The Cusp-Airy Process

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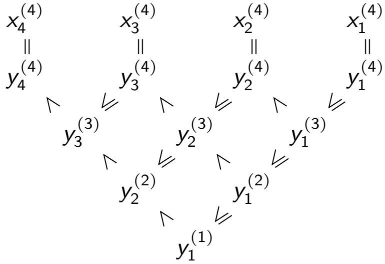
Our Model: Uniformly Random Discrete Gelfand-Tsetlin Patterns

Discrete Gelfand-Tsetlin Patterns

- For each $n \geqslant 1$, fix $x^{(n)} \in \mathbb{Z}^n$ with $x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)}$.
- Consider all Gelfand-Tsetlin patterns, with particle positions in $\mathbb{Z}$, which satisfy an asymmetric interlacing constraint, and with the particles on the top row in the $x^{(n)}$ positions:



- Impose uniform distribution. Call this the **uniformly random discrete interlacing process**.

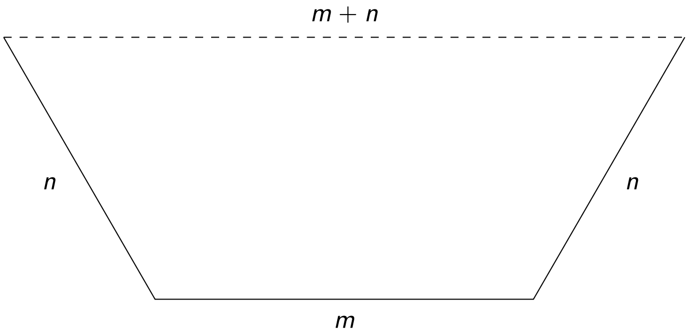
Asymptotic assumptions

- Assume that

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i^{(n)}/n} \to \mu \text{ weakly as } n \to \infty,$$

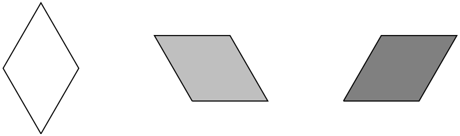
  where $\mathrm{Supp}(\mu) \subset [a, b]$ compact.
- Note, μ has density less than or equal to 1.
- Additionally assume that μ is not Lebesgue measure on an interval of length 1, a degenerate case.

Origin of the discrete process: Tilings of a ‘half-hexagon’



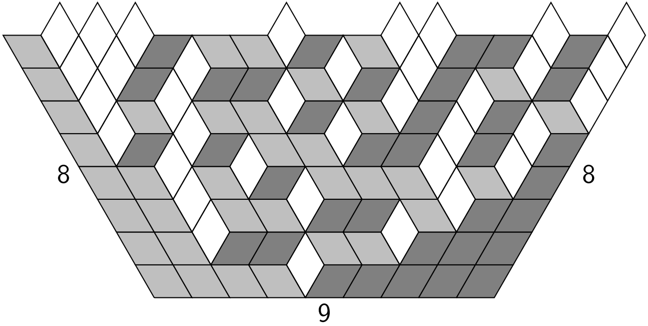
- A ‘half-hexagon’ with sides of length $n \geqslant 1$ and $m \geqslant 1$. The dotted line representing the upper boundary is considered to be ‘open’.

Three different types of lozenges



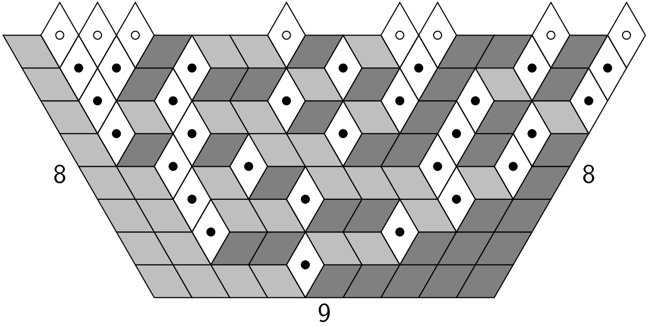
- All sides are of lenght 1.

An example tiling



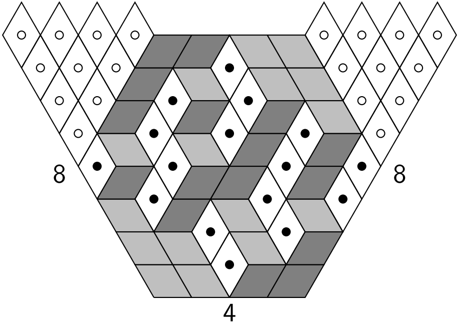
- Fix n vertical tiles on the upper boundary.
- Impose the uniform probability measure on the set of all possible tilings with the tiles on the upper boundary in these deterministic positions.

Equivalent interlaced particle system.



- Place particles in the center of each vertical tile. The particles on the top row are deterministic, and the others are random. The set of all particle positions completely determine the tiling.

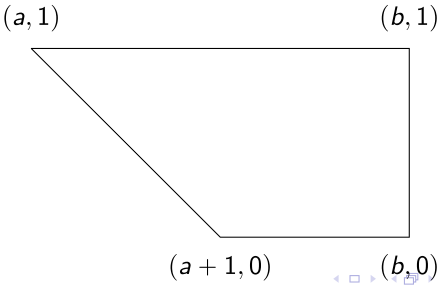
Regular hexagon example.



- In this case $\mu = \frac{1}{2}\lambda_{[0,\frac{1}{2}]} + \frac{1}{2}\lambda_{[1,\frac{3}{2}]}$, where λ is Lebesgue measure.

Asymptotic behaviour of the discrete process

- We showed that the discrete process is determinantal with a correlation kernel $K_n : (\mathbb{R} \times \{1, 2, \ldots, n\})^2 \to \mathbb{C}$ and derived an integral representation for it. This was also done independently by L. Petrov in [9].
- Rescaling vertical and horizontal by $\frac{1}{n}$, interlacing implies that the bulk of the rescaled particles asymptotically lie in $\{(\chi, \eta) \in [a, b] \times [0, 1] : b \geqslant \chi \geqslant \chi + \eta - 1 \geqslant a\}$:



Local asymptotic behaviour

- Fix (χ, η) in this shape.
- We examined the local asymptotic behaviour as $n \to \infty$, in a neighbourhood of (χ, η) by considering $K_n((x_1^{(n)}, y_1^{(n)}), (x_2^{(n)}, y_2^{(n)}))$ where $\frac{1}{n}(x_1^{(n)}, y_1^{(n)}) \to (\chi, \eta)$ and $\frac{1}{n}(x_2^{(n)}, y_2^{(n)}) \to (\chi, \eta)$.

Correlation kernel

- $K_n((x_1,y_1),(x_2,y_2)) = n^{y_2-y_1}\left(\frac{n-y_2}{n}\right)\frac{1}{(2\pi i)^2}J_n + R_n$, where J_n equals

$$\int_{c_n} dw \int_{C_n} dz \, \frac{\prod_{j=x_1+y_1-n+1}^{x_1-1}(z-\frac{j}{n})}{\prod_{j=x_2+y_2-n}^{y_2}(w-\frac{j}{n})} \, \frac{1}{w-z} \, \prod_{i=1}^{n}\left(\frac{w-x_i^{(n)}/n}{z-x_i^{(n)}/n}\right),$$

and

  - c_n contains $\frac{1}{n}\{x_2+y_2-n, x_2+y_2-n+1, \ldots, x_2\}$.
  - C_n contains all of $\{\frac{1}{n}x_j^{(n)} : x_j^{(n)} \geqslant x_1\}$, but none of $\{\frac{1}{n}x_j^{(n)} : x_j^{(n)} \leqslant x_1+y_1-n\}$.
  - c_n and C_n do not-intersect, and c_n contains C_n.

Integrand

- Integrand equals $\frac{1}{w-z}\exp(nf_n(w) - n\tilde{f}_n(z))$, where

$$f_n(w) := \frac{1}{n}\sum_{i=1}^{n}\log\left(w - \frac{x_i^{(n)}}{n}\right) - \frac{1}{n}\sum_{j=v_n+s_n-n}^{v_n}\log\left(w - \frac{j}{n}\right),$$

$$\tilde{f}_n(z) := \frac{1}{n}\sum_{i=1}^{n}\log\left(z - \frac{x_i^{(n)}}{n}\right) - \frac{1}{n}\sum_{j=u_n+r_n-n+1}^{u_n-1}\log\left(z - \frac{j}{n}\right).$$

- To first order, this approximates to $\frac{1}{w-z}\exp(nf_{(\chi,\eta)}(w) - nf_{(\chi,\eta)}(z))$, where

$$f_{(\chi,\eta)}(w) := \int_a^b \log(w-x)\mu[dx] - \int_{\chi+\eta-1}^{\chi}\log(w-x)dx.$$

Steepest descent analysis

- We used **steepest descent analysis** to examine the asymptotic behaviour: It should depend only on the roots of,

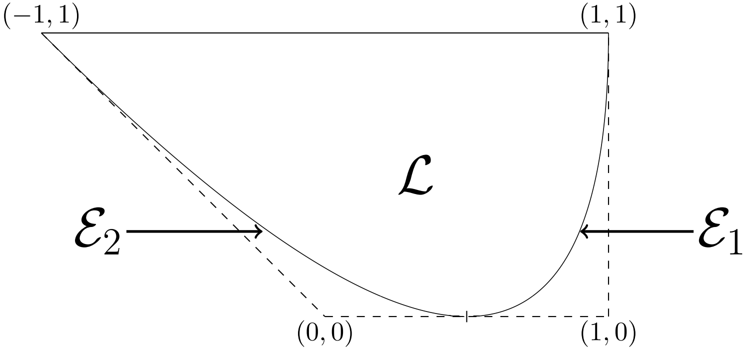
$$f'_{(\chi,\eta)}(w) = \int_a^b \frac{\mu[dx]}{w-x} - \int_{\chi+\eta-1}^{\chi} \frac{dx}{w-x}$$
$$= \int_\chi^b \frac{\mu[dx]}{w-x} - \int_{\chi+\eta-1}^{\chi} \frac{(\lambda-\mu)[dx]}{w-x} + \int_a^{\chi+\eta-1} \frac{\mu[dx]}{w-x}.$$

- λ is Lebesgue measure.
- The cancellation between λ and μ on the interval $[\chi+\eta+1,\chi]$ will be important!

Liquid region

- Definition: The **liquid region**, $\mathcal{L}$, is the set of all (χ, η) for which $f'_{(\chi,\eta)}$ has a unique root in $\mathbb{H} := \{w \in \mathbb{C} : \text{Im}(w) > 0\}$.
- Theorem: The map from $\mathcal{L}$ to the unique root in $\mathbb{H}$ is a homeomorphism. (In [4])
- We found the inverse homeomorphism, and used this to get a complete description of $\partial\mathcal{L}$ for a broad class of μ.
- Too many cases to list, so we go to examples.
- Let $\varphi : [a, b] \to [0, 1]$ represent the density of μ.

Example 1: $\varphi(x) = \frac{1}{2}$ for all $x \in [-1, 1]$.



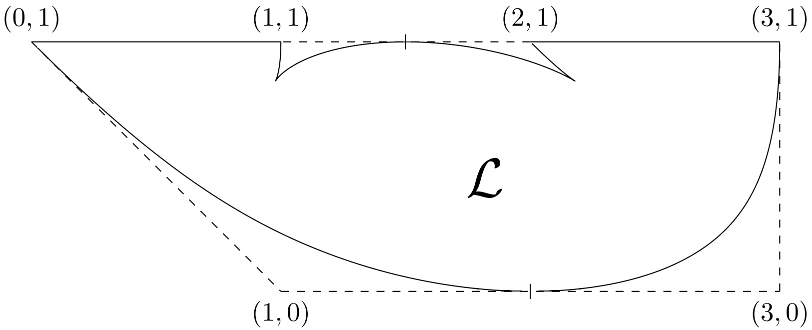
The edge in example 1

- $\mathcal{E}_1$ is the set of all (χ, η) for which $f'_{(\chi,\eta)}$ has a unique root of multiplicity 2 in $(1, +\infty)$ ($f'_{(\chi,\eta)} = f''_{(\chi,\eta)} = 0$ and $f'''_{(\chi,\eta)} \neq 0$).
- Theorem: The map from $\mathcal{E}_1$ to the unique repeated root is bijective, and is an extension of the bulk homeomorphism.
- The inverse is defined by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (1, +\infty)$, where

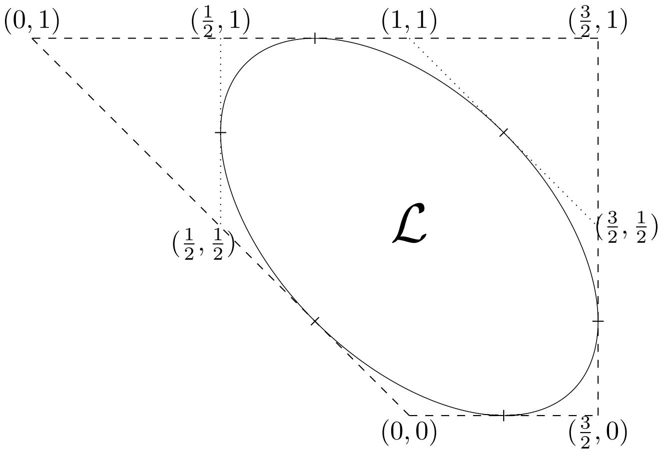
$$\chi_{\mathcal{E}}(t) := t + \frac{e^{C(t)} - 1}{e^{C(t)} C'(t)} \quad \text{and} \quad \eta_{\mathcal{E}}(t) := 1 + \frac{(e^{C(t)} - 1)^2}{e^{C(t)} C'(t)},$$

  where $C : \mathbb{C} \backslash \text{Supp}(\mu) \to \mathbb{C}$ is the **Cauchy transform** of μ.
- Similarly for $\mathcal{E}_2$ and $(-\infty, -1)$.

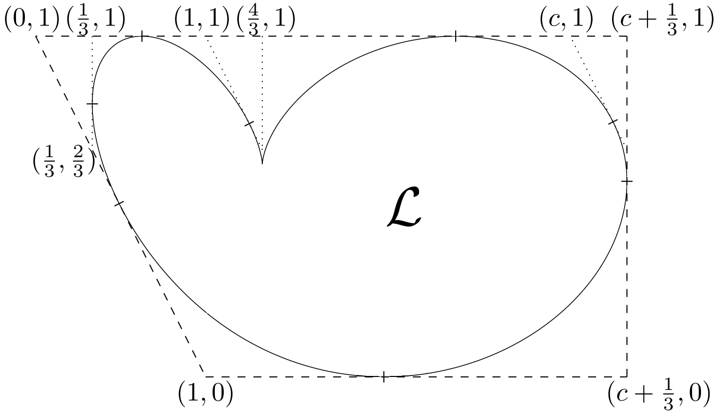
Example 2: $\varphi(x) = \frac{1}{2}$ for all $x \in [0,1] \cup [2,3]$.



Example 3: $\varphi(x) = 1$ for all $x \in [0, \frac{1}{2}] \cup [1, \frac{3}{2}]$.



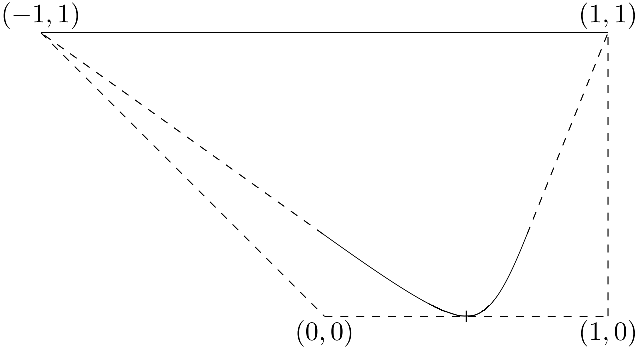
Example 4: $\varphi(x) := 1$ for all $x \in [0, \frac{1}{3}] \cup [1, \frac{4}{3}] \cup [c, c + \frac{1}{3}]$, where $c := \frac{1}{12}(23 + \sqrt{217})$.



Example 4: Jump points of $\varphi(x)$

- At $x = 1/3$ and $x = 4/3$ the density φ jumps from 1 to 0 and at $x = 1$ and $x = c$ the density jumps from 0 to 1.
- More generally, if $t \in \mathrm{Supp}(\mu)$ and there exists an interval $[x-\delta, x+\delta]$, for some $\delta > 0$, such that $\varphi\big|_{[x-\delta,x+\delta]} = \chi_{[x-\delta,x]}(t)$, we say that $x \in R_1$. Similarly, if x is such that $\varphi\big|_{[x-\delta,x+\delta]} = \chi_{[x,x+\delta]}(t)$, we say that $x \in R_2$.

Example 5: $\varphi(x) = \frac{15}{16}(x-1)^2(x+1)^2$ for all $x \in [-1, 1]$.



Example 5: Singular Parts of The Boundary

In Example 5 we note that the parameterization given by $t \to (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ from $[-\infty, -1) \cup (1, \infty] \to \partial\mathcal{L}$ does not parametrize the entire part of

$$\{(\chi, \eta) \in (a, b) \times (0, 1) : a < \chi + \eta - 1 \leqslant \chi < b\} \bigcap \partial\mathcal{L}$$

with $a = -1$ and $b = 1$ in this example. The remaining part

$$\partial\mathcal{L}_{sing} := (\{(\chi, \eta) \in (a, b) \times (0, 1) : a < \chi + \eta - 1 \leqslant \chi < b\} \bigcap \partial\mathcal{L}) \backslash \mathcal{E},$$

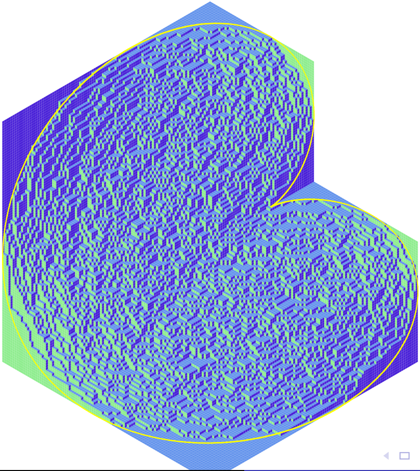
we call the singular part of the boundary of the liquid region.

Example 5: Singular Parts of The Boundary

In [5], we show that the geometry of $\partial\mathcal{L}_{sing}$ can be very complicated. In fact we show that in general $\partial\mathcal{L}$ is not homeomorphic to S^1, and we may have $\mathcal{H}^1(\partial\mathcal{L}_{sing}) = +\infty$, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure.

Moreover, we conjecture that one does not have any universal edge behaviour on $\partial\mathcal{L}_{sing}$.

Example: A simulation of a random lozenge tiling of the cut-corner hexagon model. (Picture from Kenyon)



Definition of the Cusp-Airy Kernel

For $r, s \in \mathbb{Z}$ and $\xi, \tau \in \mathbb{R}$ we define the *Cusp-Airy kernel* by

$$\mathcal{K}_{CA}((\xi, r), (\tau, s)) := -1_{\tau \geqslant \xi} 1_{s > r} \frac{(\tau - \xi)^{s-r-1}}{(s-r-1)!} + \frac{1}{(2\pi i)^2} \int_{\mathscr{L}_L + \mathscr{C}_{out}} dz \int_{\mathscr{L}_R + \mathscr{C}_{in}} dw \frac{1}{w-z} \frac{w^r}{z^s} e^{\frac{1}{3}w^3 - \frac{1}{3}z^3 - \xi w + \tau z},$$

where the contours are defined in the figure below, and $1_{a<b}$ is the indicator function for $a < b$.

Definition of the Cusp-Airy Kernel

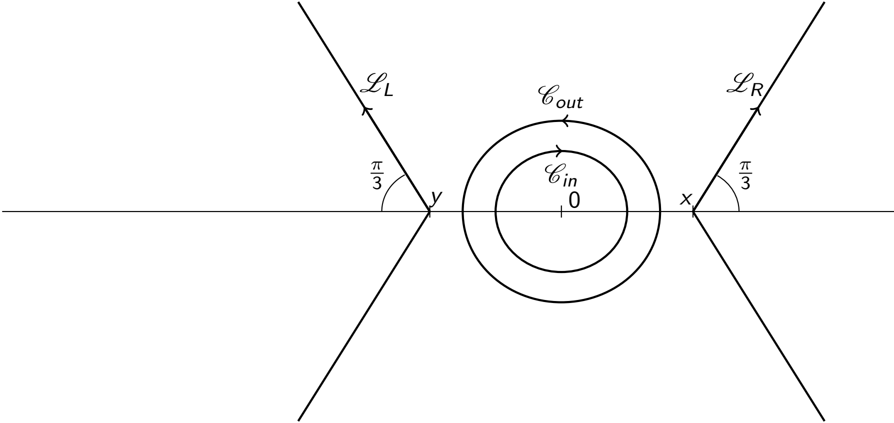


Figure: Integration contours for the Cusp-Airy kernel.

Basic Notation

Let $\{a_n\}_n$, $\{b_n\}_n$, $\{t_c^{(n)}\}_n$, $\{t_1^{(n)}\}_n$, $\{t_2^{(n)}\}_n$, $\{x_c^{(n)}\}_n$, and $\{y_c^{(n)}\}_n$ be suitably chosen sequences convergent to a, b, t_c, t_1, t_2, χ_c and η_c respectively.

Consider the signed measure

$$d\nu(x) = (\chi_{[a,t_2]}(x) + \chi_{[\chi_c,b](x)})\varphi(x)dx - (1-\varphi(x))\chi_{[t_1,\chi_c]}(x)dx,$$

where φ is the density of μ. Then at the cusp $\chi_c + \eta_c - 1 = t_c$, the asymptotic function can be written as

$$f(w;\chi_c,\eta_c) = f(w;\chi_c) = \int_{\mathbb{R}} \log(w-x)d\nu(x).$$

Assumption 1

Assume that

$$\mu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i^{(n)}/n} \rightharpoonup \mu$$

as $n \to \infty$, in the sense of weak convergence of measures, where μ is a positive Borel measure on $\mathbb{R}$.

Assumption 2

Let $t_c \in R_2$ and let $(\chi_c, \eta_c) := (\chi_{\mathcal{E}}(t_c), \eta_{\mathcal{E}}(t_c))$. Assume that $f'(t_c; \chi_c, \eta_c) = f''(t_c; \chi_c, \eta_c) = 0$ so that (χ_c, η_c) is the asymptotic cusp point.

Assumption 3

Assume that for every $\varepsilon > 0$ and n large enough we have for $t_c \in R_2$,

$$\mu_n\Big|_{[t_2+\varepsilon, t_1-\varepsilon]} = \frac{1}{n} \sum_{\lfloor nt_c \rfloor \leqslant k \leqslant n(t_1-\varepsilon)} \delta_{k/n}. \qquad (1)$$

where $\lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leqslant x\}$.

Assumption 3

Assume that there is a neighbourhood U of t_c such that

$$\lim_{n\to\infty} n^{2/3}(f_n'(z) - f'(z)) = 0 \tag{2}$$

uniformly in U.

Rescaled Coordinate System

Fix $r, s \in \mathbb{Z}$ and define the rescaled variables $\xi_n, \tau_n \in \mathbb{R}$, by

$$\begin{cases} x_1 = nx_c^{(n)} + \frac{1}{2}\left(r - c_0 n^{1/3}\xi_n\right) \\ y_1 = ny_c^{(n)} + \frac{1}{2}\left(r + c_0 n^{1/3}\xi_n\right) \\ x_2 = nx_c^{(n)} + \frac{1}{2}\left(s - c_0 n^{1/3}\tau_n\right) \\ y_2 = ny_c^{(n)} + \frac{1}{2}\left(s + c_0 n^{1/3}\tau_n\right) \end{cases},$$

where c_0 is a suitable constant. We assume that

$$\lim_{n\to\infty} \xi_n = \xi, \quad \lim_{n\to\infty} \tau_n = \tau.$$

Rescaled Coordinate System

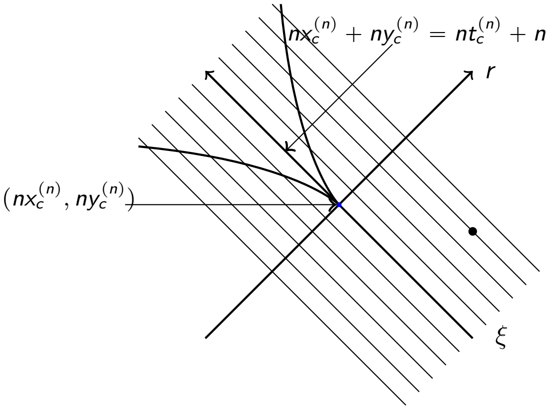


Figure: Rescaled coordinate system at the cusp.

Theorem

Under our assumptions and the scaling above the following result holds:

$$\lim_{n\to\infty} \frac{p_n(x_2,y_2)}{p_n(x_1,y_1)} \frac{c_0}{2} n^{1/3} K_{\mathcal{R}}^{(n)}((x_1,y_1),(x_2,y_2)) = \mathcal{K}_{CA}((\xi,r),(\tau,s))$$

uniformly for ξ and τ in some fixed compact subset of $\mathbb{R}$, and where

$$\frac{p_n(x_2,y_2)}{p_n(x_1,y_1)}$$

is a suitable conjugation factor.

Theorem

Let (ξ_j^r, r) be the rescaled coordinates for particles on line r. Fix $r_1, \ldots, r_M \in \mathbb{Z}$ and let $\phi : \mathbb{R} \times \{r_1, \ldots, r_M\} \to [0,1]$ be a bounded measurable function with compact support. Let $\mathbb{E}_{x^{(n)}}$ denote the expectation with respect to the determinantal point process with kernel $K_{\mathcal{R}}^{(n)}$. Then,

$$\lim_{n\to\infty} \mathbb{E}_{x^{(n)}}\left[\prod_{r\in\{r_1,\ldots,r_m\}} \prod_j (1-\phi(\xi_j^r, r))\right] = \det(I - \phi\mathcal{K}_{CA})_{L^2(\mathbb{R}\times\{r_1,\ldots,r_M\})}. \tag{3}$$

Cusp-Airy Process in Random Skew Plane Partition Models

This type of cusp situation in a random lozenge tiling model was also discovered and discussed briefly by Okounkov and Reshetikhin in [8] who called it a *Cuspidal turning point*. However, the integration contours in their formula are not correct. Also, no proof is given in [8].

Non-exact Discrete Cancellation

We note that asymptotically, we have a perfect cancellation between μ and λ on the interval $[t_c, t_2]$. For finite n, this cancellation is not exact, and will depend on our fixed discrete parameters r and s. More precisely, with

$$q_n(w; r) = 1_{r>0} \prod_{k=nt_c^{(n)}}^{nt_c^{(n)}+r-1} (w-k) + 1_{r=0} + 1_{r<0} \prod_{k=nt_c^{(n)}+r}^{nt_c^{(n)}-1} (w-k)^{-1},$$

we can write the integrand as

$$\frac{q_n(w; r)}{q_n(z; s)} \frac{e^{n(g_{n,1}(w)-n(g_{n,2}(z))}}{w-z}$$

Changing The Integration Contours

To preform a steepest descent analysis of $e^{n(g_{n,1}(w)-n(g_{n,2}(z))}$, we must change the integration contours. This is done by using the residue theorem.

Global descent and ascent contours

We show that away from the critical point t_c, we can chose the descent and ascent contours of $g_{n,1}$ and $g_{n,2}$ to be those of the the asymptotic function $f(w;\chi_c)$. The existence of these contours are proven abstractly using the structure of the support of the signed measure $\mu - \lambda|_{[a,b]}$ and properties of harmonic functions.

Local descent and ascent contours

The local contours around critical point t_c of $g_{n,1}$ and $g_{n,2}$ are glued together with those of $f(w;\chi_c)$ outside some ball of fixed radius . The local analysis inside the ball is done using Taylor expansion and the convergence speed assumption of Assumption 3.

Reflection Symmetry

The Cusp-Airy kernel satisfies

$$\mathcal{K}_{CA}((\xi,-r),(\tau,-s)) = (-1)^{s-r}\mathcal{K}_{CA}((\tau,s),(\xi,r)).$$

In particular, this implies that the correlation functions satisfies the reflection symmetry

$$\rho_n((\xi_1,-r_1),(\xi_2,-r_2),...,(\xi_n,-r_n)) = \rho_n((\xi_1,r_1),(\xi_2,r_2),...,(\xi_n,r_n))$$

for all n.

Representation of The Cusp-Airy Kernel

In this section we give an alternative representation of the Cusp-Airy kernel involving the so called r-Airy integrals and certain polynomials.

Define the *r-Airy integrals*,

$$A_r^{\pm}(u) = \frac{1}{2\pi} \int_\ell e^{\frac{1}{3} ia^3 + iua} (\mp ia)^{\pm r} \, da,$$

where $r \geqslant 0$ and ℓ is a contour from $\infty e^{5\pi i/6}$ to $\infty e^{\pi i/6}$ such that 0 lies above the contour, see [1]; compare also with the functions $s^{(m)}$ and $t^{(m)}$ in [3]. Note that $A_0^{\pm}(u) = \mathrm{Ai}\,(u)$, the standard Airy function.

Representation of The Cusp-Airy Kernel

Define the polynomials $P_n(w, \xi)$ and $p_n(\xi)$ through

$$P_n(w, \xi) := e^{-\frac{1}{3}w^3 + uw} \frac{d^n}{dw^n} e^{\frac{1}{3}w^3 - uw}$$

and

$$p_n(u) := P_n(0, u).$$

Representation of The Cusp-Airy Kernel

We can now give a different formula for the Cusp-Airy kernel in terms of the r-Airy integrals.

$$\mathcal{K}_{CA}((\xi, r), (\tau, s)) = -1_{\tau \geqslant \xi} 1_{s > r} \frac{(\tau - \xi)^{s-r-1}}{(s-r-1)!} + \tilde{\mathcal{K}}_{CA}((\xi, r), (\tau, s)),$$

where $\tilde{\mathcal{K}}_{CA}$ is given

Representation of The Cusp-Airy Kernel

(i) for $r, s \geqslant 0$, by

$$\tilde{\mathcal{K}}_{CA}((\xi, r), (\tau, s)) = \int_0^\infty A_s^-(\tau + \lambda) A_r^+(\xi + \lambda) \, d\lambda,$$

(ii) for $r \geqslant 0$, $s < 0$, by

$$\tilde{\mathcal{K}}_{CA}((\xi, r), (\tau, s)) = (-1)^s \int_0^\infty A_{-s}^+(\tau + \lambda) A_r^+(\xi + \lambda) \, d\lambda,$$

Representation of The Cusp-Airy Kernel

(iii) for $r < 0$, $s \geqslant 0$, by

$$\tilde{\mathcal{K}}_{CA}((\xi,r),(\tau,s)) = (-1)^r \int_0^\infty A_s^-(\tau+\lambda)A_{-r}^-(\xi+\lambda)\,d\lambda$$
$$+ (-1)^{s-r}\int_0^\infty p_{s-1}(\tau+\lambda)A_{-r}^-(\xi+\lambda)\,d\lambda$$
$$+ (-1)^{s-r}\sum_{k=0}^{s-1} p_k(\tau)A_{-r+s-k}(\xi) + \sum_{k=0}^{s-r-1}(-1)^k p_k(\tau)p_{s-r-1-k}(\xi),$$

(iv) and for $r, s < 0$, by

$$\tilde{\mathcal{K}}_{CA}((\xi,r),(\tau,s)) = (-1)^{s-r}\int_0^\infty A_{-s}^+(\tau+\lambda)A_{-r}^-(\xi+\lambda)\,d\lambda.$$

Similar Processes

- The GUE corner process.
- The “The Cusp-Airy version of the Tacnode process”. Uses skew-Young Tableaux. Can also be thought of a double-sided version of the GUE corner process with an additional overlap parameter r. Is studied in [2].

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