Generalized Kostka Polynomials

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23 February 2017

First, Kostka Numbers.

$\lambda, \mu \vdash n$. $K_{\lambda,\mu}$ = # of SSYT of shape λ with μ_1 1's, μ_2 2's, etc.

Example. $\lambda = (4,3,1)$ $\quad$ $K_{\lambda,\mu} = 5$.
$\mu = (3,2,2,1)$

$$\begin{array}{llll} 1 & 1 & 1 & 2 \\ 2 & 3 & 3 & \\ 4 & & & \end{array} \qquad \begin{array}{llll} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & \\ 3 & & & \end{array} \qquad \begin{array}{llll} 1 & 1 & 1 & 3 \\ 2 & 2 & 3 & \\ 4 & & & \end{array} \qquad \begin{array}{llll} 1 & 1 & 1 & 3 \\ 2 & 2 & 4 & \\ 3 & & & \end{array} \qquad \begin{array}{llll} 1 & 1 & 1 & 4 \\ 2 & 2 & 3 & \\ 3 & & & \end{array}$$

$$s_\lambda = \sum_{\mu \trianglelefteq \lambda} K_{\lambda,\mu} \, m_\mu$$

$$s_\lambda = \frac{\det[x_i^{\lambda_j + n - j}]}{\det[x_i^{n-j}]}$$

$$m_\lambda = \sum_{\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

all rearr. of λ

An alternative characterization of s_λ.

Inner product on $\Lambda_n = \mathbb{C}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$: $\quad \langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$.

Schur polynomials : (i) $s_\lambda = m_\lambda + \sum_{\mu \lhd \lambda} \underbrace{a_{\lambda,\mu}}_{\in \mathbb{C}} m_\mu$,

and (ii) $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$.

A 2-parameter deformation.

Another inner product on Λ_n : $\quad \langle p_\lambda, p_\mu \rangle_{qt} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$.

Macdonald polynomials : (i) $P_\lambda = m_\lambda + \sum_{\mu \lhd \lambda} \underbrace{a_{\lambda,\mu}}_{\in \mathbb{C}(q,t)} m_\mu$,

and (ii) $\langle P_\lambda, P_\mu \rangle_{qt} = 0, \quad \lambda \neq \mu$.

Motivation for our problem.

$$s_\lambda = \sum_{\mu \trianglelefteq \lambda} K_{\lambda,\mu}\, m_\mu$$

$$P_\lambda(X;q,q) = \sum_{\mu \trianglelefteq \lambda} K_{\lambda,\mu}\, P_\lambda(X;q,1)$$

Problem: study the combinatorics of

$\in \mathbb{C}(q)$

$$P_\lambda(X;q,q^{r+1}) = \sum_{\mu \trianglelefteq \lambda} K_{\lambda,\mu,r}(q)\, P_\lambda(X;q,q^r), \qquad r = 0,1,2,\dots$$

or, simultaneously for all r,

$\in \mathbb{C}(q,t)$

$$P_\lambda(X;q,qt) = \sum_{\mu \trianglelefteq \lambda} K_{\lambda,\mu}(q,t)\, P_\lambda(X;q,t).$$

Kostka-Foulkes:

Hall-Littlewood

$$s_\lambda = \sum_{\mu \trianglelefteq \lambda} K_{\lambda,\mu}(t)\, P_\mu(X;0,t)$$

Lascoux + Schützenberger:

$$K_{\lambda,\mu}(t) = \sum_{T \in SSYT(\lambda,\mu)} t^{\mathrm{charge}(T)}$$

Example. $K_{21,111}(t)$:

$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}$ $\quad rw = 3_1\, 1_0\, 2_0 \quad$ charge $= 1$

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$ $\quad rw = 2_1\, 1_0\, 3_1 \quad$ charge $= 2$

$$K_{21,111}(t) = t + t^2.$$

K(t) matrix:

$$\begin{bmatrix} s_3 \\ s_{21} \\ s_{111} \end{bmatrix} = \begin{bmatrix} 1 & t & t^3 \\ & 1 & t+t^2 \\ & & 1 \end{bmatrix} \begin{bmatrix} P_3(t) \\ P_{21}(t) \\ P_{111}(t) \end{bmatrix}$$

Have

$$\begin{bmatrix} P_3(q,qt) \\ P_{21}(q,qt) \\ P_{111}(q,qt) \end{bmatrix} = \begin{bmatrix} 1 & t\frac{1-q^3}{1-q^3t}\frac{1-q^2}{1-q^2t} & t^2\frac{1-q^3}{1-q^3t}\frac{1-q^2}{1-q^2t}\frac{1-q}{1-qt}\frac{t-q}{1-qt^2} \\ & 1 & \star \\ & & 1 \end{bmatrix} \begin{bmatrix} P_3(q,t) \\ P_{21}(q,t) \\ P_{111}(q,t) \end{bmatrix}$$

$$\star = t\,\frac{1-q}{1-q^3t^2}\,\frac{1-q}{1-qt^2}\,\frac{1-q^3t^3}{1-qt} + t^2\,\frac{1-q}{1-q^3t^2}\,\frac{1-q}{1-qt^2}\,\frac{1-q^3}{1-q}$$

Setting $q=0$ recovers K(t).

Weyl character formula.

$$s_\lambda = \frac{\det[x_i^{\lambda_j+n-j}]}{\det[x_i^{n-j}]} = \frac{a_{\lambda+\rho}}{a_\rho}$$

Vandermonde

$$\prod_{i<j} x_i - x_j$$

$$\rho = (n-1, n-2, \cdots, 2, 1, 0)$$

The qt - version:

$$P_\lambda(X; q, qt) = \frac{A_{\lambda+\rho}(X; q, t)}{A_\rho(X; t)}$$

t-Vandermonde

$$\prod_{\alpha \in R^+} (t^{1/2} X^{\alpha/2} - t^{-1/2} X^{-\alpha/2})$$

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$$

Explicit formula for rank one root system.

$$P_{k\omega}(X;q,qt) = P_{k\omega} + \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} t^i \prod_{j=0}^{i} \frac{1-q^{k-j}}{1-q^{k-j}t} P_{k\omega - i\alpha}$$

Example. $P_{5\omega}(q,qt) = P_{5\omega}(q,t) + \frac{1-q^5}{1-q^5t}\frac{1-q^4}{1-q^4t}\, t\, P_{3\omega}(q,t) + \frac{1-q^5}{1-q^5t}\frac{1-q^4}{1-q^4t}\frac{1-q^3}{1-q^3t}\frac{1-q^2}{1-q^2t}\, t^2 P_{\omega}(q,t)$

$t=1$:

$$P_{5\omega}(q,q) = P_{5\omega}(q,1) + P_{3\omega}(q,1) + P_{\omega}(q,1)$$

$$s_5 = m_5 + m_{41} + m_{32}$$

$t=q$:

$$P_{5\omega}(q,q^2) = P_{5\omega}(q,q) + \frac{1-q^4}{1-q^6}\, q\, P_{3\omega}(q,q) + \frac{1-q^2}{1-q^6}\, q^2 P_{\omega}(q,q)$$

$$= s_5 + \frac{1-q^4}{1-q^6}\, q\, s_{41} + \frac{1-q^2}{1-q^6}\, q^2 s_{32}$$

$t=q^2$:

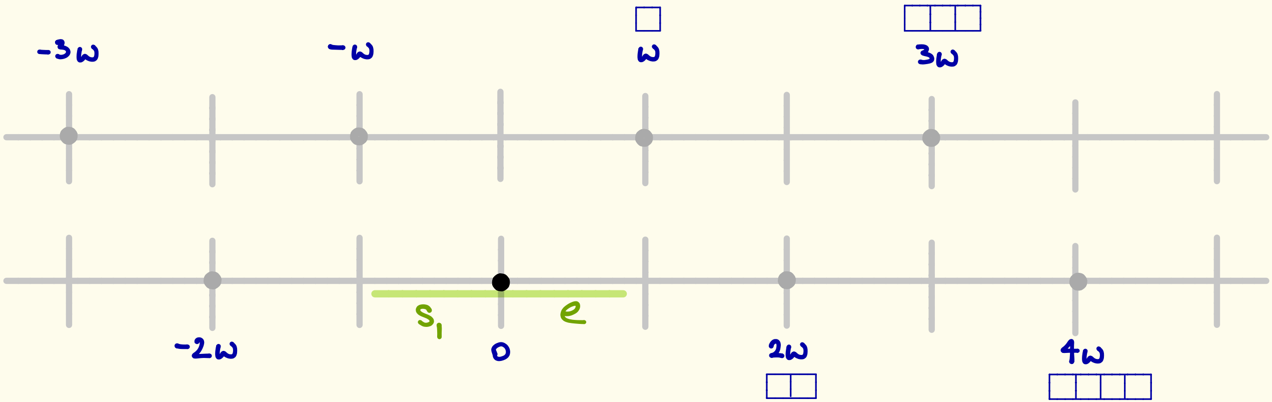
$$P_{5\omega}(q,q^3) = P_{5\omega}(q,q^2) + \frac{1-q^5}{1-q^7}\frac{1-q^4}{1-q^6}\, q^2 P_{3\omega}(q,q^2) + \frac{1-q^3}{1-q^7}\frac{1-q^2}{1-q^6}\, q^4 P_{\omega}(q,q^2)$$

$$= s_5 + \left(\frac{1-q^5}{1-q^7}\frac{1-q^4}{1-q^6}\, q^2 + \frac{1-q^4}{1-q^6}\, q\right) s_{41}$$

$$+ \left(\frac{1-q^3}{1-q^7}\frac{1-q^2}{1-q^6}\, q^4 + \frac{1-q^4}{1-q^7}\frac{1-q}{1-q^6}\, q^3 + \frac{1-q^2}{1-q^6}\, q^2\right) s_{32}$$

Macdonald polynomials and the double affine Hecke algebra $\tilde{H}$.

Rank one:



Pos. root $\alpha = e_1 - e_2$

Fund. wt. $\omega = \frac{1}{2}\alpha$

Weight lattice $L = \mathbb{Z}\omega$

Finite Weyl group $W_0 = \mathfrak{S}_2 = \langle s_1 \mid s_1^2 = e \rangle$

Macdonald polynomials and the double affine Hecke algebra $\tilde{H}$.

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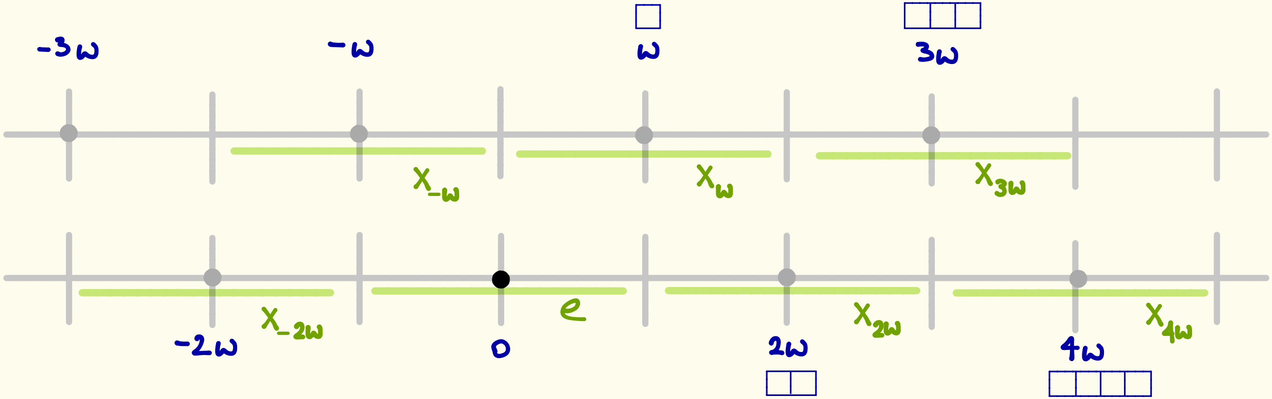

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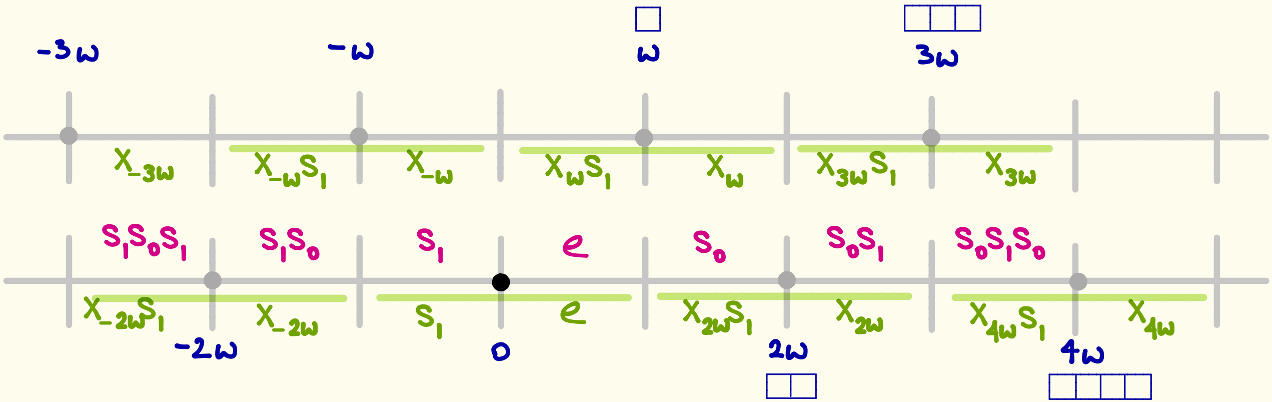

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Affine Weyl gp. $W_a = \hat{\mathfrak{S}}_2$
$= \langle s_1, s_0 \mid s_0^2 = s_1^2 = e,\ s_0s_1s_0 = s_1s_0s_1 \rangle$

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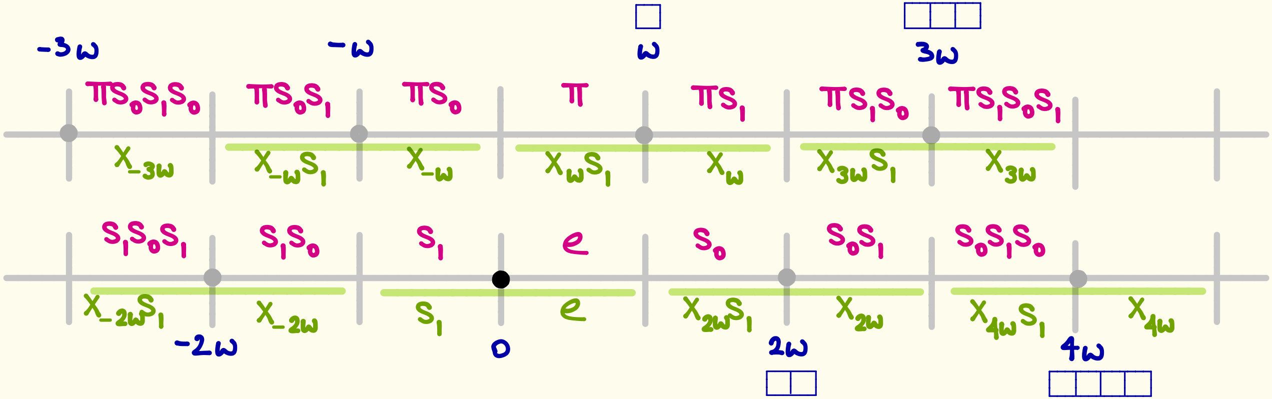

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Extended aff Weyl gp. $W = L \rtimes W_0$

Macdonald polynomials and the double affine Hecke algebra $\tilde{H}$.

Rank one generators: T_0, T_1

$X^\mu, \quad \mu \in L = \mathbb{Z}\omega,$

π

relations: $\pi^2 = e$

$T_0 T_1 T_0 = T_1 T_0 T_1$

$T_i^2 = (t^{1/2} - t^{-1/2}) T_i + 1 \quad$ for $i = 0, 1,$

$X^\lambda X^\mu = X^{\lambda+\mu} \quad$ for $\lambda, \mu \in L,$

$T_0 X^{k\omega} T_0 = q^k X^{-k\omega} \quad$ if $\langle \mu, \alpha_i^\vee \rangle = 0,$

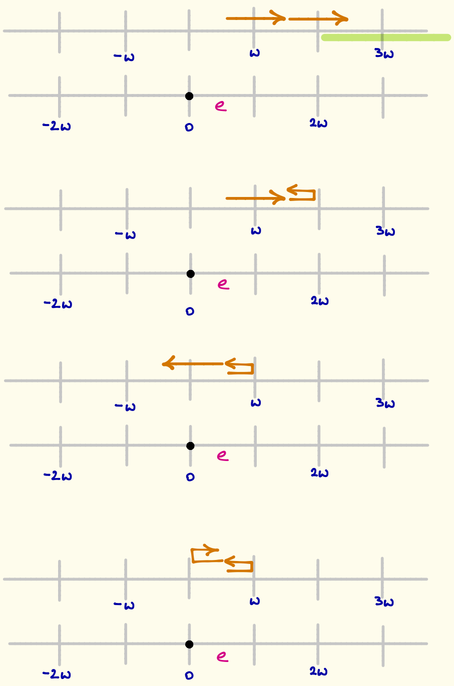
$T_1 X^{k\omega} T_1 = X^{-k\omega} \quad$ if $\langle \mu, \alpha_i^\vee \rangle = 1,$

$\pi T_1 = T_0 \pi$

$\pi X^{k\omega} = q^{k/2} X^{-k\omega} \pi \quad$ for $k \in \mathbb{Z}.$

Macdonald polynomials and the double affine Hecke algebra $\tilde{H}$.

P_λ can be computed by symmetrizing E_λ — nonsymmetric Macdonald



Example.

intertwining ops. in $\tilde{H}$

$$\tau_1^\vee = T_1 + \frac{t^{-1/2} - t^{1/2}}{1 - Y^{-\alpha^\vee}}$$

$$\tau_0^\vee = X^\alpha T_1 + \frac{(t^{-1/2} - t^{1/2}) q Y^{\alpha^\vee}}{1 - q Y^{\alpha^\vee}}$$

$$\pi^\vee = X^\omega T_1$$

$$E_{3\omega}(X; q, t) = \pi^\vee \tau_1^\vee \tau_0^\vee \, 1$$

$$= t^{1/2} X^{3\omega} + t^{1/2} \frac{(1-t)q}{1-qt} X^{\omega} + t^{1/2} \frac{(1-t)q^2}{1-q^2 t} X^{-\omega} + t^{1/2} \frac{(1-t)}{1-qt} \frac{(1-t)q^2}{1-q^2 t} X^{\omega}$$

Macdonald polynomials and the double affine Hecke algebra $\tilde{H}$.

P_λ can be computed by symmetrizing E_λ

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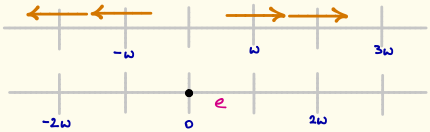


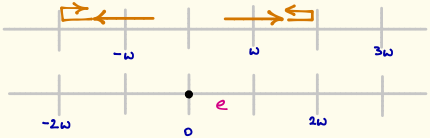
Example.

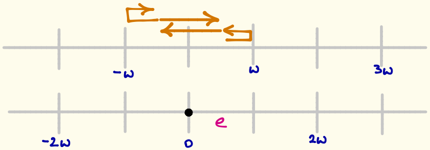
symmetrizing op in $\tilde{H}$

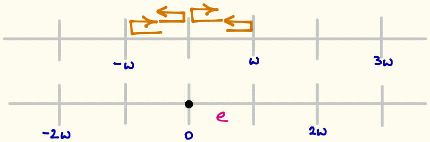
$$\mathbb{1}_0 = \sum_{w \in W_0} t^{-\ell(w_0 w)/2} T_w$$

$$P_{3\omega}(X;q,t) = \mathbb{1}_0 E_{3\omega} \mathbb{1}$$

$=$ 8 terms arising from paths

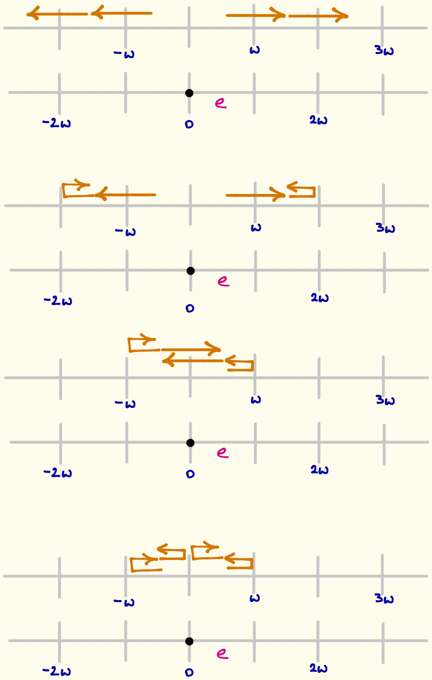






Macdonald polynomials and the double affine Hecke algebra $\tilde{H}$.

A_λ can be computed by skewsymmetrizing E_λ.



Example.

skew-symmetrizing op in $\tilde{H}$

$$\varepsilon_0 = \sum_{w \in W_0} (-1)^{\ell(w)} t^{\ell(w_0 w)/2} T_w$$

$$A_{3\omega}(X; q, t) = \varepsilon_0 E_{3\omega} 1$$

$$= \text{8 terms arising from paths}$$

Back to the problem.

WCF : $A_\rho(X;t)\, P_\lambda(X;q,qt) = A_{\lambda+\rho}(X;q,t)$

write as a sum of
$A_\rho P_\mu(X;q,t)$ $\mu \trianglelefteq \lambda$

Easier plan: Find a formula for
$A_\rho P_\lambda(X;q,t)$ as a lin. comb. of $A_\mu(X;q,t)$.

Lemma. $$\boxed{A_\rho P_{k\omega} = A_{(k+1)\omega} - \frac{1-q^k}{1-q^k t}\frac{1-q^{k-1}}{1-q^{k-1} t}\, t\, A_{(k-1)\omega} \quad , k \geq 1}$$

Lemma. $$\boxed{A_\rho P_{k\omega} = A_{(k+1)\omega} - \frac{1-q^k}{1-q^k t}\,\frac{1-q^{k-1}}{1-q^{k-1}t}\; t\, A_{(k-1)\omega}, \quad k \geq 1}$$

Idea: Use product formula for intertwining operators in $\tilde{H}$.

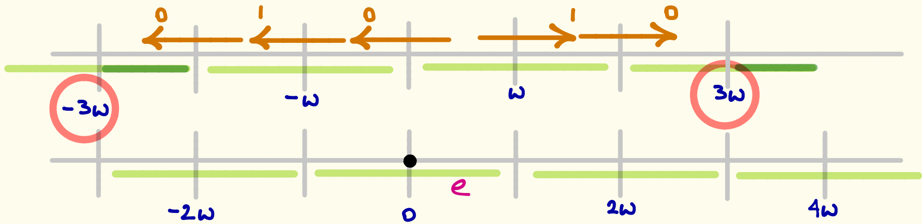
$$A_\rho P_{k\omega} = A_\rho \mathbb{1}_0 \tau^\vee_{k\omega} 1$$

Lemma. $$\mathbb{1}_0 \tau^\vee_\mu = \sum_{v \in W^\mu} \prod_{a^\vee \in m_\mu^{-1} d(v^{-1}, v_\mu^{-1})} t^{1/2} \frac{1-q^{s(a^\vee)} t^{h(a^\vee)-1}}{1-q^{s(a^\vee)} t^{h(a^\vee)}} \tau^\vee_v \tau^\vee_\mu$$

measures position of endpoint to $x^{v\mu}$

Example.

$$P_{3\omega} = E_{-3\omega} + \frac{1-q^3}{1-q^3 t} t^{1/2} E_{3\omega}$$



Lemma. $$\boxed{A_\rho P_{k\omega} = A_{(k+1)\omega} - \frac{1-q^k}{1-q^k t}\,\frac{1-q^{k-1}}{1-q^{k-1}t} + A_{(k-1)\omega}\quad , \ k \geq 1}$$

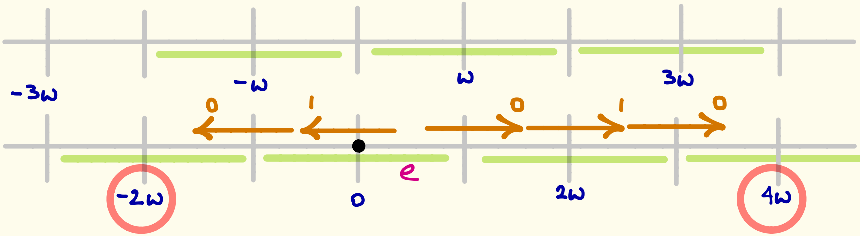
Idea: Use product formula for intertwining operators in $\tilde{H}$.

$$A_\rho P_{k\omega} = \varepsilon_0 t^{1/2} X^\rho \mathbb{1}_0 \tau^\vee_{k\omega} 1$$

Theorem. [Y. 2010] $$X^\mu \mathbb{1}_0 \tau^\vee_\lambda = \sum_{\text{paths } p} a_p(q,t)\, \tau^\vee_{wt(p)}.$$

Example. "$t^{1/2} X^\rho = \pi^\vee$"

$$\pi^\vee P_{3\omega} = E_{4\omega} - \frac{1-q^3}{1-q^3 t}\, t^{1/2} E_{-2\omega}$$



Lemma. $$\boxed{A_\rho P_{k\omega} = A_{(k+1)\omega} - \frac{1-q^k}{1-q^k t}\,\frac{1-q^{k-1}}{1-q^{k-1}t} + A_{(k-1)\omega}, \quad k\geq 1}$$

Idea: Use product formula for intertwining operators in $\tilde{H}$.

$$A_\rho P_{k\omega} = \varepsilon_0 X^\rho \mathbb{1}_0 \tau^\vee_{k\omega} \mathbf{1}$$

Lemma. $$\varepsilon_0 \tau^\vee_w \tau^\vee_\mu = \left(\prod_{a^\vee \in m_\mu^{-1}\alpha(1,w^{-1})} t^{1/2} \frac{1-q^{s(a^\vee)}t^{h(a^\vee)-1}}{1-q^{s(a^\vee)}t^{h(a^\vee)}} \right) \varepsilon_0 \tau^\vee_\mu .$$

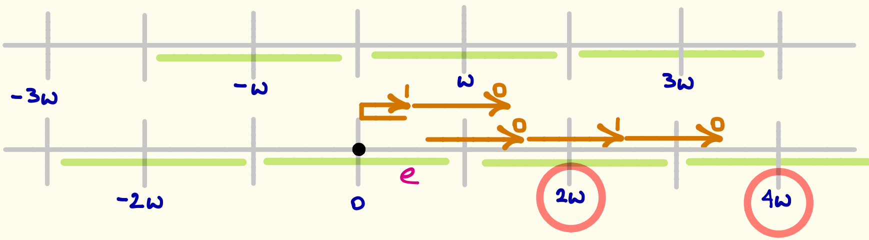
↘ measures position of endpoint to x^μ

Example. Apply straightening law.

$\varepsilon_0 \pi^\vee P_{3\omega}$

$$= \varepsilon_0 E_{4\omega} - \frac{1-q^3}{1-q^3 t} t^{1/2} \varepsilon_0 \tau^\vee_1 E_{2\omega}$$

$$= A_{4\omega} - \frac{1-q^3}{1-q^3 t} t^{1/2} \frac{1-q^2}{1-q^2 t} t^{1/2} A_{2\omega}$$



Now that $A_{(k+1)\omega} = A_p P_{k\omega} + \frac{1-q^k}{1-q^k t} \frac{1-q^{k-1}}{1-q^{k-1} t} \underbrace{t A_{(k-1)\omega}}_{\text{lower terms}}$,

apply recursion to finish.

$$\boxed{P_{k\omega}(X; q, qt) = P_{k\omega} + \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} t^i \prod_{j=0}^{i} \frac{1-q^{k-j}}{1-q^{k-j} t} P_{k\omega - i\alpha}.}$$

Theorem.

$$A_\rho P_\lambda(X;q,t) = A_{\lambda+\rho} + \sum_{p} b_p \; e_p \left(\prod_{\substack{\text{steps} \\ s\in p}} n_s \, g_s \right) A_{-w_0 wt(p)}$$

sum over paths of type m_ρ^{-1}, beginning at $m_\lambda^{-1} w^{-1}$, $w \in W^\lambda$, contained in the dom. chamber, $-w_0 wt(p)$ is regular dominant.

where it begins

where it ends (signs)

short crossings

where it folds (signs)

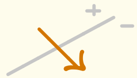
$$b_p = \prod_h t^{1/2} \frac{1-q^h t^{h-1}}{1-q^h t^h}$$

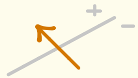
h separates $i(p)$ and $x^{i(p)}$

$$e_p = \prod_h \left(-t^{1/2}\right) \frac{1-q^h t^{h-1}}{1-q^h t^h}$$

h separates $e(p)$ and $m_{wt(p)}$

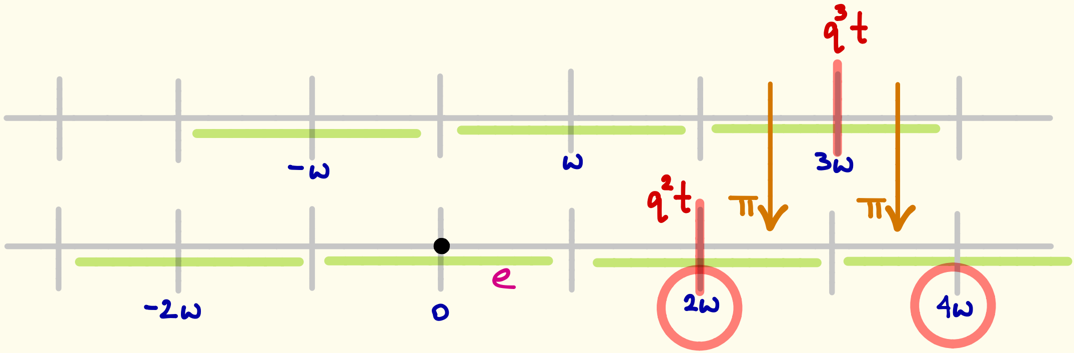
$$n_s = \begin{cases} \dfrac{1-q^s t^{s-1}}{1-q^s t^s} \, \dfrac{1-q^s t^{s+1}}{1-q^s t^s} & \text{if} \\ 1 & \text{if} \end{cases}$$





Revisit old example.

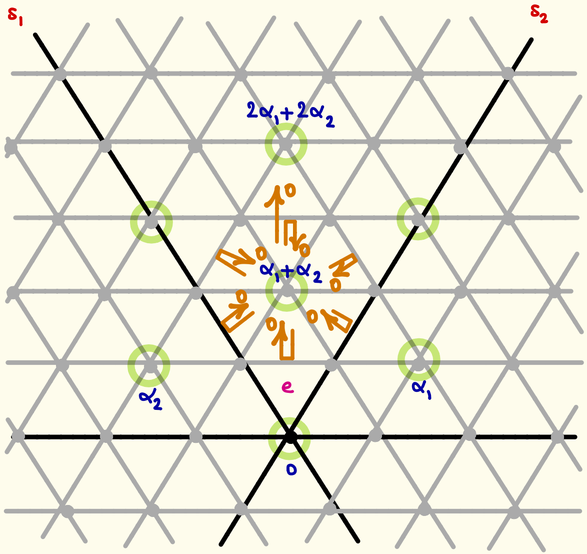
$$A_e P_{3\omega} = \varepsilon_0 \pi^\vee \mathbb{1}_0 \tau^\vee_{k\omega} 1 = A_{4\omega} - \underbrace{\frac{1-q^3}{1-q^3 t}}_{b_p} t^{1/2} \underbrace{\frac{1-q^2}{1-q^2 t}}_{e_p} t^{1/2} A_{2\omega}$$



Recursive formula in general case.

Theorem. $$P_\lambda(X;q,qt) = P_\lambda(X;q,t) - \sum_p b_p e_p \left(\prod_{\substack{\text{steps}\\ s\in p}} n_s g_s \right) P_{-w_0 wt(p)-\rho}(X;q,qt)$$

Example.



paths of type $m_p^{-1} = s_0$ beginning at $s_0 w^{-1}$, $w \in W_0$, contained in the dom. chamber, $-w_0 wt(p)$ is regular dominant.

$$P_{\alpha_1+\alpha_2}(X;q,qt) = P_{\alpha_1+\alpha_2}(X;q,t) + (6 \text{ terms})\, P_0(X;q,qt)$$

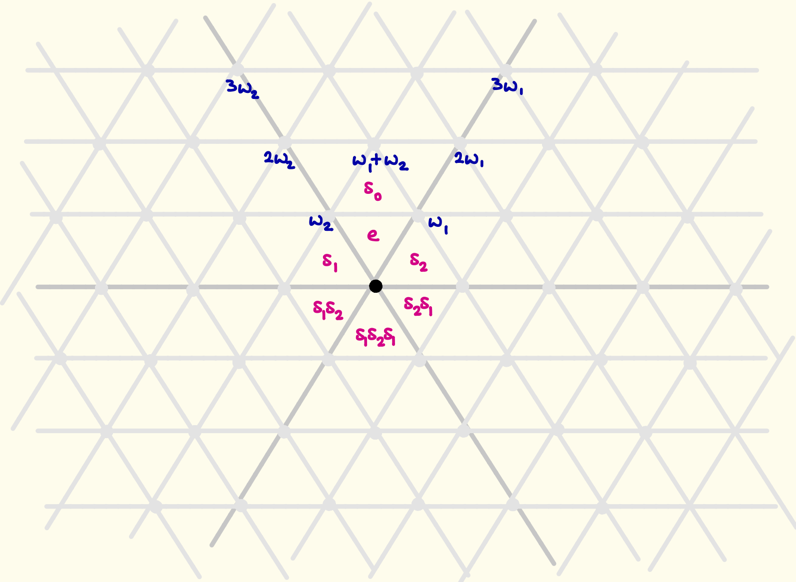
	b	e	n	g
1	1	$-t^{3/2}\frac{1-q}{1-qt}\frac{1-q^2t}{1-q^2t^2}\frac{1-q}{1-qt}$	1	$-t^{-1/2}\frac{1-t}{1-q^3t^2}q^3t^2$
s_1	$t^{1/2}\frac{1-q}{1-qt}$	$t\frac{1-q}{1-qt}\frac{1-q^2t}{1-q^2t^2}$	1	$-t^{-1/2}\frac{1-t}{1-q^2t}q^2t$
s_2	$t^{1/2}\frac{1-q}{1-qt}$	$t\frac{1-q}{1-qt}\frac{1-q^2t}{1-q^2t^2}$	1	$-t^{-1/2}\frac{1-t}{1-q^2t}q^2t$
s_1s_2	$t\frac{1-q}{1-qt}\frac{1-q^2t}{1-q^2t^2}$	$-t^{1/2}\frac{1-q}{1-qt}$	1	$t^{-1/2}$
s_2s_1	$t\frac{1-q}{1-qt}\frac{1-q^2t}{1-q^2t^2}$	$-t^{1/2}\frac{1-q}{1-qt}$	1	$t^{-1/2}$
$s_1s_2s_1$	$t^{3/2}\frac{1-q}{1-qt}\frac{1-q^2t}{1-q^2t^2}\frac{1-q}{1-qt}$	1	1	$t^{-1/2}\frac{1-t}{1-qt^2}$

$$\sum = t\,\frac{1-q}{1-q^3t^2}\,\frac{1-q}{1-qt^2}\,\frac{1-q^3t^3}{1-qt} + t^2\,\frac{1-q}{1-q^3t^2}\,\frac{1-q}{1-qt^2}\,\frac{1-q^3}{1-q}$$

Further work.

- Find better combinatorial models (no recursion, no cancellation)
- Understand charge in 2-parameter setting ?

The A_2 picture.



roots $\quad \alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3,$

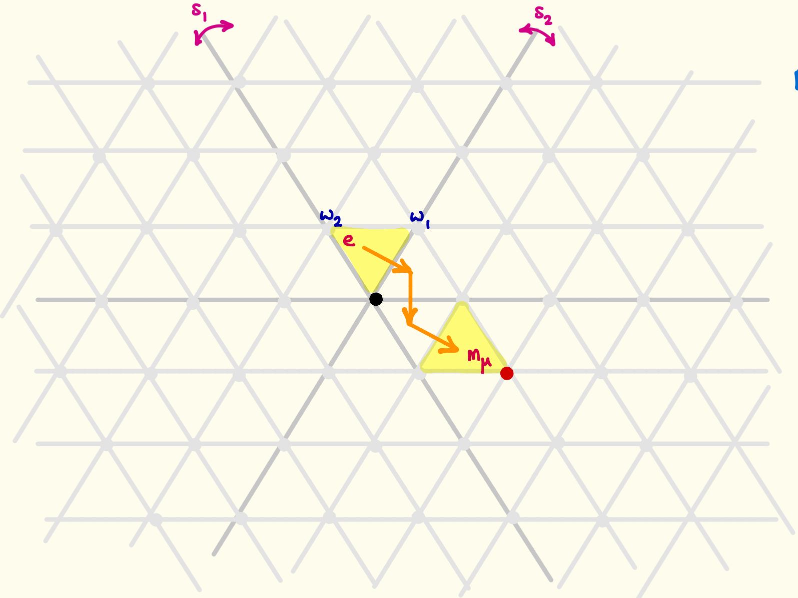
wts $\quad \omega_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \quad \omega_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2,$

$\square \qquad \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$

lattices
$$Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$$
$$L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$$
$$\Pi \cong L/Q = \langle \pi \mid \pi^3 = e \rangle$$

Weyl gp
$$W_0 = S_3 = \langle s_1, s_2 \mid \cdots \rangle$$
$$W_a = \hat{S}_3 = \langle s_1, s_2, s_0 \mid \cdots \rangle$$
$$W_e = \Pi \ltimes W_a$$

The A_2 picture.



wt. $\mu = -\alpha_2 = \omega_1 - 2\omega_2 \in L$

mlcr. $m_\mu = s_2 s_1 s_0 \in W_e$

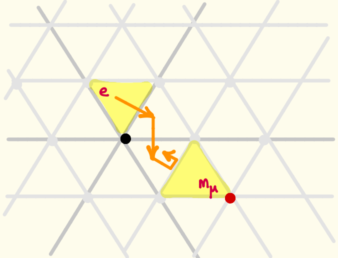
$$E_\mu = \tau_2^\vee \tau_1^\vee \tau_0^\vee \mathbb{1}$$

$$\tau_2^\vee = T_2 + \frac{t^{-1/2} - t^{1/2}}{1 - Y^{-\alpha_2^\vee}}$$

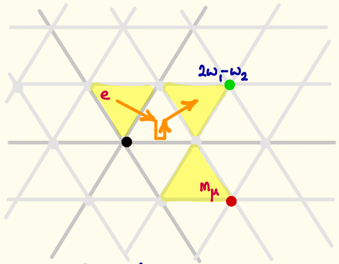
$$\tau_1^\vee = T_1 + \frac{t^{-1/2} - t^{1/2}}{1 - Y^{-\alpha_1^\vee}}$$

$$\tau_0^\vee = X^{\alpha_1 + \alpha_2} T_1 T_2 T_1 + \frac{(t^{-1/2} - t^{1/2}) q Y^{\alpha_1^\vee + \alpha_2^\vee}}{1 - q Y^{\alpha_1^\vee + \alpha_2^\vee}}$$

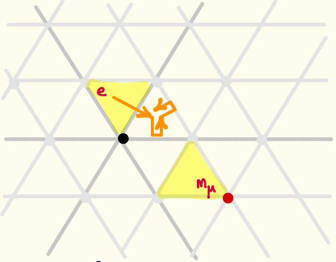
$t^{1/2} x_1 x_3^2$



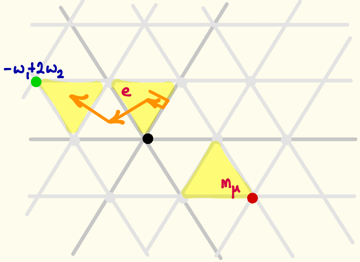
$t^{1/2} \frac{1-t}{1-qt^2} x_1 x_2 x_3$



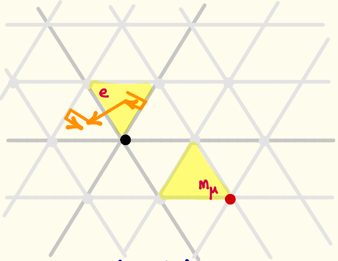
$t^{1/2} \frac{1-t}{1-qt} x_1^2 x_3$



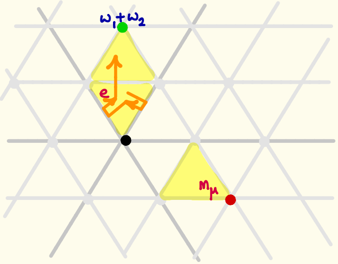
$t^{-1/2} q t^2 \frac{1-t}{1-qt^2} \frac{1-t}{1-qt} x_1 x_2 x_3$



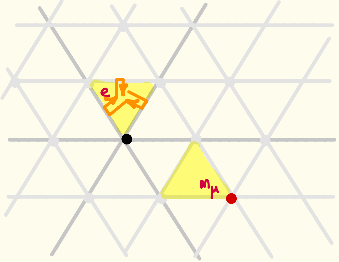
$t^{1/2} \frac{1-t}{1-q^2t^2} x_1 x_2^2$



$t^{-1/2} q t^2 \frac{1-t}{1-qt^2} \frac{1-t}{1-q^2t^2} x_1 x_2 x_3$



$t^{1/2} \frac{1-t}{1-qt} \frac{1-t}{1-q^2t^2} x_1^2 x_2$



$t^{-1/2} q t^2 \frac{1-t}{1-qt^2} \frac{1-t}{1-qt} \frac{1-t}{1-q^2t^2} x_1 x_2 x_3$

The double affine Hecke algebra $\tilde{H}$ is generated by

$$\begin{array}{ll} T_0, T_1, \ldots, T_n, & \\ X^\mu, & \mu \in L, \\ \pi, & \pi \in \Pi, \end{array}$$

with relations

$$\begin{array}{ll} T_i T_j \cdots = T_j T_i \cdots & \text{if } s_i s_j \cdots = s_j s_i \cdots \text{ in } W_a, \\ T_i^2 = (t^{1/2} - t^{-1/2}) T_i + 1 & \text{for } i = 0, \ldots, n, \end{array}$$

$$\begin{array}{ll} X^\lambda X^\mu = X^{\lambda+\mu} & \text{for } \lambda, \mu \in L, \\ T_i X^\mu = X^\mu T_i & \text{if } \langle \mu, \alpha_i^\vee \rangle = 0, \\ T_i X^\mu T_i = X^{s_i \mu} & \text{if } \langle \mu, \alpha_i^\vee \rangle = 1, \end{array}$$

$$\begin{array}{ll} \pi T_i = T_j \pi & \text{if } \pi s_i = s_j \pi \text{ in } W_e, \\ \pi X^\mu = X^{\pi\mu} \pi & \text{if } \pi x_\mu = x_{\pi\mu} \pi \text{ in } W_e. \end{array}$$