Exercises on Hurwitz numbers

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1. Construct the character table of the symmetric group S_3: the rows correspond to the irreducible representations, the columns to conjugacy classes.

2. Recall the definition of the Schur polynomial corresponding to a partition λ of K:

$$s_\lambda(p_1, p_2, \dots) = \frac{1}{K!} \sum_{\sigma \in S_K} \chi_\lambda(\sigma) p(\sigma).$$

Write out the three Schur polynomials corresponding to partitions of 3.

3. Consider the Laurent series

$$\begin{array}{rclcccccc} \varphi_1(z) & = & 2z^{-1} & - & 3 & + & z, & & \\ \varphi_2(z) & = & & & 1 & + & 5z & + & z^2, \end{array}$$

$\varphi_k(z) = z^k$ for $k \geq 3$. Write out the solution τ of the Hirota hierarchy and the solution F of the KP hierarchy corresponding to the vector $\varphi_1 \wedge \varphi_2 \wedge \dots$. Besides the Schur polynomials you have already computed you will need

$$s_{2,2}(p_1, p_2, p_3, p_4) = \frac{1}{12} p_1^4 + \frac{1}{4} p_2^2 - \frac{1}{3} p_1 p_3.$$

4. Use the ELSV formula

$$h_{g;k_1,\dots,k_n} = (2g - 2 + n + \sum k_i)! \prod_{i=1}^{n} \frac{k_i^{k_i}}{k_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \lambda_2 - \dots + (-1)^g \lambda_g}{(1 - k_1\psi_1) \cdots (1 - k_n\psi_n)}$$

for $g = 2$, $n = 1$ to find the integrals

$$x = \int_{\overline{\mathcal{M}}_{2,1}} \psi_1^4, \qquad y = \int_{\overline{\mathcal{M}}_{2,1}} \psi_1^3 \lambda_1, \qquad z = \int_{\overline{\mathcal{M}}_{2,1}} \psi_1^2 \lambda_2.$$

5. Prove the following Abel identity:

$$\sum_{p+q=n} \frac{n!}{p!\,q!} p^p q^{q-1} = (n-1) n^{n-1}.$$