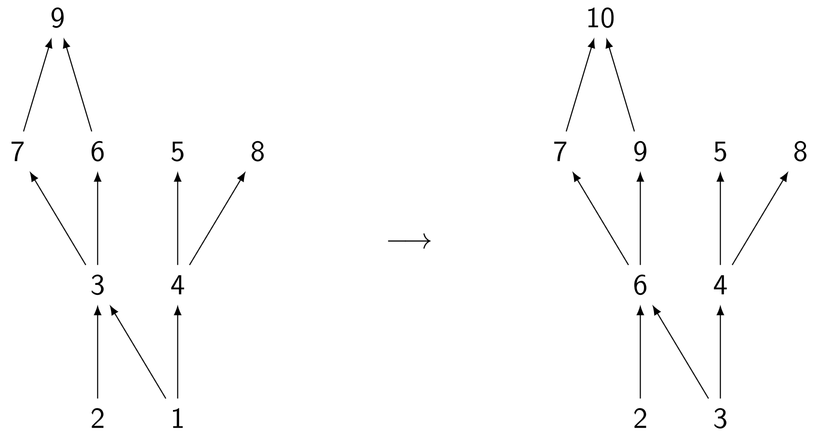

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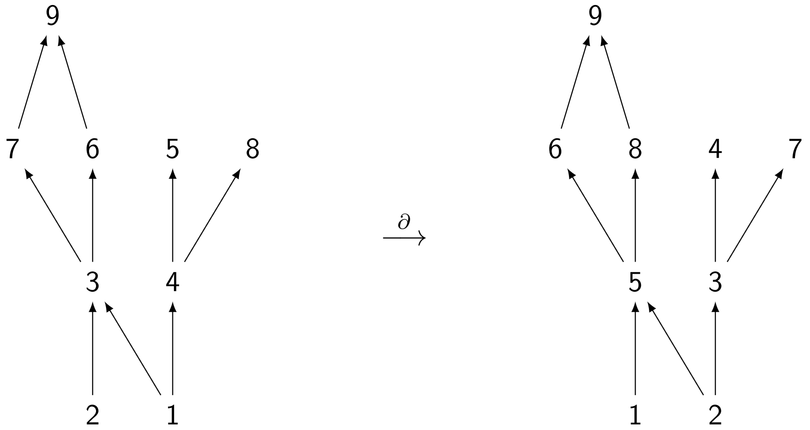

Algebraic formulation in terms of transposition

- Define τ_i on $\mathcal{L}(P)$

$$\tau_i \pi = \begin{cases} \pi_1 \cdots \pi_{i-1}\pi_{i+1}\pi_i \cdots \pi_n & \text{if } \pi_i \text{ and } \pi_{i+1} \text{ are not comparable in } P, \\ \pi_1 \cdots \pi_n & \text{otherwise.} \end{cases}$$

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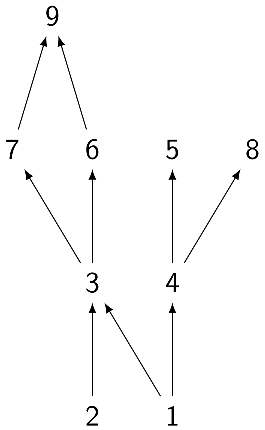
The case $j = 1$ is the previous promotion operator.

Algebraic formulation on previous example



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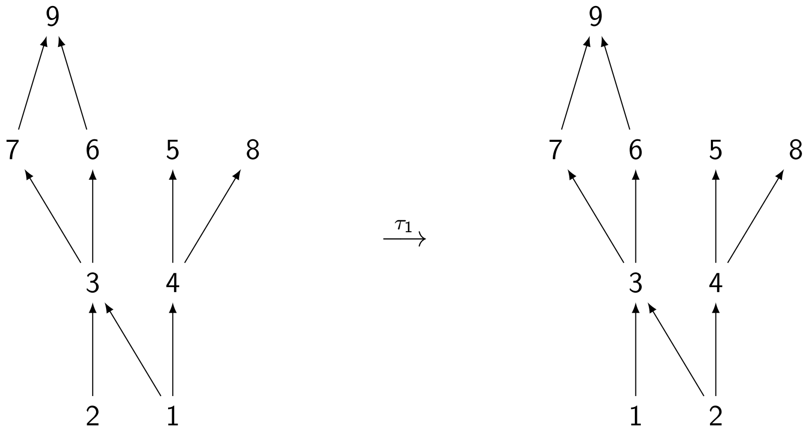

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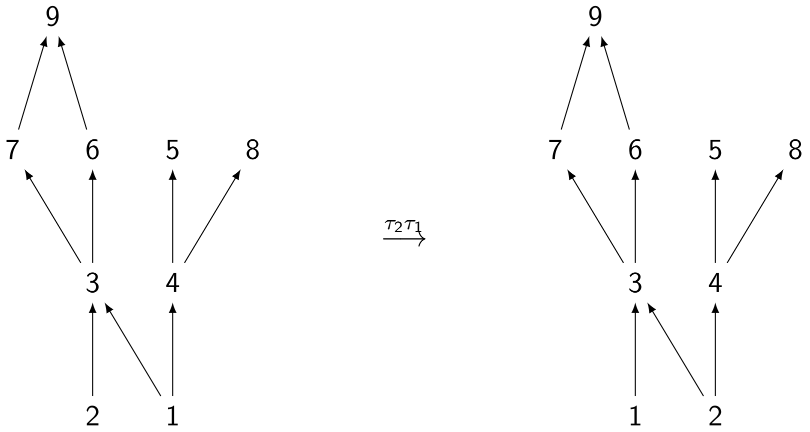

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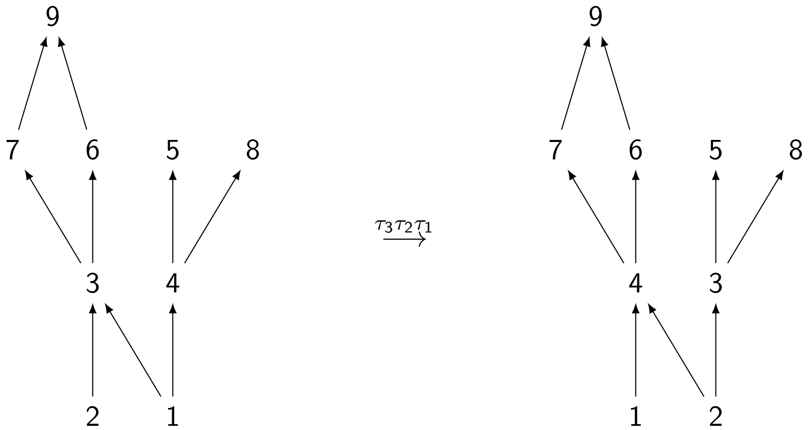

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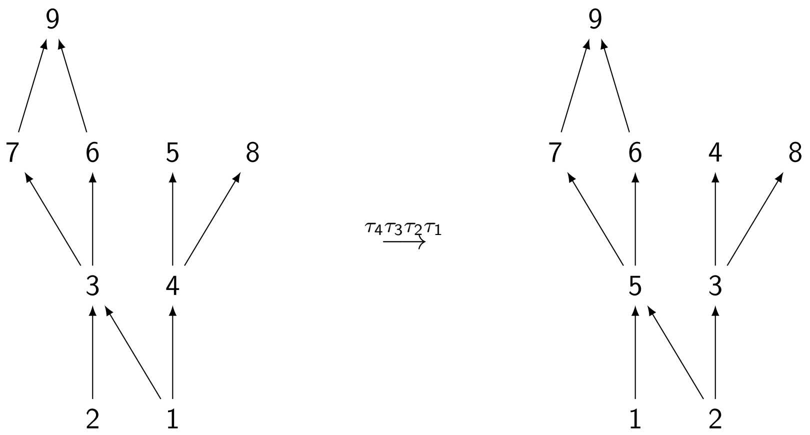

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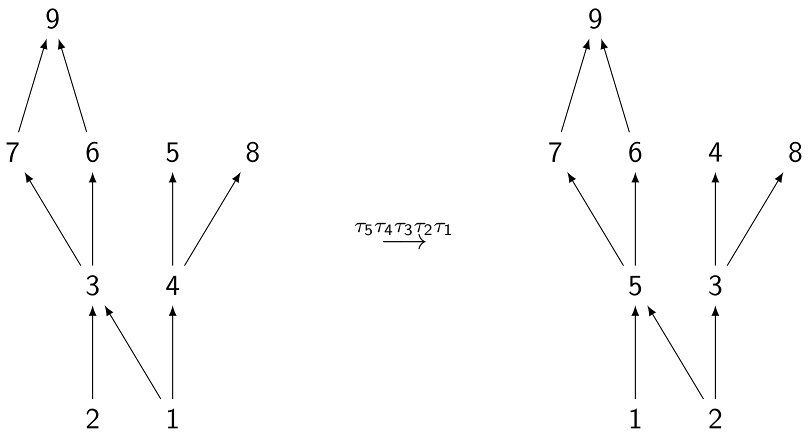

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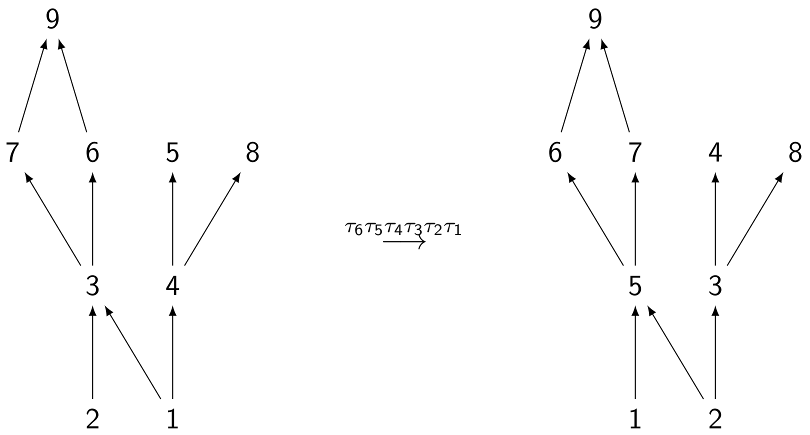

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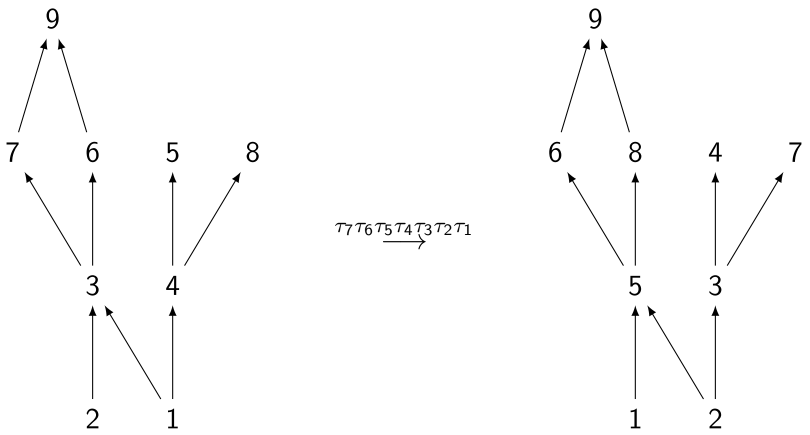

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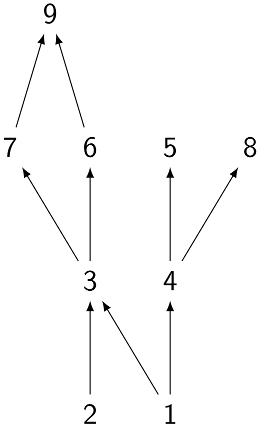
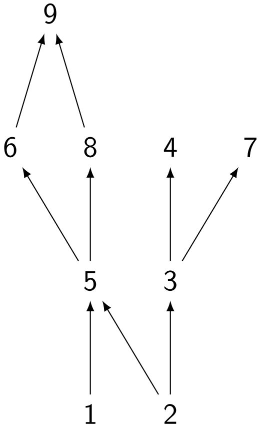


$$\xrightarrow{\tau_8\tau_7\tau_6\tau_5\tau_4\tau_3\tau_2\tau_1}$$



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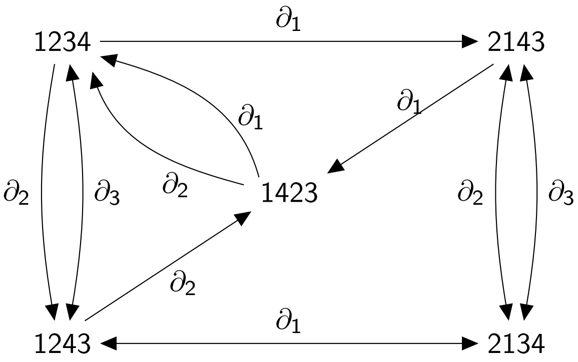

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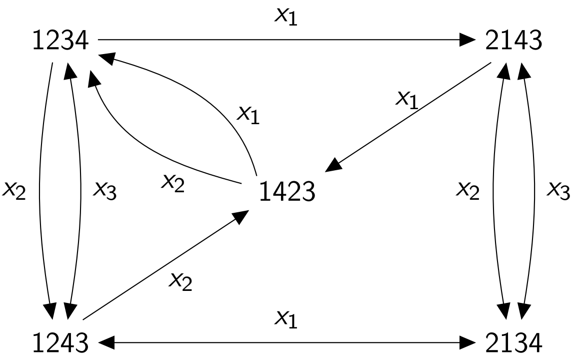
- Follows from the fact that $\partial_i^k = \partial_i$ for large enough k.

Example



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The stationary state weight $w(\pi)$ of the linear extension $\pi \in \mathcal{L}(P)$ for the continuous time Markov chain for the promotion graph is given by

$$w(\pi) = \prod_{i=1}^{n} \frac{x_1 + \cdots + x_i}{x_{\pi_1} + \cdots + x_{\pi_i}} \,,$$

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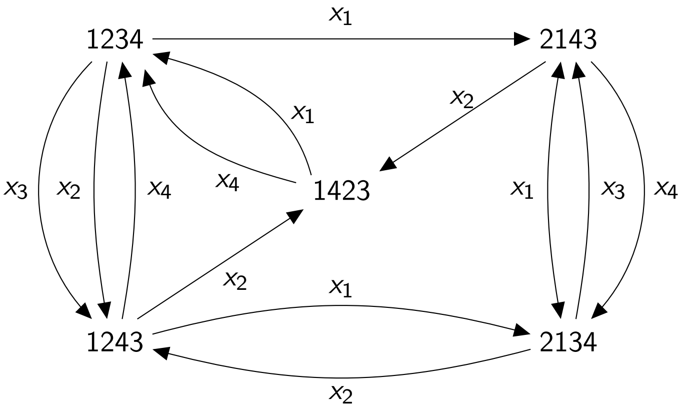
- Note that $w(\pi)$ is independent of P.
- Proved by induction.

Example



Example (continued)

The transition matrix this time is given by

$$\begin{pmatrix} x_4 & x_4 & x_1+x_4 & 0 & 0 \\ x_2+x_3 & x_3 & 0 & x_2 & 0 \\ 0 & x_2 & x_2+x_3 & 0 & x_2 \\ 0 & x_1 & 0 & x_4 & x_1+x_4 \\ x_1 & 0 & 0 & x_1+x_3 & x_3 \end{pmatrix}$$

Notice that row sums are no longer one.
The stationary distribution is

$$\left(1, \quad \frac{x_1+x_2+x_3}{x_1+x_2+x_4}, \quad \frac{(x_1+x_2)(x_1+x_2+x_3)}{(x_1+x_2)(x_1+x_2+x_4)}, \quad \frac{x_1}{x_2}, \quad \frac{x_1(x_1+x_2+x_3)}{x_2(x_1+x_2+x_4)}\right).$$

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- A union of chains is also a rooted forest.

More terminology

- An upper set S in P is a subset of $[n]$ such that if $x \in S$ and $y \succeq x$, then also $y \in S$.
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- $f([y, \hat{1}])$ is the number of maximal chains in the interval $[y, \hat{1}]$.
- Brown defined, for each element $x \in L$, a derangement number d_x

$$d_x = \sum_{y \succeq x} \mu(x, y) f([y, \hat{1}]) \, .$$

Spectrum of the Transition Matrix

Theorem (AKS 2014)

Let P be a rooted forest, M the transition matrix of the promotion graph. Then

$$\det(M - \lambda \mathbb{1}) = \prod_{\substack{S \subseteq [n] \\ S \text{ upper set in } P}} (\lambda - x_S)^{d_S},$$

where $x_S = \sum_{i \in S} x_i$ and d_S is the derangement number in the lattice L (by inclusion) of upper sets in P.

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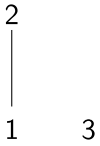
In other words, for each upper set $S \subseteq [n]$, there is

- an eigenvalue x_S
- with multiplicity d_S.

Running Example

$P =$ 

- $\mathcal{L}(P) = \{123, 132, 312\}$
- Upper sets: ϕ, $\{2\}$, $\{3\}$, $\{2,3\}$, $\{1,2\}$, $\{1,2,3\}$
- Eigenvalues of M: $x_1 + x_2 + x_3$, x_2, 0.

A special case: the Tsetlin library

The Tsetlin library:

- n books on a shelf

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- Once the book is chosen, it is moved to the back:

$$\boxed{B_1}\boxed{B_2}\cdots\boxed{B_i}\cdots\boxed{B_n} \quad \to \quad \boxed{B_1}\boxed{B_2}\cdots\boxed{B_n}\boxed{B_i}$$

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Same as promotion Markov chain on the antichain!

In this case $\mathcal{L}(P) = S_n$.

A Markov chain on permutations

- Let $\pi \in S_n$ be a permutation.
- The stationary distribution of the Tsetlin library is given by [Hendricks '72]

$$\mathbb{P}(\pi) = \prod_{i=1}^{n} \frac{x_{\pi_i}}{x_{\pi_1} + \cdots + x_{\pi_i}}.$$

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- A derangement is a permutation without fixed points. Let d_m be the number of derangements in S_m.
- Let M_n be the transition matrix. Then [Phatarfod '91]

$$\det(M_n - \lambda \mathbb{1}) = \prod_{S \subset [n]} (\lambda - x_S)^{d_{[n] \setminus |S|}}$$

where $x_S = \sum_{i \in S} x_i$.

Example

- The case of $n = 3$:

$$M_3 = \begin{pmatrix} x_3 & x_3 & 0 & 0 & x_3 & 0 \\ x_2 & x_2 & x_2 & 0 & 0 & 0 \\ 0 & 0 & x_3 & x_3 & 0 & x_3 \\ x_1 & 0 & x_1 & x_1 & 0 & 0 \\ 0 & 0 & 0 & x_2 & x_2 & x_2 \\ 0 & x_1 & 0 & 0 & x_1 & x_1 \end{pmatrix} \begin{bmatrix} 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{bmatrix}$$

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- $$\mathbb{P}(231) = \frac{x_3 x_1}{(x_2 + x_3)(x_1 + x_2 + x_3)}$$

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- Eigenvalues: $1, x_3, x_2, x_1$ and 0 twice.

Generalizations

- Umpteen generalizations!
- Different moves, more shelves.
- Infinite libraries.
- Hyperplane arrangements [Bidigare, Hanlon, Rockmore '99]
- Left regular bands (monoids) [Brown '00]
- $\mathcal{R}$-trivial monoids [Steinberg '06]

Outline

Abelian sandpile models / chip-firing games

- A graph G
- Configuration: distribution of grains of sand at each site
- Grains fall in at random
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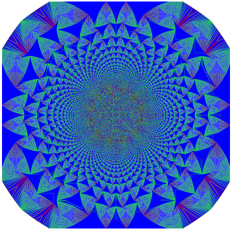

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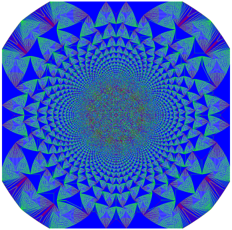

- Prototypical model for the phenomenon of self-organized criticality, like a heap of sand

Arborescences or upward rooted trees

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Arborescences or upward rooted trees

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- Set of leaves L: vertices with in-degree zero.

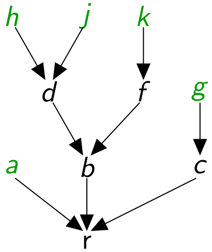

Figure: An arborescence with leaves at a, g, h, j, k.

Configurations

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- Variable t_v: the number of grains of sand at $v \in V$.

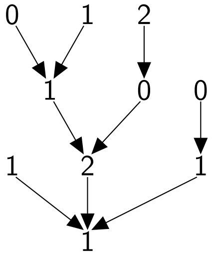

Figure: A configuration with all thresholds 2.

Markov Chains

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- Unlike in the abelian sandpile model, sand grains only enter at leaves.
- The operators defining the entrance of sand grains are the same in both models.

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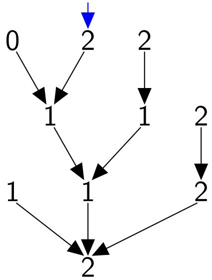
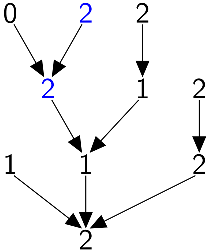
Follow the path $\ell^{\downarrow}$ from ℓ to the root r

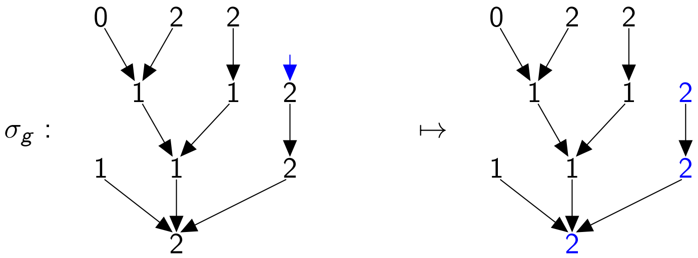
- Add a grain to the first vertex along the way that has not yet reached its threshold, if such a vertex exists.
- If no such vertex exists, then the grain is interpreted to have left the tree at the root and $\sigma_\ell(t) = t$.

σ_j :



$\mapsto$




σ_g :
0
2
2
1
1
2
1
1
2
2
↦
0
2
2
1
1
2
1
1
2
2

Topple operators

Definition (Trickle-down sandpile model)

$$\theta_v : \Omega(\mathcal{T}) \to \Omega(\mathcal{T})$$

θ_v moves one grain from $v \in V$ to the first available site along $v^{\downarrow}$. If no such site exists, the grain exits the system.

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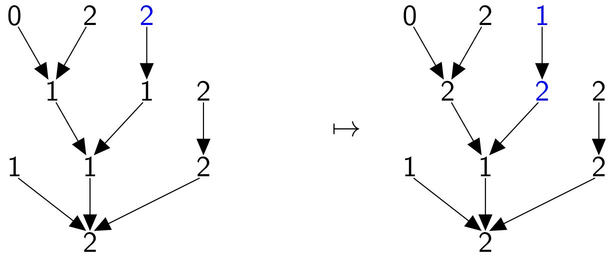
Remark

If $t_v = 0$ (no grain at site v), then $\theta_v(t) = \tau_v(t) = t$.

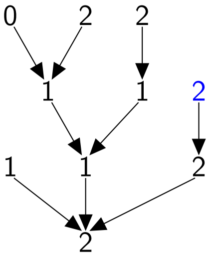
Toppling in the Trickle-down sandpile model

θ_k :



Toppling in the Trickle-down sandpile model

θ_g :



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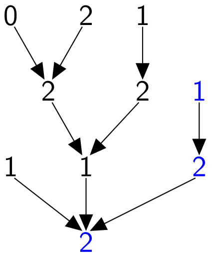

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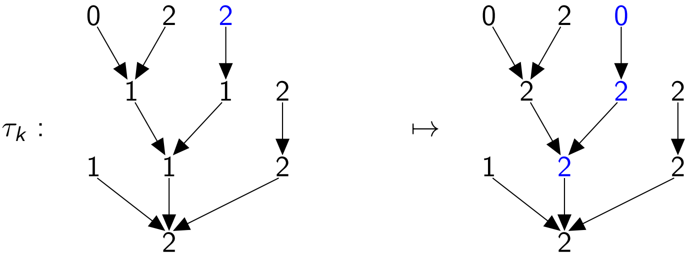

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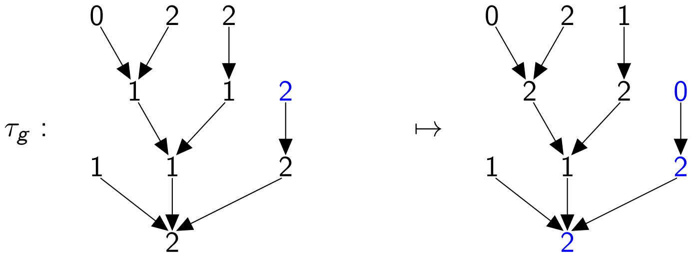

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  We assume that
  1. $$0 < x_v, y_\ell \leq 1$$
  2. $$\sum_{v \in V} x_v + \sum_{\ell \in L} y_\ell = 1$$

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- Sources on all vertices: Allow source operators at all vertices, not just leaves!

Ergodicity

Proposition (ASST 2015)

G_θ: directed graph with

- *vertex set $\Omega(\mathcal{T})$*
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Then G_θ is strongly connected and hence the Trickle-down sandpile model is ergodic.

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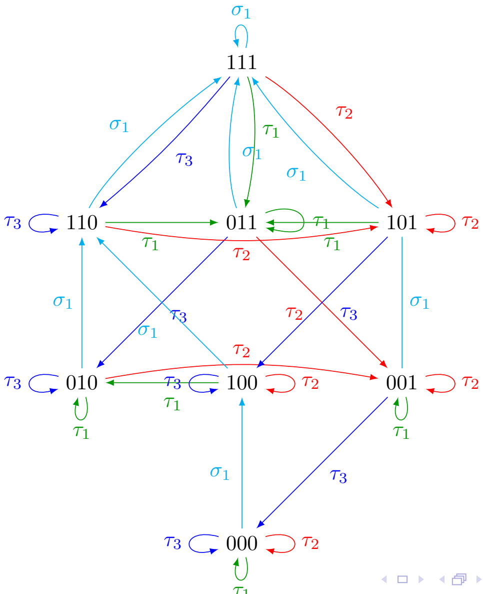
Markov chains on a line with thresholds 1



Trickle-down sandpile model: Stationary distribution

- $L_v := \{\ell \in L \mid v \text{ is a vertex of } \ell^{\downarrow}\}$
- $Y_v := \sum_{\ell \in L_v} y_\ell$
- For $0 \leq h \leq T_v$

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Theorem (ASST 2015)

The stationary distribution of the Trickle-down sandpile Markov chain defined on G_θ is given by the product measure

$$\mathbb{P}(t) = \prod_{v \in V} \rho_v(t_v).$$

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$$\mu_v(h) := \begin{cases} \dfrac{Y_v^h x_v}{(Y_v + x_v)^{h+1}} & \text{if } h < T_v \\[2ex] \dfrac{Y_v^{T_v}}{(Y_v + x_v)^{T_v}} & \text{if } h = T_v \end{cases}$$

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Theorem (AST 2015)

Let $T_v = 1$ for all $v \in V$, $v \neq \mathrm{r}$ and $T_{\mathrm{r}} = m$ for some positive integer m. Then the stationary distribution of the Landslide sandpile model defined on G_τ is given by the product measure

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Landslide sandpile model: Spectrum

For subsets $S \subseteq V$ and $\ell^{\downarrow}$ the set of vertices on path from ℓ to r:

$$y_S = \sum_{\ell \in L, \ell^{\downarrow} \subseteq S} y_\ell \quad \text{and} \quad x_S = \sum_{v \in S} x_v .$$

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Multiplicities: T_{S^c}

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Rate of convergence: Total variation distance from stationarity after k steps $||P^k - \pi||$.

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Mixing time is at most $\frac{2(n+c-1)}{p}$.

Outline

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Example

$w = 231231 \in \mathfrak{R}$ for S_4. Then $\partial_1(w) = 123121$ since $123123 = 121323 = 212323$ is not reduced!

Methods from the representation theory of monoids

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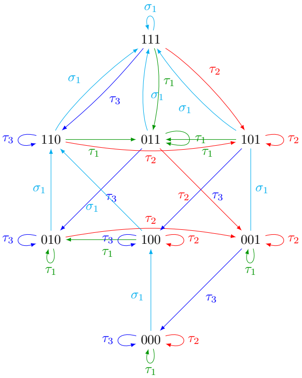
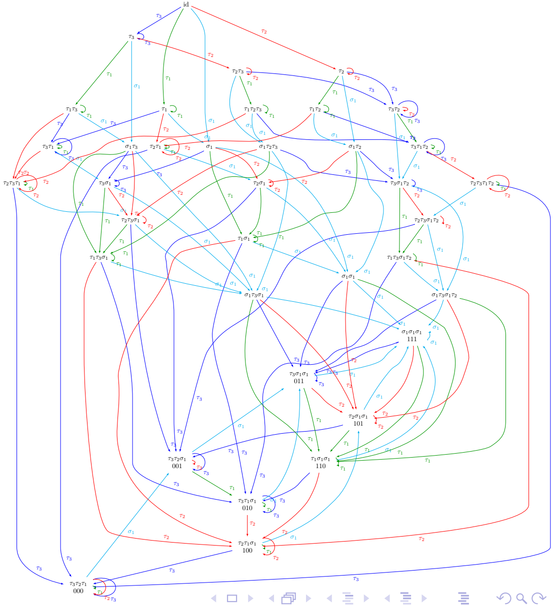
Alternatively from transition matrix M of Markov chain:

$$m_i = M_{x_i=1; x_1=\cdots=x_{i-1}=x_{i+1}=\cdots=x_n=0}$$

The left Cayley graph for the 1D sandpile model





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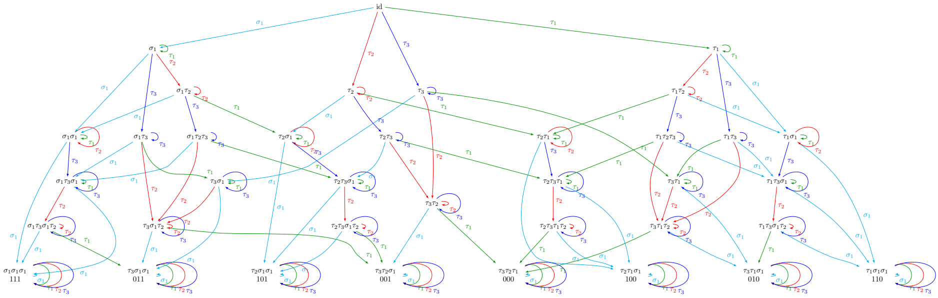

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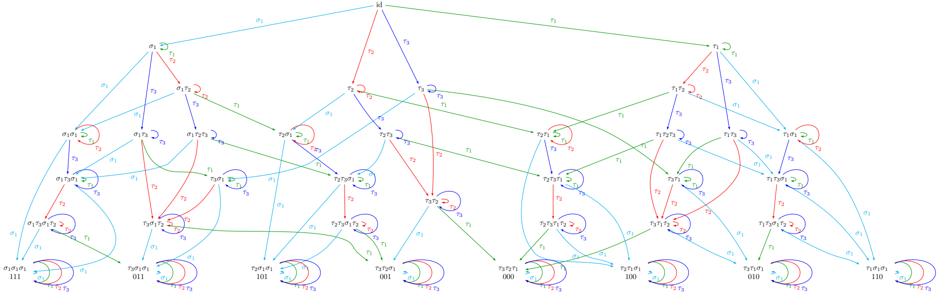

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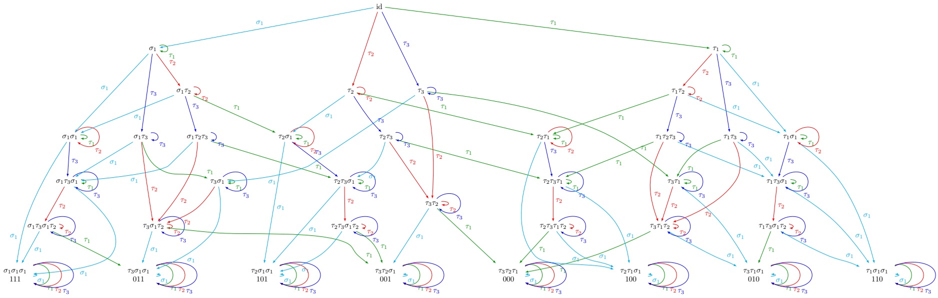

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Definitions: Green relations

- Left and right preorders on $\mathcal{M}$:

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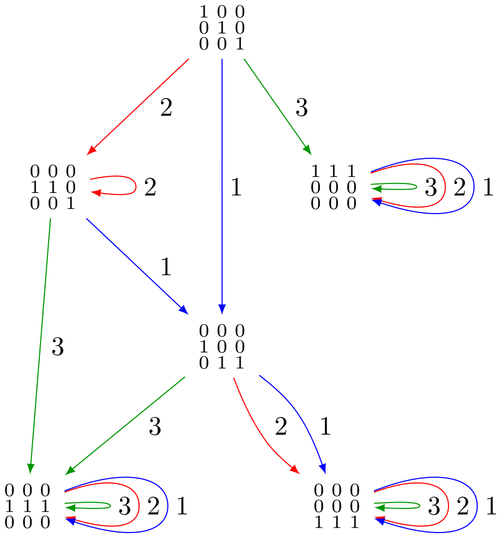
Definition

$\mathcal{M}$ is $\mathscr{R}$-trivial ($\mathscr{L}$-trivial) if all $\mathscr{R}$-classes ($\mathscr{L}$-classes) are singletons. Equivalently, if the preorders are partial orders.

$\mathscr{R}$-trivial monoid for promotion example



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Using representation theory of right regular band

- Tsetlin library, Hyperplane arrangements, ...
- Bidigare, Hanlon, Rockmore '99, Brown '00, Saliola, ...
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Outlook

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Mixing time for linear extensions [AST17]:

- Counting linear extensions is an important problem in practice.
- Define random-to-random promotion operator on posets via $\tau_i \tau_{i+1} \cdots \tau_j$.
- Explicit conjecture for second largest eigenvalue for random-to-random shuffling on posets.
- Conjectured mixing time is $O(n^2 \log n)$ (as opposed to Bubley-Dyer's result of $O(n^3 \log n)$)

Thank you !