Jack characters: dual combinatorics of Jack polynomials

Piotr Śniady

Polska Akademia Nauk

motivations for Jack characters:

- deformation of characters of the symmetric groups $\mathfrak{S}(n)$,
- Jack polynomials,
- extra deformation parameter α,
  good for algebraic combinatorics
- amazing combinatorial conjectures
  related to $\longrightarrow$ maps,
- extra deformation parameter α,
  new scaling and universality for random matrices,
  β-ensembles
  $\longrightarrow$ talk of Dołęga

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asymptotic representation theory of symmetric groups

the symmetric group $\mathfrak{S}(n)$
is the group of permutations of $\{1, \dots, n\}$

$$\mathfrak{S}(1) \subset \mathfrak{S}(2) \subset \cdots \subset \mathfrak{S}(\infty)$$

can we study (representations of) the symmetric groups $\mathfrak{S}(n)$

- in the limit $n \to \infty$?
- for all n at the same time?
- for $n = \infty$?

$\longrightarrow$ Vershik

characters of the symmetric group $\mathfrak{S}(n)$:

$$\chi^{(1)}_\lambda(\pi) := \frac{\operatorname{Tr} \rho_\lambda(\pi)}{\operatorname{Tr} \rho_\lambda(\mathrm{Id})}$$

λ - Young diagram with $|\lambda| = n$ boxes

π - partition of n

π - fixed partition

λ - Young diagram with arbitrary number of boxes

$$\mathrm{Ch}^{(1)}_\pi(\lambda) := \begin{cases} \underbrace{|\lambda| \cdot (|\lambda|-1) \cdots (|\lambda|-|\pi|+1)}_{|\pi| \text{ factors}} \; \chi^{(1)}_\lambda(\pi, 1, \dots, 1) & \text{if } |\pi| \leq |\lambda| \\ 0 & \text{otherwise} \end{cases}$$

⟶ Kerov & Olshanski

characters of the symmetric group $\mathfrak{S}(n)$:

λ - Young diagram with
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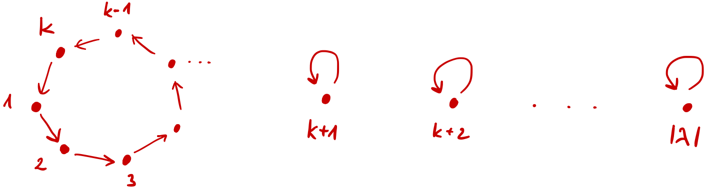
π - partition of n

π = (k)



$$\mathrm{Ch}_k^{(1)}(\lambda) := \begin{cases} \underbrace{|\lambda|\cdot(|\lambda|-1)\cdots(|\lambda|-k+1)}_{k \text{ factors}} \; \chi_\lambda^{(1)}(k,1,\dots,1) & \text{if } k \leq |\lambda| \\ 0 & \text{otherwise} \end{cases}$$

⟶ Kerov & Olshanski

Jack characters:

$|\pi| = |\lambda| = n$

$$J_\lambda^{(\alpha)} = \sum_\pi \chi_\lambda^{(\alpha)}(\pi)\, p_\pi \frac{n!}{z_\pi}$$

$\alpha > 0$

$$\mathrm{Ch}_\pi(\lambda) = \mathrm{Ch}_\pi^{(\alpha)}(\lambda) =$$

$$\alpha^{-\frac{|\pi|-\ell(\pi)}{2}} \underbrace{|\lambda| \cdot (|\lambda|-1) \cdots (|\lambda|-|\pi|+1)}_{|\pi| \text{ factors}} \chi_\lambda^{(\alpha)}(\pi, 1, 1, \ldots, 1)$$

if $|\pi| \leq |\lambda|$

⟶ Lassalle; Dołęga & Féray

examples

$$\mathrm{Ch}_1(\lambda) = \sum_i \lambda_i,$$

$$\mathrm{Ch}_2(\lambda) = \sqrt{\alpha} \sum_i \left(\lambda_i^2 - \lambda_i\right) - \frac{1}{\sqrt{\alpha}} \sum_i 2(i-1)\lambda_i$$

polynomials in $\lambda_1, \lambda_2, \ldots$

set $x_i = \lambda_i - \frac{i}{\alpha}$

$$\mathrm{Ch}_2(\lambda) = \sqrt{\alpha} \sum_i \left(x_i^2 - \left(\frac{-i}{\alpha} \right)^2 \right) + \\ + \left(-\sqrt{\alpha} + \frac{2}{\sqrt{\alpha}} \right) \sum_i \left(x_i - \left(\frac{-i}{\alpha} \right) \right)$$

symmetric polynomials in $x_1, x_2, \ldots$

proof $\longrightarrow$ Lassalle

$\longrightarrow$ algebra of α-polynomial functions of Kerov & Olshanski for each π and each $\alpha > 0$ $\longrightarrow$ FÉRAY

$\mathrm{Ch}_\pi(\lambda_1, \lambda_2, \dots)$ is the unique polynomial such that:

- $\mathrm{Ch}_\pi\left(x_1 + \frac{1}{\alpha}, x_2 + \frac{2}{\alpha}, \dots\right)$ is symmetric in $x_1, x_2, \dots$;
- polynomial $\mathrm{Ch}_\pi(\lambda_1, \lambda_2, \dots)$ is of degree $|\pi|$; its top-degree homogeneous part is equal to the power-sum symmetric function
$$\alpha^{\frac{|\pi|-\ell(\pi)}{2}} \ p_\pi;$$
- for all partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $|\lambda| < |\pi|$
$$\mathrm{Ch}_\pi(\lambda_1, \lambda_2, \dots) = 0$$

structure coefficients for Jack characters:

$$\mathrm{Ch}_2 \mathrm{Ch}_2 = 2\delta \mathrm{Ch}_2 + 2 \mathrm{Ch}_{1,1} + 4 \mathrm{Ch}_3 + \mathrm{Ch}_{2,2},$$

$$\mathrm{Ch}_3 \mathrm{Ch}_2 = 6\delta \mathrm{Ch}_3 + \mathrm{Ch}_{3,2} + 6 \mathrm{Ch}_{2,1} + 6 \mathrm{Ch}_4,$$

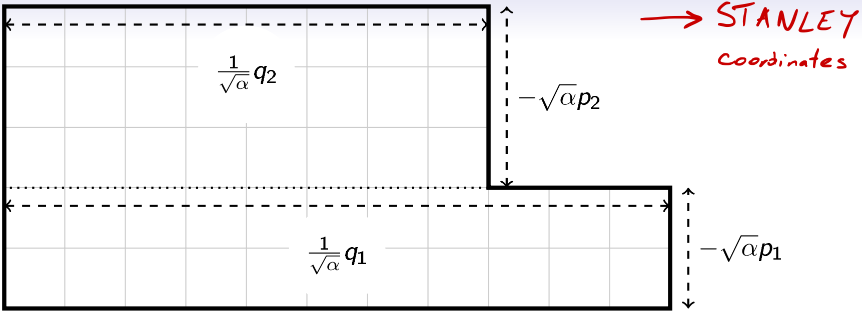
$$\begin{aligned}\mathrm{Ch}_3 \mathrm{Ch}_3 = {} & (6\delta^2 + 3) \mathrm{Ch}_3 + 9\delta \mathrm{Ch}_{2,1} + 18\delta \mathrm{Ch}_4 + 3 \mathrm{Ch}_{1,1,1} + \\ & + 9 \mathrm{Ch}_{3,1} + 9 \mathrm{Ch}_{2,2} + 9 \mathrm{Ch}_5 + \mathrm{Ch}_{3,3},\end{aligned}$$

$$\mathrm{Ch}_{2,2} \mathrm{Ch}_2 = 4\delta \mathrm{Ch}_{2,2} + 8 \mathrm{Ch}_4 + 4 \mathrm{Ch}_{2,1,1} + 8 \mathrm{Ch}_{3,2} + \mathrm{Ch}_{2,2,2}$$

where

$$\delta = \sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}$$

more conjectures $\longrightarrow$ Śniady `arXiv:1603.04268`;
partial results $\longrightarrow$ Burchardt

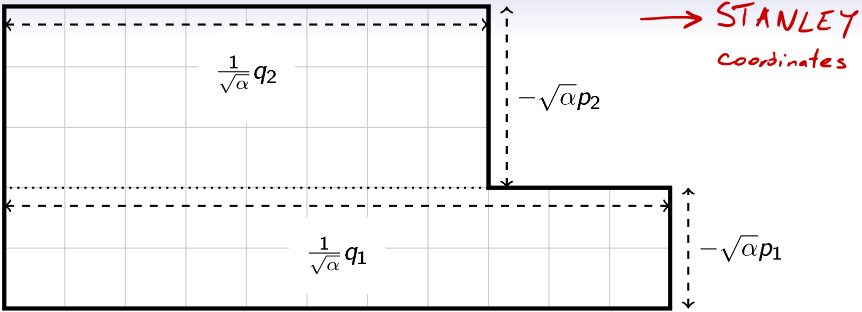


$$
\begin{aligned}
-\operatorname{Ch}_3 \quad &= p_1^3 q_1 + 3 p_1^2 q_1^2 + p_1 q_1^3 + 3 p_1^2 p_2 q_2 + 3 p_1 p_2^2 q_2 \\
&+ p_2^3 q_2 + 3 p_1 p_2 q_1 q_2 + 3 p_1 p_2 q_2^2 + 3 p_2^2 q_2^2 + p_2 q_2^3 \\
&+ 3 p_1^2 q_1 \gamma + 3 p_1 q_1^2 \gamma + 6 p_1 p_2 q_2 \gamma + 3 p_2^2 q_2 \gamma \\
&+ 3 p_2 q_2^2 \gamma + 2 p_1 q_1 \gamma^2 + 2 p_2 q_2 \gamma^2 + p_1 q_1 + p_2 q_2
\end{aligned}
$$

where

$$\gamma = -\delta = -\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}$$

$\longrightarrow$ STANLEY polynomials; see also $\longrightarrow$ KEROV polynomials;



$$
\begin{aligned}
-\mathrm{Ch}_3^{\mathrm{top}} = {} & p_1^3 q_1 + 3p_1^2 q_1^2 + p_1 q_1^3 + 3p_1^2 p_2 q_2 + 3p_1 p_2^2 q_2 \\
& + p_2^3 q_2 + 3p_1 p_2 q_1 q_2 + 3p_1 p_2 q_2^2 + 3p_2^2 q_2^2 + p_2 q_2^3 \\
& + 3p_1^2 q_1 \gamma + 3p_1 q_1^2 \gamma + 6p_1 p_2 q_2 \gamma + 3p_2^2 q_2 \gamma \\
& + 3p_2 q_2^2 \gamma + 2p_1 q_1 \gamma^2 + 2p_2 q_2 \gamma^2
\end{aligned}
$$

where

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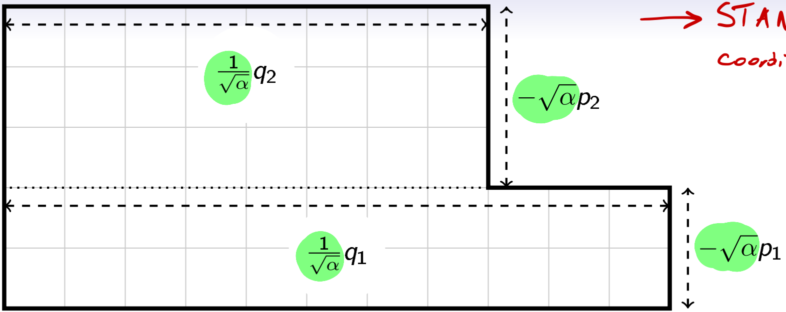
$\longrightarrow$ Stanley polynomials; see also $\longrightarrow$ Kerov polynomials;



→ STANLEY coordinates

$$\begin{aligned} -\operatorname{Ch}_3^{\text{top}} = {} & p_1^3 q_1 + 3p_1^2 q_1^2 + p_1 q_1^3 + 3p_1^2 p_2 q_2 + 3p_1 p_2^2 q_2 \\ & + p_2^3 q_2 + 3p_1 p_2 q_1 q_2 + 3p_1 p_2 q_2^2 + 3p_2^2 q_2^2 + p_2 q_2^3 \\ & + 3p_1^2 q_1 \gamma + 3p_1 q_1^2 \gamma + 6p_1 p_2 q_2 \gamma + 3p_2^2 q_2 \gamma \\ & + 3p_2 q_2^2 \gamma + 2p_1 q_1 \gamma^2 + 2p_2 q_2 \gamma^2 \end{aligned}$$

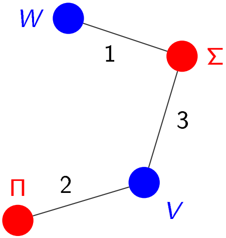
where

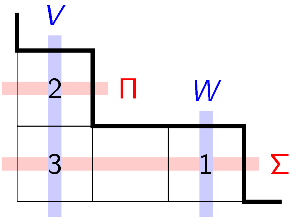
$$\gamma = -\delta = -\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}$$

$\longrightarrow$ STANLEY polynomials; see also $\longrightarrow$ KEROV polynomials;

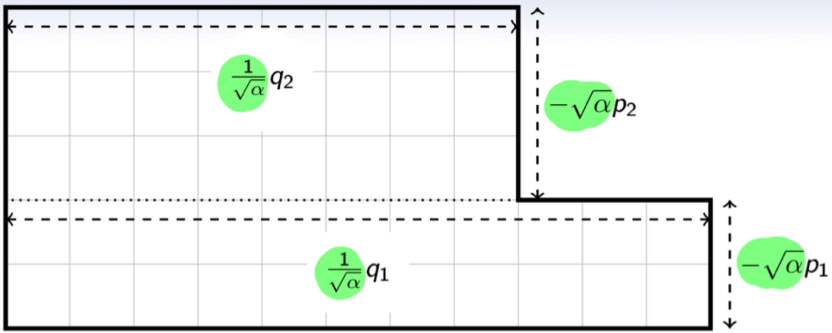
$$\mathrm{Ch}_\pi = ?$$

embeddings of a graph to a Young diagram





$$\mathfrak{N}_G(\lambda) = \sqrt{\alpha}^{|\text{blue vertices}|} \left(-\frac{1}{\sqrt{\alpha}}\right)^{|\text{red vertices}|} \#\text{embeddings of } G \text{ to } \lambda$$



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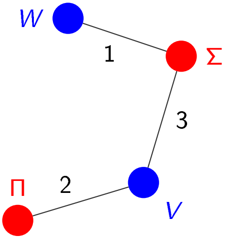
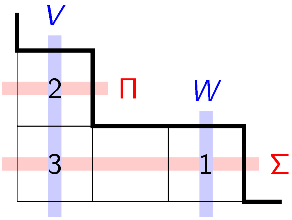
polynomial in $p_1, p_2, \dots, q_1, q_2, \dots$

with non-negative integer coefficients

embeddings of a graph to a Young diagram





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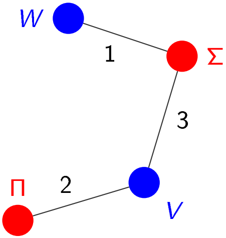
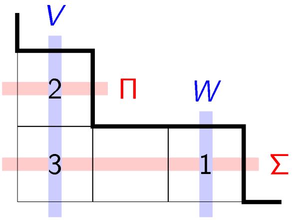
Problem

find some nice family of graphs such that

$$\mathrm{Ch}_\pi(\lambda) = \sum_G c_G \, \mathfrak{N}_G(\lambda)$$

embeddings of a graph to a Young diagram





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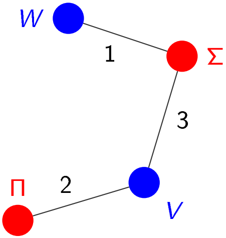

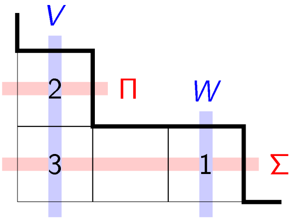

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Problem

find some nice family of maps such that

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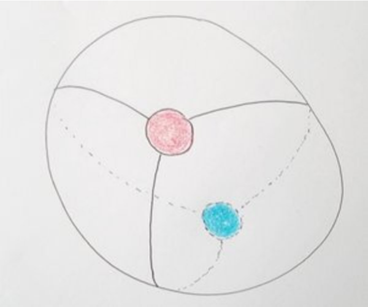
for $\alpha \in \{1, \frac{1}{2}, 2\}$ we know the answer!

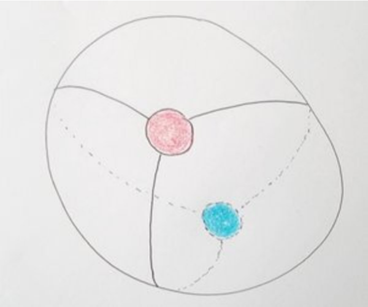
map is a graph on a surface

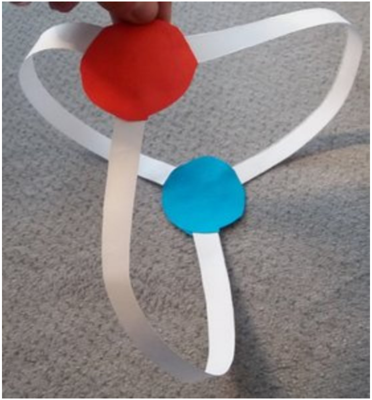


map on a sphere

map is a graph on a surface

each map can be visualized as a ribbon graph





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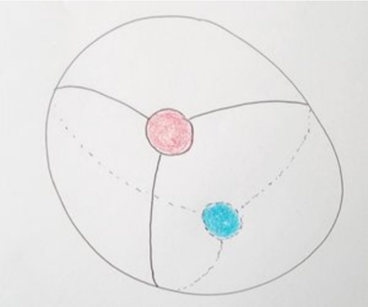
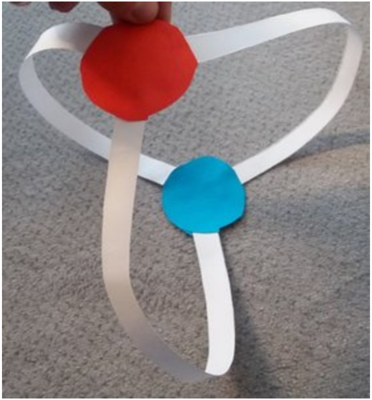
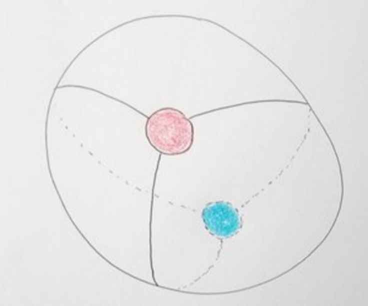
today: all maps are bicolored (red and blue vertices)





map on a sphere

we require that if we cut the surface along the edges
the surface breaks into a number of faces (=polygons)



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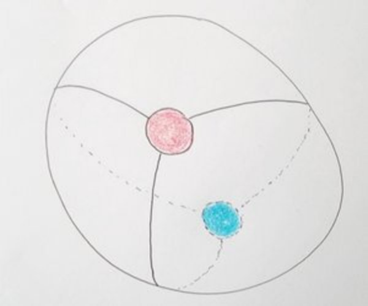

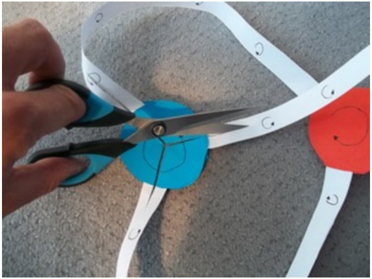

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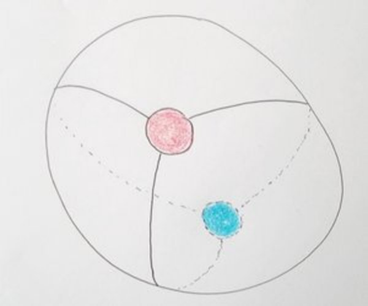

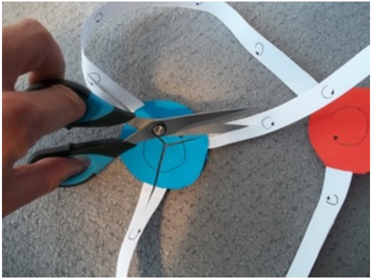

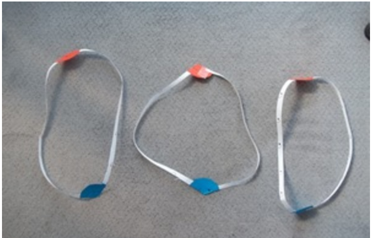

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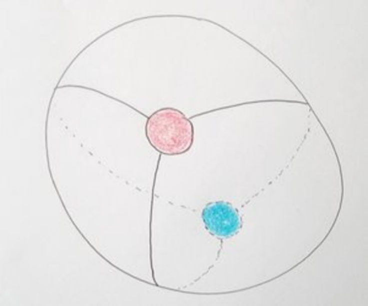

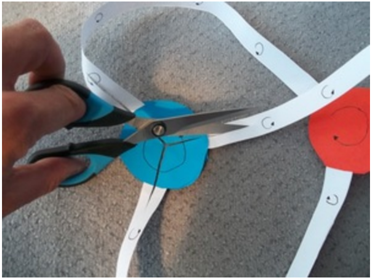

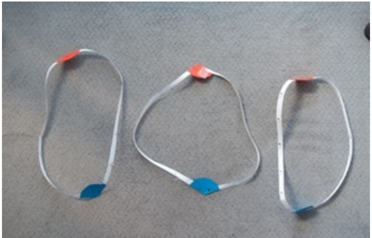

a map with three faces

one $2 \cdot 1$-gon, one $2 \cdot 1$-gon,
one $2 \cdot 1$-gon, so
face-type $(1, 1, 1)$

conjecture

there exists some nice family of coefficients $\text{mon}_M \in \mathbb{Q}[\gamma]$ such that

$$\text{Ch}_\pi(\lambda) = \sum_M \text{mon}_M \, \mathfrak{N}_M(\lambda),$$

where the sum runs over maps M with face-type π

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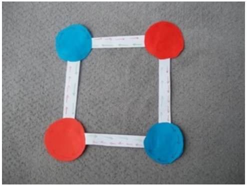
hint from Goulden & Jackson:
how (non)orientable is the map M?

mon= measure of nonorientability

for an edge E of a map M...

remove the edge E; what happens to the number of faces of $M \setminus E$?

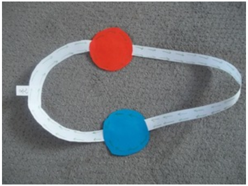
we say that E is:



border

if $\#\text{faces}(M \setminus E) = \#\text{faces}(M) - 1$;

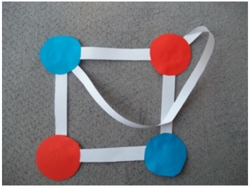
factor $:= \frac{1}{2}$



twisted

if $\#\text{faces}(M \setminus E) = \#\text{faces}(M)$;

factor $:= \gamma$



handle

if $\#\text{faces}(M \setminus E) = \#\text{faces}(M) + 1$;

factor $:= 1$

how non-orientable is your map?

1. choose random order of the edges!

how non-orientable is your map?

1. choose random order of the edges!
2. take the first edge;
   is it twisted / border / handle?
   calculate the corresponding factor!

how non-orientable is your map?

1. choose random order of the edges!
2. take the first edge;
   is it twisted / border / handle?
   calculate the corresponding factor!
3. remove this edge,

how non-orientable is your map?

1. choose random order of the edges!
2. take the next edge;
   is it twisted / border / handle?
   calculate the corresponding factor!
3. remove this edge,
4. take the next edge, repeat,

how non-orientable is your map?

1. choose random order of the edges!
2. take the next edge;
   is it twisted / border / handle?
   calculate the corresponding factor!
3. remove this edge,
4. take the next edge, repeat,

all edges removed? multiply all factors!

how non-orientable is your map?

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take the mean value of the product

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   is it twisted / border / handle?
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all edges removed? multiply all factors!

take the mean value of the product

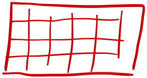
this is the measure of non-orientability $\mathrm{mon}(M)$ of a map M

conjecture

$$\mathrm{Ch}_\pi(\lambda) = \sum_M \mathrm{mon}_M \, \mathfrak{N}_M(\lambda),$$

where the sum runs over maps M with face-type π

- true for $\alpha \in \{\frac{1}{2}, 2\}$ any λ

- true for $\lambda =$  any α

bad news

$$\mathrm{Ch}_\pi(\lambda) \neq \sum_M \mathrm{mon}_M \, \mathfrak{N}_M(\lambda),$$

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good news

$$\mathrm{Ch}^{\mathrm{top}}_\pi(\lambda) = \sum_M \left([\text{top-degree part in } \gamma] \, \mathrm{mon}_M \right) \mathfrak{N}_M(\lambda),$$

where the sum runs over maps M with face-type π

bad news

$$\mathrm{Ch}_\pi(\lambda) \neq \sum_M \mathrm{mon}_M \, \mathfrak{N}_M(\lambda),$$

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$$\mathrm{Ch}^{\mathrm{top}}_\pi(\lambda) = \sum_M \left([\text{top-degree part in } \gamma] \, \mathrm{mon}_M \right) \mathfrak{N}_M(\lambda),$$

where the sum runs over maps M with face-type π

$$\begin{aligned} -\mathrm{Ch}_3 \quad &= p_1^3 q_1 + 3p_1^2 q_1^2 + p_1 q_1^3 + 3p_1^2 p_2 q_2 + 3p_1 p_2^2 q_2 \\ &+ p_2^3 q_2 + 3p_1 p_2 q_1 q_2 + 3p_1 p_2 q_2^2 + 3p_2^2 q_2^2 + p_2 q_2^3 \\ &+ 3p_1^2 q_1 \gamma + 3p_1 q_1^2 \gamma + 6p_1 p_2 q_2 \gamma + 3p_2^2 q_2 \gamma \\ &+ 3p_2 q_2^2 \gamma + 2p_1 q_1 \gamma^2 + 2p_2 q_2 \gamma^2 + p_1 q_1 + p_2 q_2 \end{aligned}$$

bad news

$$\mathrm{Ch}_\pi(\lambda) \neq \sum_M \mathrm{mon}_M \, \mathfrak{N}_M(\lambda),$$

where the sum runs over maps M with face-type π

good news

$$\mathrm{Ch}^{\mathrm{top}}_\pi(\lambda) = \sum_M \left([\text{top-degree part in } \gamma] \, \mathrm{mon}_M \right) \mathfrak{N}_M(\lambda),$$

where the sum runs over maps M with face-type π

$$\begin{aligned} -\mathrm{Ch}^{\mathrm{top}}_3 = {} & p_1^3 q_1 + 3p_1^2 q_1^2 + p_1 q_1^3 + 3p_1^2 p_2 q_2 + 3p_1 p_2^2 q_2 \\ & + p_2^3 q_2 + 3p_1 p_2 q_1 q_2 + 3p_1 p_2 q_2^2 + 3p_2^2 q_2^2 + p_2 q_2^3 \\ & + 3p_1^2 q_1 \gamma + 3p_1 q_1^2 \gamma + 6p_1 p_2 q_2 \gamma + 3p_2^2 q_2 \gamma \\ & + 3p_2 q_2^2 \gamma + 2p_1 q_1 \gamma^2 + 2p_2 q_2 \gamma^2 \end{aligned}$$

bad news

$$\mathrm{Ch}_\pi(\lambda) \neq \sum_M \mathrm{mon}_M \, \mathfrak{N}_M(\lambda),$$

where the sum runs over maps M with face-type π

good news

$$\mathrm{Ch}_k^{\mathrm{top}}(\lambda) = \sum_M \left([\text{top-degree part in } \gamma] \, \mathrm{mon}_M \right) \mathfrak{N}_M(\lambda),$$

where the sum runs over maps M with one face, k edges

$$\begin{aligned} -\mathrm{Ch}_3^{\mathrm{top}} = {} & p_1^3 q_1 + 3p_1^2 q_1^2 + p_1 q_1^3 + 3p_1^2 p_2 q_2 + 3p_1 p_2^2 q_2 \\ & + p_2^3 q_2 + 3p_1 p_2 q_1 q_2 + 3p_1 p_2 q_2^2 + 3p_2^2 q_2^2 + p_2 q_2^3 \\ & + 3p_1^2 q_1 \gamma + 3p_1 q_1^2 \gamma + 6p_1 p_2 q_2 \gamma + 3p_2^2 q_2 \gamma \\ & + 3p_2 q_2^2 \gamma + 2p_1 q_1 \gamma^2 + 2p_2 q_2 \gamma^2 \end{aligned}$$

new scaling: $\longrightarrow$ β-ensembles for $\beta \to 0$, $\beta \to \infty$

degree of a map = (number of vertices) + (exponent of γ)

main contribution from maps (with an order on egdes) such that
(a) during the edge removal there are no border edges $\iff$
(b) during the edge removal each connected component = one face

such maps are called top-twisted

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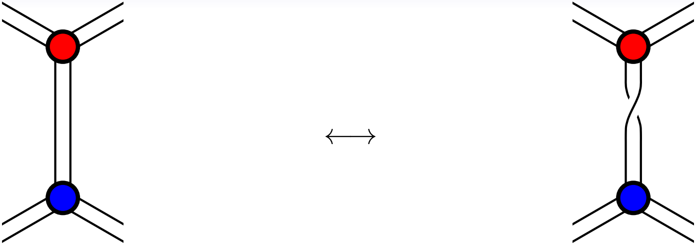
there is a bijection between: Czyżewska-Jankowska, Śniady

- pairs $(M, \prec)$, where
  M is a non-oriented, rooted map with k edges, one face;
  $\prec$ is an order on the edges which makes M top-twisted; } top-twisted
- pairs $(M, \prec)$, where
  M is an oriented, rooted map with k edges, connected,
  arbitrary number of faces;
  $\prec$ an arbitrary order on the edges of M;



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$$\mathrm{Ch}_k^{\mathrm{top}}(\lambda) =$$

$$\sum_{\substack{\text{oriented map } M \\ \text{with } k \text{ edges}}} \gamma^{k+1-|\text{vertices}|}\, \mathfrak{N}_M(\lambda) =$$

$$\sum_{\substack{\text{non-oriented map } M \\ \text{with face-type } (k)}} \left([\text{top-degree part in } \gamma]\, \mathrm{mon}_M \right) \mathfrak{N}_M(\lambda)$$

proof: abstract characterization of Jack characters

farewell questions

- for special values of α there are known formulas for Ch_π:
  - $\alpha = 1$, $\gamma = 0$:
    summation over oriented maps,
  - $\alpha \in \{\frac{1}{2}, 2\}$, $\gamma = \pm\frac{1}{\sqrt{2}}$:
    summation over non-oriented maps,
  - $\alpha \to 0$, $\alpha \to \infty$, $|\gamma| \to \infty$:
    summation over top-twisted maps
  - general α?
- what does it tell us about random matrices?
- what is the representation theory behind Jack characters?

Maciej Dołęga, Valentin Féray, Piotr Śniady
Jack polynomials and orientability generating series of maps
Séminaire Lotharingien de Combinatoire 70 (2014), Article B70j

Piotr Śniady
Top degree of Jack characters and enumeration of maps
Preprint arXiv:1506.06361

Piotr Śniady
Structure coefficients for Jack characters: approximate factorization property
Preprint arXiv:1603.04268

Agnieszka Czyżewska-Jankowska, Piotr Śniady
Jack characters and enumeration of maps.
Preprint arXiv:1611.02446

$$\mathrm{Ch}_1 = \underbrace{R_2}_{\mathrm{Ch}_1^{\mathrm{top}}},$$

$$\mathrm{Ch}_2 = \underbrace{R_3 + R_2\gamma}_{\mathrm{Ch}_2^{\mathrm{top}}},$$

$$\mathrm{Ch}_3 = \underbrace{R_4 + 3R_3\gamma + 2R_2\gamma^2}_{\mathrm{Ch}_3^{\mathrm{top}}} + R_2,$$

$$\mathrm{Ch}_4 = \underbrace{R_5 + 6R_4\gamma + R_2^2\gamma + 11R_3\gamma^2 + 6R_2\gamma^3}_{\mathrm{Ch}_4^{\mathrm{top}}} + 5R_3 + 7R_2\gamma.$$

$$\alpha t \frac{\partial}{\partial t} \log \left(\sum_{\lambda} \frac{J_\lambda(\mathbf{x})\ J_\lambda(\mathbf{y})\ J_\lambda(\mathbf{z})\ t^{|\lambda|}}{\langle J_\lambda, J_\lambda \rangle_\alpha} \right) = \sum_{n \geq 1} t^n \left(\sum_{\mu,\nu,\tau \vdash n} h^{\tau}_{\mu,\nu}(\alpha - 1)\ p_\mu(\mathbf{x})\ p_\nu(\mathbf{y})\ p_\tau(\mathbf{z}) \right)$$

conjecture [Goulden & Jackson 1996]

there exists a function η such that

$$h^{\tau}_{\mu,\nu}(\beta) = \sum_{M} \beta^{\eta(M)}$$

where the summation runs over connected, rooted maps
with face-type τ,
blue vertex distribution μ, and red vertex distribution ν,
and $\eta(M) \in \{0, 1, 2, \dots\}$

TODO:

roles of n and k should be interchanged.