# Asymptotic study of the graph of zigzag diagrams

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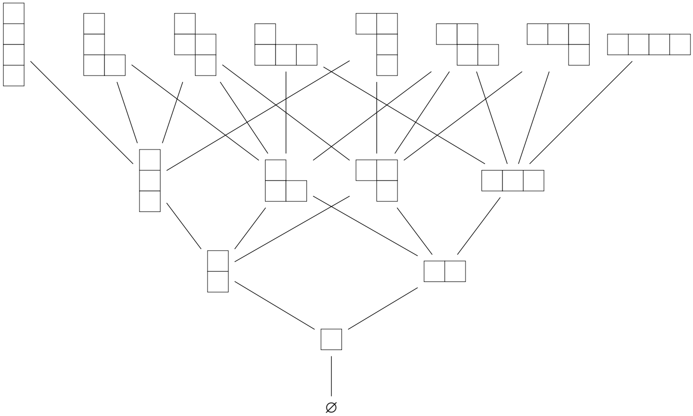



Figure: Initial part of the graph of zigzag didagrams.

# Asymptotic study of a branching graph

Interesting questions on a rooted graded graph $\mathcal{G}$:

1. Describe the Choquet simplex $\mathcal{H}(\mathcal{G})$ of harmonic measures on $\mathcal{G}$ (with the topology of weak convergence): functions $p : \mathcal{G} \longrightarrow \mathbb{R}^+$ such that $p(\varnothing) = 1$ and $p(\lambda) = \sum_{\lambda \nearrow \mu} p(\mu)$ for all $\lambda \in \mathcal{G}$.
   The boundary $\partial_{\text{min}}\mathcal{G}$ of $\mathcal{H}(\mathcal{G})$ is called the minimal boundary of the graph.
2. Describe the Martin boundary $\partial_M \mathcal{G}$ of $\mathcal{G}$: functions $p : \mathcal{G} \longrightarrow \mathbb{R}^+$ obtained as a weak limit of Martin Kernels $K_\nu$, with $K_\nu(\mu) = \frac{\#\{\text{paths from } \mu \text{ to } \nu\}}{\#\{\text{paths from } \varnothing \text{ to } \nu\}}$. $\partial_{\text{min}}\mathcal{G} \subset \partial_M \mathcal{G} \subset \mathcal{H}(\mathcal{G})$ (Doob's theory).
3. Describe the behavior of a random path following a harmonic measure (implies a geometric embedding of the graph).

# Young graph

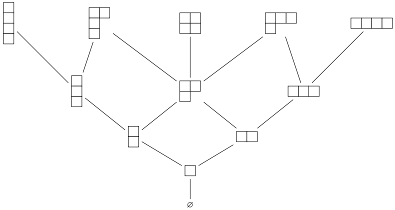



Figure: First levels of the Young graph

## Definition

The Young graph $\mathcal{Y} = \sqcup_{n\geq 0}\mathcal{Y}_n$ is the infinite graded graph such that

- vertices of rank $n$ are partitions $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$ of $n$, and
- the edge structure is given by the Pieri rule.

# Results on the Young graph (Vershik-Kerov, Logan-Shepp

1,2: $\partial_{\min}\mathcal{Y} \simeq_\phi \Delta^{(2)} := \{\alpha_1 \geq \alpha_2 \geq \dots, \beta_1 \geq \beta_2 \geq \dots \,|\, \sum \alpha_i + \beta_i \leq 1\}$, and $\partial_{\min}\mathcal{Y} = \partial_M \mathcal{Y}$ .

3: Embed $\lambda \vdash n$ in $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ as

$$L(\lambda) = (\frac{\lambda_1}{n}, \frac{\lambda_2}{n}, \dots) \times (\frac{\lambda'_1}{n}, \frac{\lambda'_2}{n}, \dots)$$

(where $\lambda'$ is the conjugate partition of $\lambda$). Then for a path $(\lambda^n)_{n\geq 1}$ following the harmonic measure $\phi(\alpha, \beta)$, a.s

$$L(\lambda^n) \longrightarrow (\alpha, \beta).$$

If $(\lambda^n)_{n\geq 1}$ follows the Plancherel measure $\phi(0,0)$, then a rotated and rescaled version of $\lambda$ converges almost surely to a deterministic curve $\Omega$ defined by

$$\Omega(x) = \begin{cases} \frac{2}{\pi}(u \arcsin(u/2) + \sqrt{4-u^2}, & |u| \leq 2, \\ |u|, & |u| \geq 2. \end{cases}$$

# Definition of the graph $\mathcal{Z}$

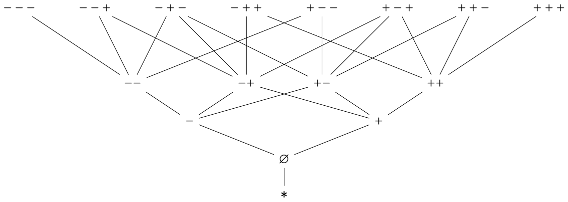



Figure: First levels of the graph $\mathcal{Z}$.

## Definition (Viennot)

The graph of zigzag diagrams $\mathcal{Z} = \sqcup_{n \geq 0} \mathcal{Z}_n$ is the infinite graded graph such that

- vertices of rank $n$ are words in $\{+, -\}$ of length $n - 1$, and
- the edge structure is given by the subword order.

# Paths on $\mathcal{Z}$

There is a bijection between paths $(w_k)_{0 \leq k \leq n}$ on $\mathcal{Z}$ and sequences of permutations $(\sigma_k)_{0 \leq k \leq n}$ such that:

- $\sigma_n \in S_n$.
- $\sigma_n$ is obtained from $\sigma_{n+1}$ by deleting the symbol $n+1$.
- $\sigma_n(i) > \sigma_n(i+1)$ if and only if $w_n(i) = -$: $w_n$ is called the descent word of $\sigma_n$, also denoted by $w(\sigma_n)$.

$$* \longrightarrow \varnothing \longrightarrow - \longrightarrow +- \longrightarrow -+- \longrightarrow -+--$$

$$\Big\downarrow$$

$$* \longrightarrow (\mathbf{1}) \longrightarrow (\mathbf{2}1) \longrightarrow (2\mathbf{3}1) \longrightarrow (\mathbf{4}231) \longrightarrow (42\mathbf{5}31)$$

Figure: An example of the equivalence between paths on $\mathcal{Z}$ and sequences of coherent permutations.

# Parameter space of the minimal boundary (Gnedin-Olshanski, 2006)

Denote by $U^{(2)}$ the set of pairs $(U_\uparrow, U_\downarrow)$ of open subset of $[0,1]$ such that $U_\uparrow \cap U_\downarrow = \varnothing$, with the distance

$$d\Big((U_\uparrow, U_\downarrow),(V_\uparrow, V_\downarrow)\Big) = \sup\Big(d_{\text{Haus}}(U_\uparrow^c, V_\uparrow^c), d_{\text{Haus}}(U_\downarrow^c, V_\downarrow^c)\Big).$$

Mapping each element of $U^{(2)}$ to the double decreasing sequence of lengths of the interval components of $U_\uparrow$ and $U_\downarrow$ yields a surjective continuous map $T$ from $U^{(2)}$ to $\Delta^{(2)}$.
Each vertex $w$ of $\mathcal{Z}$ yields an element $U(w) = (U_\uparrow(w), U_\downarrow(w))$ by considering connected components of $+$ and $-$. For example $U_\uparrow(++--+) = \left]0, \frac{2}{5}\right[ \cup \left]\frac{4}{5}, 1\right[$ and $U_\downarrow(++--+) = \left]\frac{2}{5}, \frac{4}{5}\right[$. In this case,

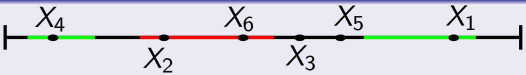
$$T(U_\uparrow(++--+), U_\downarrow(++--+)) = ((\frac{2}{5}, \frac{1}{5}, 0, \dots), (\frac{2}{5}, 0, \dots)).$$

# Oriented Paintbox construction (Gnedin-Olshanski,2006)

Let $(X_1, X_2, \ldots, X_k)$ be a sequence in $[0,1]$. The random permutation $\sigma_U(X_1, \ldots, X_k)$ is defined by the following rule: $i$ is left to $j$ in the word representation of $\sigma_U(X_1, \ldots, X_k)$ iff

- $X_i$ and $X_j$ are not in the same component of $U$ and $X_i < X_j$, or
- $X_i$ and $X_j$ are in the same component of $U_\uparrow$ and $i < j$, or
- $X_i$ and $X_j$ are in the same component of $U_\downarrow$ and $j < i$.

## Example




$\sigma_{(U_\uparrow, U_\downarrow)}(X_1, X_2, X_3, X_4, X_5, X_6) = (4, 6, 2, 3, 5, 1).$

# Minimal boundary of $\mathcal{Z}$ (Gnedin-Olshanski,2006)

Let $(X_1, X_2, \ldots)$ be a sequence of i.i.d random variables in $[0,1]$, and set $\sigma_{(U_\uparrow,U_\downarrow)}(k) = \sigma_{(U_\uparrow,U_\downarrow)}(X_1,\ldots,X_k)$.
Then, $\left(w\left(\sigma_{(U_\uparrow,U_\downarrow)}(n)\right)\right)_{n\geq 0}$ is a random path on $\mathcal{Z}$ following a harmonic measure denoted by $\psi(U_\uparrow, U_\downarrow)$.

**Theorem (Gnedin, Olshanski)**

- *The map $\psi$ yields a homeomorphism from $\mathcal{U}^{(2)}$ to $\partial_{\min}\mathcal{Z}$.*
- *If $(w_n)_{n\geq 0}$ follows the measure $\psi(U_\uparrow, U_\downarrow)$, then a.s*

$$U(w_n) \underset{n\to+\infty}{\longrightarrow} (U_\uparrow, U_\downarrow).$$

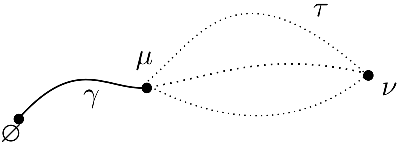
# Statements of results

1. If $(w_n)_{n\geq 1}$ is a path such that $U(w_n)$ converges to $(U_\uparrow, U_\downarrow)$, then $K_{w_n}$ converges to $p_{\psi(U_\uparrow, U_\downarrow)}$. In particular, $\partial_{\min}(\mathcal{Z}) = \partial_M(\mathcal{Z})$.
2. For $w \in \mathcal{Z}_n$, define the piece-wise linear function $f_w \in \mathcal{C}([0,1])$ by $f_w(0) = 0$ and $f_w'(x) = w(i)$ for $x \in ]\frac{i-1}{n-1}, \frac{i}{n-1}[$. Then, for $(w_n)_{n\geq 0}$ following the harmonic measure $\psi(\varnothing, \varnothing)$,
$$\frac{1}{\sqrt{n}} f_{w_n} \underset{n\to+\infty}{\longrightarrow} \sqrt{\frac{1}{3}} \mathcal{B}_{[0,1]}.$$
3. There exists a surjective map $\pi$ from paths on $\mathcal{Z}$ to paths on $\mathcal{Y}$ such that the following square commutes:
$$\begin{array}{ccc} \mathcal{U}^{(2)} & \underset{T}{\longrightarrow} & \Delta^{(2)} \\ \downarrow \psi & & \downarrow \phi \\ \partial_{\min} Z & \underset{\pi^*}{\longrightarrow} & \partial_{\min} \mathcal{Y} \end{array}.$$

# Interpretation of the Martin Kernel

$$K_\nu(\mu) = \frac{\#\{\text{paths from } \mu \text{ to } \nu\}}{\#\{\text{paths from } \varnothing \text{ to } \nu\}}$$

Equivalently, $K_\nu(\mu)$ is the probability that the initial part of a uniform random path ending at $\nu$ is a chosen path ending at $\mu$.




$$\mathbb{P}(\tau_{[0,m]} = \gamma) = K_\nu(\mu)$$

In our case, for $w \in \mathcal{Z}_n$ and $w \in \mathcal{Z}_k$,

$$K_w(w') = \mathbb{P}(\sigma^w_{\downarrow k} = \sigma),$$

where $\sigma^w$ is a uniformly distributed permutation with descent set $w$, $\sigma^w_{\downarrow k}$ denotes the permutation $\sigma^w$ where all letters bigger than $k$ have been deleted, and $w(\sigma) = w'$.

# Recovering the oriented Paintbox

Goal: Show that if $U(w_n) \longrightarrow (U_\uparrow, U_\downarrow)$, then for $k \geq 1$,

$$\sigma_{\downarrow k}^{w_n} \underset{n\to+\infty}{\longrightarrow} \sigma_{U_\uparrow, U_\downarrow}(X_1, \ldots, X_k).$$

## Sketch of proof:

- There exist $(U_\uparrow^{w_n}, U_\downarrow^{w_n}) \in \mathcal{U}^{(2)}$ and a random vector $(X_1^{w_n}, \ldots, X_n^{w_n})$ in $[0,1]^k$ such, that for $1 \leq k \leq n$,

$$\sigma_{\downarrow k}^{w_n} \simeq_{\text{Law}} \sigma_{(U_\uparrow^{w_n}, U_\downarrow^{w_n})}(X_1^{w_n}, \ldots, X_k^{w_n}).$$

- As $n$ goes to infinity, $(U_\uparrow^{w_n}, U_\downarrow^{w_n})_{n\geq 1}$ goes to $(U_\uparrow, U_\downarrow)$ and the vector $(X_1^{w_n}, \ldots, X_k^{w_n})_{n\geq 1}$ converges to $(X_1, \ldots, X_{k)}$.
- The convergence of $(U_\uparrow^{w_n}, U_\downarrow^{w_n})_{n\geq 1}$ and $(X_1^{w_n}, \ldots, X_k^{w_n})_{n\geq 1}$ implies the convergence of $\sigma_{(U_\uparrow^{w_n}, U_\downarrow^{w_n})}(X_1^{w_n}, \ldots, X_k^{w_n})$.

# Oshanin and Voituriez's probabilistic approach

Plancherel measure:$=\psi(\varnothing,\varnothing)$.
In this case, the oriented Paintbox construction $\sigma_{(\varnothing,\varnothing)}(X_1,\ldots,X_k)$ is simply the ranking function on $(X_1,\ldots,X_k)$. When $(X_1,\ldots,X_k)$ is a uniform random vector on $[0,1]^k$,

$$\sigma_{(\varnothing,\varnothing)}(X_1,\ldots,X_k) \simeq \mathrm{Unif}(S_k).$$

What is the asymptotic law of $w[\sigma_{(\varnothing,\varnothing)}(X_1,\ldots,X_k)]$ ?
**Idea of Oshanin and Voituriez:** Rather than look at $w(\sigma)$, look at $w(\sigma^{-1})$.

$$\sigma^{-1}(i) > \sigma^{-1}(i+1) \Leftrightarrow i \text{ right to } (i+1) \text{ in } \sigma \Leftrightarrow X_i > X_{i+1}.$$

Define the word $\tilde{w}(\sigma) \in \mathcal{Z}_{n-1}$ as $[\tilde{w}(\sigma)](i) = -$ if and only if $X_i > X_{i+1}$. Then,

- $\tilde{w}[\sigma_{(\varnothing,\varnothing)}(X_1,\ldots,X_k)] \simeq w[\sigma_{(\varnothing,\varnothing)}(X_1,\ldots,X_k)]$., and
- $(\tilde{w}[\sigma_{(\varnothing,\varnothing)}(X_1,\ldots,X_k)], X_k)_{k\geq 1}$ is a Markov process.

# CLT for the Plancherel measure

Asymptotic law of $f_{w_n}$:

1. By the Markov property $f_{w_n}(\frac{i_2}{n-1}) - f_{w_n}(\frac{i_1}{n-1})$ is independent of $f_{w_n}(\frac{j_2}{n-1}) - f_{w_n}(\frac{j_1}{n-1})$ for $i_1 \leq i_2 < j_1 \leq j_2$, and $f_{w_n}(\frac{i_2}{n-1}) - f_{w_n}(\frac{i_1}{n-1})$ is distributed as $f_{w_{i_2-i_1+1}}(1)$.
2. $f_{w_n}(1) = \#\{\text{ascent of } \sigma_n\} - \#\{\text{descent of } \sigma_n\} = 2\#\{\text{ascent of } \sigma_n\} - (n-1)$. Moreover, the distribution of the number of ascents of a uniform permutation is known, which allows the limit formula

$$\frac{1}{\sqrt{n}} f_{w_n}(1) \longrightarrow \mathcal{N}(0, 1/3).$$

3. Standard tightness results conlude the proof.

# RSK algorithm

1. A finite path on $\mathcal{Y}$ ending on $\lambda$ is equivalent to a standard filling of $\lambda$.
2. A finite path on $\mathcal{Z}$ ending on $w$ is equivalent to a permutation $\sigma$ such that $w(\sigma) = w$.

How to relate both structures ?

**Robinson-Schensted-Knuth algorithm**: canonical procedure associating a pair $(P(\sigma), Q(\sigma))$ of standard filling of a Young diagram $\lambda(\sigma)$ to a permutation $\sigma$. This yields a map

| | | |
|---|---|---|
| Finite path $\gamma$ on $\mathcal{Z}$ | $\rightarrow$ | Permutation $\sigma(\gamma)$ |
| | | $\downarrow$ |
| Pair of paths on $\mathcal{Y}$ ending at $\lambda[\sigma(\gamma)]$ | $\leftarrow$ | Pair of standard Young tableaux $(P[\sigma(\gamma)], Q[\sigma(\gamma)])$ |

# From paths on $\mathcal{Z}$ to paths on $\mathcal{Y}$

**Fact 1:** If $\pi$ is the map sending $\gamma$ to $P(\sigma(\gamma))$, then the map respects both graph structures. **Why ?**

- For $\sigma \in S_n$, the position of the first $k$ integers in $P(\sigma)$ only depends on their relative positions.
- If $\gamma \subset \gamma'$ in $\mathcal{Z}$, then $\sigma(\gamma)$ is obtained from $\sigma(\gamma')$ by deleting integers larger than the length $k$ of $\gamma$. Thus, the relative position of the first $k$ integers is the same in $\sigma(\gamma')$ and $\sigma(\gamma)$.

**Fact 2:**$\pi^*$ sends $\psi(U_\uparrow, U_\downarrow)$ to $\phi(T(U_\uparrow), T(U_\downarrow))$. **Why ?**

- If $(\sigma_n)_{n\geq 0}$ is a harmonic random path on $\mathcal{Z}$, $\pi((\sigma_n)_{n\geq 0})$ is a harmonic random path on $\mathcal{Y}$ (because $w(\sigma_n)$ can be read on $Q(\sigma_n)$).
- If $U(w(\sigma_n))$ converges to $(U_\uparrow, U_\downarrow)$, then $L(\lambda(\sigma_n))$ converges to $T((U_\uparrow, U_\downarrow)$ (requires Greene's theorem).

# Perspectives

- What are the fluctuations of $U(w_n)$ when $(w_n)_{n\geq 1}$ follows a harmonic measure different from $\psi(\varnothing, \varnothing)$.
- Can we deduce the Vershik-Kerov-Logan-Shepp shape from the central limit theorem for $\psi(\varnothing, \varnothing)$ ?
- Let us consider Hall-Littlewood quasi-symmetric functions (introduced by Hivert): graph structure ? minimal boundary ?
- Relation with the infinite Jeu de Taquin process (idea suggested by Piotr Sniady).

Thank you for your attention !