Exercises on Hurwitz numbers

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1. Construct the character table of the symmetric group $S_3$: the rows correspond to the irreducible representations, the columns to conjugacy classes.

2. Recall the definition of the Schur polynomial corresponding to a partition $\lambda$ of $K$:

$$s_\lambda(p_1, p_2, \ldots) = \frac{1}{K!} \sum_{\sigma \in S_K} \chi_\lambda(\sigma)p(\sigma).$$

Write out the three Schur polynomials corresponding to partitions of 3.

3. Consider the Laurent series

$$\varphi_1(z) = 2z^{-1} - 3 + z, \quad \varphi_2(z) = 1 + 5z + z^2, \quad \varphi_k(z) = z^k \quad \text{for } k \geq 3.$$

Write out the solution $\tau$ of the Hirota hierarchy and the solution $F$ of the KP hierarchy corresponding to the vector $\varphi_1 \wedge \varphi_2 \wedge \ldots$. Besides the Schur polynomials you have already computed you will need

$$s_{2,2}(p_1, p_2, p_3, p_4) = \frac{1}{12}p_1^4 + \frac{1}{4}p_2^2 - \frac{1}{3}p_1p_3.$$

4. Use the ELSV formula

$$h_{g,k_1,\ldots,k_n} = (2g - 2 + n + \sum k_i)! \prod_{i=1}^n \frac{k_i^{k_i}}{k_i!} \int_{M_{g,n}} \frac{1 - \lambda_1 + \lambda_2 - \cdots + (-1)^g\lambda_g}{(1 - k_1\psi_1) \cdots (1 - k_n\psi_n)}$$

for $g = 2$, $n = 1$ to find the integrals

$$x = \int_{M_{2,1}} \psi_1^4, \quad y = \int_{M_{2,1}} \psi_1^3\lambda_1, \quad z = \int_{M_{2,1}} \psi_1^2\lambda_2.$$

5. Prove the following Abel identity:

$$\sum_{p+q=n} \frac{n!}{p!q!}p^pq^{q-1} = (n-1)n^{n-1}.$$