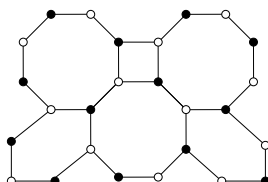Béatrice de Tilière

1. *Kasteleyn's theorem.*

   (a) Find a Kasteleyn orientation of the following graph.

   (b) Prove the following result of Kasteleyn: *Let* G *be a finite, planar graph. Suppose that edges are oriented so that all inner faces are cw odd. Then, each cycle containing in its interior an even (resp. odd) number of vertices is cw odd (resp. even).*

2. *Counting tilings.*

   (a) Count the number of tilings of an equilateral triangle of side n.

   $n = 6$
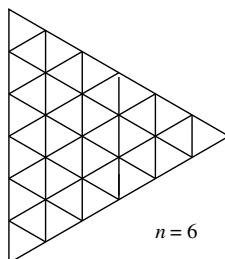
   (b) Count the number of tilings of a rectangle of side $2 \times n$.
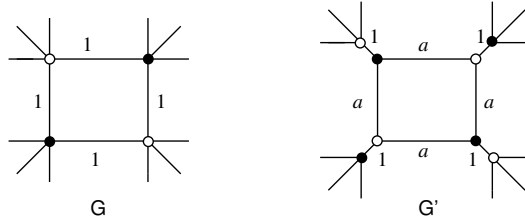

   $n = 8$
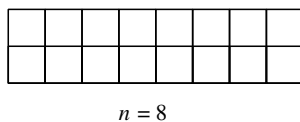
3. *Dimer surgery.*

   If a graph has a degree 2 vertex, consider the graph G$'$ obtained by merging this vertex with its two neighbors.

   (a) Exhibit a bijection between dimer configurations of G and G$'$.
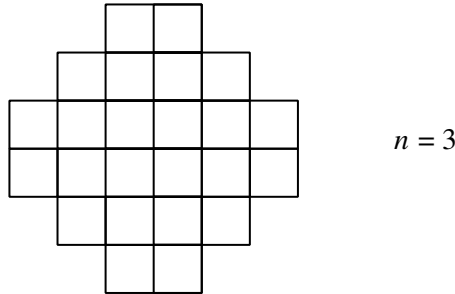   (b) Find weights on G$'$ such that this bijection is weight preserving.
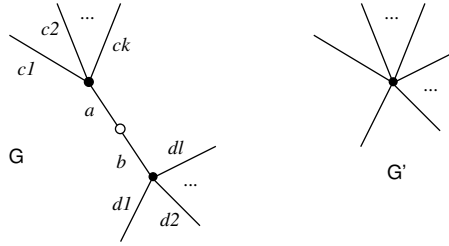
Let G and G' be two graphs differing by an "urban renewal":

(a) Exhibit a local mapping between dimer configurations of G and G'.

(b) Find a and b such that $Z(\mathsf{G}) = bZ(\mathsf{G}')$.

4. *Number of tilings of the Aztec diamond.*

Le Z_n be the number of tilings of the Aztec diamond of size n.

$n = 3$

Using "fusion" and "urban renewal", determine a relation between Z_n and Z_{n-1}, and then compute Z_n as a function of n.

5. Using the identity

$$\mathrm{cn}(u + v)\mathrm{cn}(u) = \mathrm{cn}(v) - \mathrm{sn}(u + v)\mathrm{sn}(u)\mathrm{dn}(v)$$

, prove that, for every $u \in \mathbb{T}(k)$, for every $\alpha, \beta \in \mathbb{R}$ such that $\beta - \alpha = \theta$:

$$\mathrm{cn}\left(\frac{u - \beta}{2}\right) - \mathrm{cn}\left(\frac{u - \alpha}{2}\right) = \frac{\mathrm{sn}\,\theta}{1 + \mathrm{cn}\,\theta}\left[\mathrm{sn}\left(\frac{u - \alpha}{2}\right)\mathrm{dn}\left(\frac{u - \beta}{2}\right) - \mathrm{sn}\left(\frac{u - \beta}{2}\right)\mathrm{dn}\left(\frac{u - \alpha}{2}\right)\right].$$

(The cn *and* dn *functions are even, the* sn *function is odd).*