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# Anisotropic (2+1)-d growth and Gaussian limits of $q$-Whittaker processes

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- A model of growth in $d = 2$ dimension is the KPZ equation

$$\frac{\partial h}{\partial t} = \nu \Delta h + Q(\nabla h) + \eta$$

for some quadratic form $Q$ ($\eta$: local noise term)

- Two cases behaving quite differently:

(a) Isotropic KPZ: $\mathrm{sign}(Q) = (+1, +1)$ or $\mathrm{sign}(Q) = (-1, -1)$
- Fluctuations grow as $t^{\chi}$ for some $\chi \simeq 0.240...$ (numerics)
- No analytic results

(b) Anisotropic KPZ: $\mathrm{sign}(Q) = (+1, -1)$ (including $0$ as well)

- Prediction from physics: for anisotropic KPZ, the non-linearity becomes irrelevant for the fluctuations Wolf'91
- For $Q = 0$, we have the Edwards-Wilkinson equation

$$\frac{\partial h}{\partial t} = \nu \Delta h + \eta$$

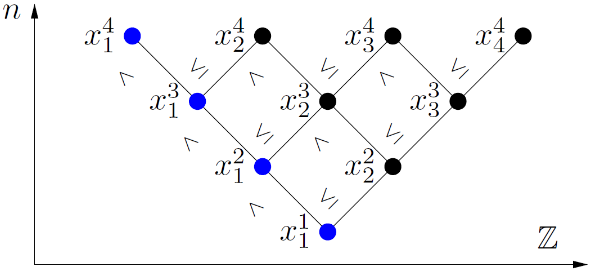
for which

$$\mathrm{Var}(h(\xi t, t)) \sim \ln(t), \quad t \to \infty$$

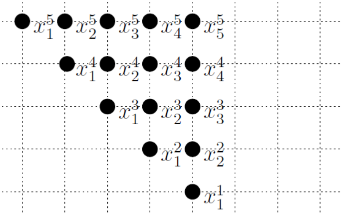
- Logarithmic fluctuations shown in a special model Prähofer,Spohn'97
- The model we consider in this talk is in the anisotropic KPZ framework

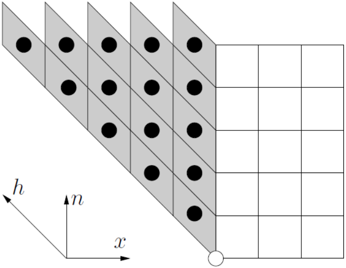
- A model of interacting particles in the anisotropic KPZ growth Borodin,Ferrari'08
- State space: interlacing particles $x_k^n \in \mathbb{Z}$, $1 \le k \le n \le N$ satisfy

$$x_1^{n+1} < x_1^n \le x_2^{n+1} < x_2^n \le \ldots < x_n^n \le x_{n+1}^{n+1}.$$



- Packed initial condition: $x_k^n = -n + k$
- Dynamics: particles try to jump to its right with rate 1. The jump of a particle $(k, n)$:
  - is blocked by $(k, n-1)$: if $x_k^n = x_k^{n-1}$,
  - pushes $(k+1, n+1)$ if $x_k^n = x_{k+1}^{n+1}$.




- Height function $h$:

$$\{h(x,n) \geq a\} = \{x_a^n \geq x\}$$

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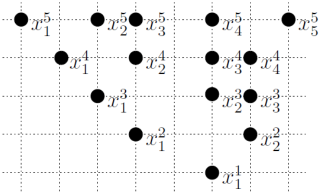


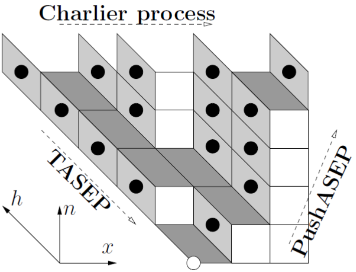


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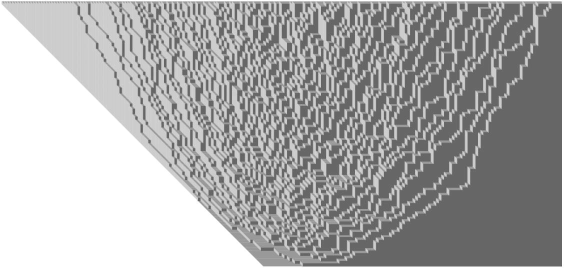


- Height function $h$:

$$\{h(x,n) \geq a\} = \{x_a^n \geq x\}$$

- Key property for the analysis: the model is expressed in terms of a Schur process and has determinantal correlations

**Theorem**

*Consider any $N$ (distinct) triples $(x_j, n_j, t_j)$ such that*

$$t_1 \leq t_2 \leq \ldots \leq t_N, \quad n_1 \geq n_2 \geq \ldots \geq n_N.$$

*Then,*

$$\mathbb{P}\big(\textit{at each } (x_j, n_j, t_j), j = 1, \ldots, N, \\ \textit{there exists a particle}\big) \\ = \det[K(x_i, n_i, t_i; x_j, n_j, t_j)]_{1 \leq i,j \leq N}$$

*for an explicit kernel $K$.*

- Height function $h = h(x, n)$
- $u_x$ = Slope along the $x$-direction
- $u_n$ = Slope along the $n$-direction

$\Rightarrow$ Speed of growth $v$ is given by

$$v(u_x, u_n) = -\frac{\sin(\pi u_x)\sin(\pi u_n)}{\pi \sin(\pi(u_x + u_n))}$$

- Using correlations at different times Borodin,Ferrari'08
- Using the definition from the dynamics Chhita,Ferrari'15

- Is the model in the anisotropic KPZ class?

$$\det(\mathrm{Hessian}(v)) = -4\pi^2 \frac{\sin(\pi u_x)^2 \sin(\pi u_n)^2}{\sin(\pi(u_x + u_n))^4} < 0$$

$\Rightarrow$ Yes.

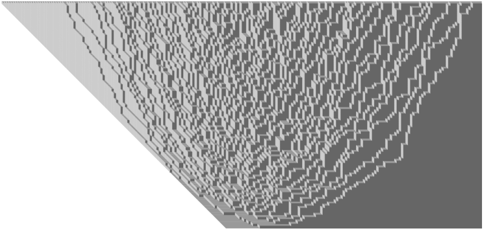
- Macroscopic parametrization

$$n = \eta L, \quad x = -\eta L + \nu L, \quad t = \tau L$$

- Random region is

$$\mathcal{D} = \{(\nu, \eta, \tau) \in \mathbb{R}^3_+, |\sqrt{\tau} - \sqrt{\eta}| < \sqrt{\nu} < \sqrt{\tau} + \sqrt{\eta}\}$$

- Kenyon's map $\Omega : \mathcal{D} \to \mathbb{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$



$(\pi_\nu/\pi, \pi_\eta/\pi, \pi_\tau/\pi)$ are the frequencies of the three types of lozenge tilings. Kenyon'04

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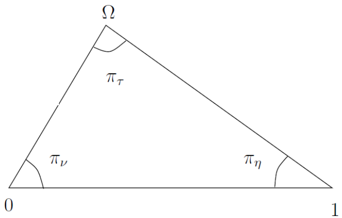



$(\pi_\nu/\pi, \pi_\eta/\pi, \pi_\tau/\pi)$ are the frequencies of the three types of lozenge tilings. Kenyon'04

### Theorem

*Consider any (disjoints) $N$ triples $\kappa_j = (\nu_j - \eta_j, \eta_j, \tau_j)$, with $(\nu_j, \eta_j, \tau_j) \in \mathcal{D}$,*

$$\tau_1 \leq \tau_2 \leq \ldots \leq \tau_N \quad \eta_1 \geq \eta_2 \geq \ldots \geq \eta_N.$$

*Set $H_L(\kappa) := \sqrt{\pi}\,(h(\kappa L) - \mathbb{E}(h(\kappa L)))$. Then,*

$$\lim_{L\to\infty} \mathbb{E}(H_L(\kappa_1)\cdots H_L(\kappa_N)) = \begin{cases} 0, & \textit{odd } N, \\ \sum_{\textit{pairings } \sigma} \prod_{j=1}^{N/2} G(\Omega_{\sigma(2j-1)}, \Omega_{\sigma(2j)}), & \textit{even } N, \end{cases}$$

*with $G(z,w) = -(2\pi)^{-1} \ln |(z-w)/(z-\bar{w})|$ is the Green function of the Laplacian on $\mathbb{H}$ with Dirichlet boundary conditions.*

- Conjecture Borodin,Ferrari'08
  The same result holds without the requirement

$$\tau_1 \leq \tau_2 \leq \ldots \leq \tau_N, \quad \eta_1 \geq \eta_2 \geq \ldots \geq \eta_N$$

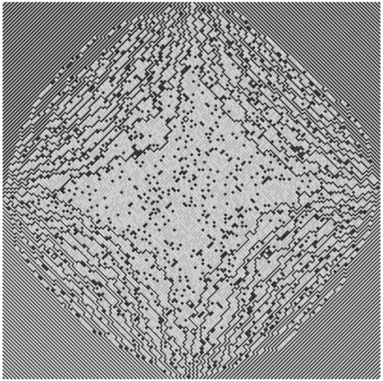
  provided all the $\Omega(\kappa_j)$'s are distinct.
- Reason: the space-time directions with constant $\Omega$ are the "characteristic lines", along which we expect the decorrelation to be much slower than along any space-time directions
- Slow decorrelations phenomena: true for the $d = 1$ KPZ
  Ferrari'08;Corwin,Ferrari,Péché'10

Is there a model in which one can prove existence of the slow decorrelation?

Can we study the full space-time covariance?

- Discrete time versions leads to "more natural" random tiling measure, like the Aztec diamond or lozenge tilings
  Nordenstam'08;Borodin,Gorin'08;Nordenstam,Young'11
- At fixed time one has a non-intersecting line ensemble description: it is a Schur processes (Okounkov,Reshetikin'03), as in the Aztec diamond case Johansson'03
- The corresponding Markov dynamics on Schur process generalizes the one coming from the shuffling algorithm for Aztec diamond (Elkies,Kuperbert,Larsen,Propp'92)
  Borodin,Ferrari'15
- These measures and the corresponding Markov dynamics can also be described in terms of dimer models on Rail Yard graphs studied in
  Betea,Bouttilier,Bouttier,Chapuy,Corteel,Ramassamy'14-'15

- Not all random tiling fits in the Schur process framework.
  E.g., periodic Aztec Chhita,Young'13;Chhita,Johansson'14
- With periodicity one can get a "gas phase" macroscopically flat. Proof of the Airy point field at the edge of the gas phase was recently obtained. Beffara,Chhita,Johansson'16

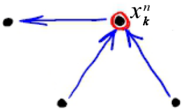


- Same state space and initial condition as in the model discussed above
  - State space: interlacing particles $x_k^n \in \mathbb{Z}$, $1 \leq k \leq n \leq N$ satisfy

$$x_1^{n+1} < x_1^n \leq x_2^{n+1} < x_2^n \leq \ldots < x_n^n \leq x_{n+1}^{n+1}.$$

  - Packed initial condition: $x_k^n = -n + k$
- Dynamics: fix $q \in (0,1)$. The jump rate of particle $(k,n)$ is given by

$$\frac{(1 - q^{x_k^{n-1} - x_k^n - 1})(1 - q^{x_k^n - x_{k-1}^n})}{1 - q^{x_k^n - x_{k-1}^{n-1}}}$$




- The $q = 0$ is the determinantal model studied before

- Key property for the analysis: the $q$-moments have explicit expressions Borodin,Corwin'11

$$\mathbb{E}\left[\prod_{i=1}^{m} q^{x_1^{n_i}(t)+\cdots+x_{r_i}^{n_i}(t)}\right]$$

$$= \prod_{i=1}^{m} \frac{1}{(2\pi \mathrm{i})^{r_i} r_i!} \oint \cdots \oint \prod_{1\le i<j\le m} \prod_{k=1}^{r_i} \prod_{\ell=1}^{r_j} \frac{q(z_{i,k}-z_{j,\ell})}{z_{i,k}-qz_{j,\ell}}$$

$$\times \prod_{i=1}^{m} \frac{(-1)^{\frac{r_i(r_i+1)}{2}} \prod_{1\le k<\ell\le r_i} (z_{i,k}-z_{i,\ell})^2}{\prod_{k=1}^{r_i} (z_{i,k})^{r_i}} \prod_{k=1}^{r_i} \frac{e^{z_{i,k}(q-1)t}}{(1-z_{i,k})^{n_i}} dz_{i,k}.$$

(The domain of integrations are nested contours)

## Some known results on the $q$-Whittaker process

- Tracy-Widom distribution for fixed $q$, let $\kappa > 1/(1-q)$. Then as $N \to \infty$,

$$\frac{x_1^N(\kappa N) - c_1(\kappa)N}{c_2(\kappa)N^{1/3}} \stackrel{(d)}{=} \xi_{\mathrm{GUE}}$$

Ferrari,Veto'13;Barraquand'14

- Polymer models: for $q = e^{-\varepsilon}$ and $t = \tau/\varepsilon^2$, as $\varepsilon \to 0$,

$$x_1^N(t) \stackrel{(d)}{=} \frac{\tau}{\varepsilon^2} - N\frac{\ln(1/\varepsilon)}{\varepsilon} - \frac{\ln Z(\tau, N)}{\varepsilon}$$

where $Z(\tau, N)$ is the partition function of the semidiscrete directed polymer model (O'Connell,Yor'01)

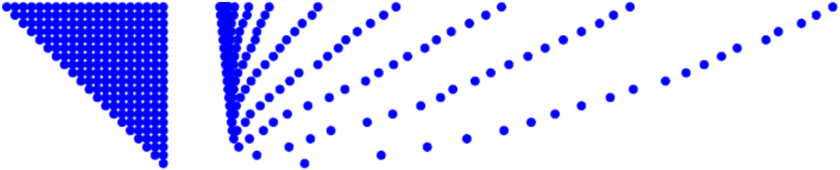
Borodin,Corwin'11

- This polymer model was further used to get distribution laws of the solution of the KPZ equation

Borodin,Corwin,Ferrari'12;+Veto'14

In our work we consider shorter time scales:

$$q = e^{-\varepsilon}, \quad t = \tau/\varepsilon.$$



Simulation of $q$-Whittaker particle system with $N = 20$ particles, $q = e^{-\varepsilon}$ and $\varepsilon = 0.01$. The centered and diffusively scaled particle process $x_k^{(n)}(\tau/\varepsilon)$ is plotted for $\tau = 1$

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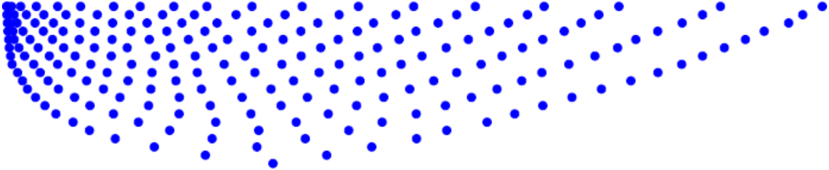

Simulation of $q$-Whittaker particle system with $N = 20$ particles, $q = e^{-\varepsilon}$ and $\varepsilon = 0.01$. The centered and diffusively scaled particle process $x_k^{(n)}(\tau/\varepsilon)$ is plotted for $\tau = 10$

## Law of large numbers

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} x_k^n(\varepsilon^{-1}\tau) = y_k^n(\tau)$$

where

$$e^{-(y_1^n(\tau)+\cdots+y_r^n(\tau))} = \oint \cdots \oint \mathfrak{F}_\tau(n, r; z_1, \ldots, z_r) dz_1 \cdots dz_r$$

with

$$\mathfrak{F}_\tau(n, r; z_1, \ldots, z_r) = \frac{(-1)^{\frac{r(r+1)}{2}} \prod_{1 \le k < \ell \le r} (z_k - z_\ell)^2}{(2\pi \mathrm{i})^r r! \prod_{k=1}^{r} (z_k)^r} \prod_{k=1}^{r} \frac{e^{-z_k \tau}}{(1 - z_k)^n}$$

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where

$$e^{-(y_1^{(n)}(\tau)+\cdots+y_r^{(n)}(\tau))} = e^{-\tau r} \det\left[G_{r,\tau}(n+1-r+j-i)\right]_{i,j=1}^{r}$$

with $G$ being polynomials given by

$$G_{r,\tau}(m) = \sum_{i \geq 0} \frac{\tau^i (r)_{m-i-1}}{i!(m-i-1)!}.$$

Gaussian fluctuations

For any fixed $N$, as $\varepsilon \to 0$,

$$x_k^n(\tau/\varepsilon) = \varepsilon^{-1} y_k^n(\tau) + \varepsilon^{-1/2} \xi_k^n(\tau)$$

where $\{\xi_k^n(\tau), 1 \leq k \leq n \leq N\}$ is a centered Gaussian process with explicit covariance (written in terms of contour integrals).

We would like to study the large time and large-$N$ limit

Large time simplification allowing large-$N$ asymptotic analysis

For any fixed $T > 0$, the limit

$$\zeta_k^n(T) = \lim_{L \to \infty} L^{-1/2} \xi_k^n(LT)$$

exists and $\zeta = \{\zeta_k^n(T), 1 \le k \le n \le N\}$ is a centered Gaussian process with covariance given through the following formula. For $n_1 \ge n_2$,

$$\mathrm{Cov}\left(\zeta_1^{n_1}(T) + \ldots + \zeta_{r_1}^{n_1}(T); \zeta_1^{n_2}(T) + \ldots + \zeta_{r_2}^{n_2}(T)\right)$$
$$= \frac{\oint\oint \frac{r_1 r_2}{z_1 - w_1}\Big(\prod_{1\le i<j\le r_1}(z_j - z_i)^2 \prod_{m=1}^{r_1} \frac{e^{Tz_m}}{(z_m)^{n_1}} dz_m\Big)\Big(\prod_{1\le i<j\le r_2}(w_j - w_i)^2 \prod_{m=1}^{r_2} \frac{e^{Tw_m}}{(w_m)^{n_2}} dw_m\Big)}{\Big(\oint \prod_{1\le i<j\le r_1}(z_j - z_i)^2 \prod_{m=1}^{r_1} \frac{e^{Tz_m}}{(z_m)^{n_1}} dz_m\Big)\Big(\oint \prod_{1\le i<j\le r_2}(w_j - w_i)^2 \prod_{m=1}^{r_2} \frac{e^{Tw_m}}{(w_m)^{n_2}} dw_m\Big)}$$

Using random matrix type algebra we rewrite

$$\mathrm{Cov}\Big(\zeta_{r_1}^{n_1}(T), \zeta_{r_2}^{n_2}(T)\Big)$$
$$= \oint_{\Gamma_0} \frac{dw}{2\pi \mathrm{i}} \oint_{\Gamma_{0,w}} \frac{dz}{2\pi \mathrm{i}} \frac{1}{z-w} \frac{e^{Tz} e^{Tw}}{z^{n_1} w^{n_2}} \frac{(p_{r_1-1}^{n_1}(z))^2}{\langle p_{r_1-1}^{n_1}, p_{r_1-1}^{n_1} \rangle_{n_1}} \frac{(p_{r_2-1}^{n_2}(w))^2}{\langle p_{r_2-1}^{n_2}, p_{r_2-1}^{n_2} \rangle_{n_2}}$$

where $p_r^n$ are orthogonal polynomials with respect to the scalar product $\langle f, g \rangle_n = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} f(z) g(z) \frac{e^{Tz}}{z^n} dz$. Finally we have

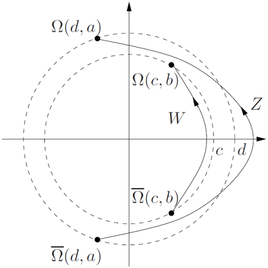
$$\mathrm{Cov}\Big(\zeta_{r_1}^{n_1}(T=1), \zeta_{r_2}^{n_2}(T=1)\Big) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dw \oint_{\Gamma_{0,w}} dz \frac{1}{z-w}$$
$$\times \left( \int_0^\infty dx (w-x)^{r_2-1} x^{n_2-r_2} e^{-x} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_w} du \frac{e^u}{(w-u)^{r_2} u^{n_2-r_2+1}} \right)$$
$$\times \left( \int_0^\infty dy (z-y)^{r_1-1} y^{n_1-r_1} e^{-y} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_z} dv \frac{e^v}{(z-v)^{r_1} v^{n_1-r_1+1}} \right).$$

## Asymptotic covariance in the bulk

Let us denote

$$\Omega(c,b) = c(1-2b+2\mathrm{i}\sqrt{b(1-b)}).$$

Take any $a,b \in (0,1)$, $d>0$ and $c \in (0,d]$ such that $\Omega(c,b) \neq \Omega(d,a)$.



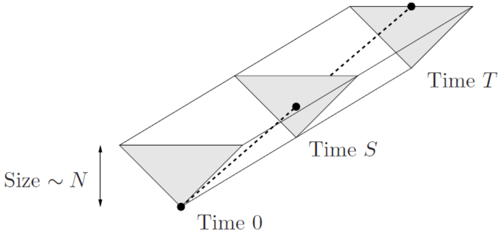

Then, the large $N$ limit of the covariance is given by

$$\lim_{N\to\infty} N\mathrm{Cov}\left(\zeta_{adN}^{dN}(T=1), \zeta_{bcN}^{cN}(T=1)\right)$$
$$= \frac{16}{(2\pi\mathrm{i})^2}\int_{\overline{\Omega}(c,b)}^{\Omega(c,b)} dW \int_{\overline{\Omega}(d,a)}^{\Omega(d,a)} dZ \frac{1}{Z-W}$$
$$\times \frac{1}{\sqrt{(W-\Omega(c,b))(W-\overline{\Omega}(c,b))}\sqrt{(Z-\Omega(d,a))(Z-\overline{\Omega}(d,a))}}$$

## Slow decorrelation

Take any $a, b \in (0,1)$, $d > 0$ and $c \in (0,d]$. Then for any $T > 1$,

$$\lim_{N\to\infty} N\mathrm{Cov}\Big(\zeta^{dNT}_{adNT}(T), \zeta^{cN}_{bcN}(T=1)\Big)$$
$$= \lim_{N\to\infty} N\mathrm{Cov}\Big(\zeta^{dN}_{adN}(T=1), \zeta^{cN}_{bcN}(T=1)\Big).$$




Here we looked at space points $\mathcal{O}(N)$ away. On which scale one has non-trivial time-time correlations?

Short distance behavior

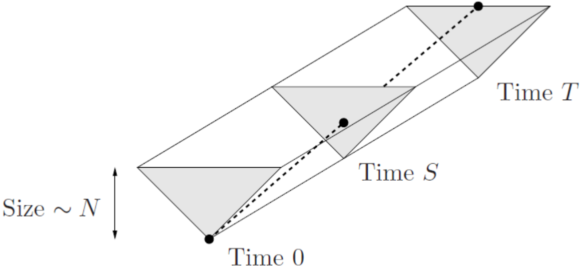
$$\lim_{N\to\infty} N\mathrm{Cov}\Big(\zeta_{adN}^{dN}(T=1), \zeta_{bcN}^{cN}(T=1)\Big) = \frac{-4}{\pi}\frac{\ln(|\Omega(d,a)-\Omega(c,b)|)}{\sqrt{\mathrm{Im}\Omega(d,a)}\sqrt{\mathrm{Im}\Omega(c,b)}} + \mathcal{O}(1)$$

as $|\Omega(d,a)-\Omega(c,b)| \to 0$.

## Time correlation at $\mathcal{O}(\sqrt{N})$ from the characteristic lines

Fix $d > 0$, $a \in (0,1)$, $T > S > 0$.
For $\eta = (\eta_1, \eta_2)$ let $\zeta(T, \eta; N) = N^{1/2} \zeta_k^n(T)$

$$n = dNT + \left(\eta_1 \sqrt{(1-a)d} + \eta_2 \sqrt{ad}\right) \sqrt{N} T,$$
$$k = adNT + \eta_2 \sqrt{ad} \sqrt{N} T.$$



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$$k = adNT + \eta_2\sqrt{ad}\sqrt{N}T.$$

Then for $\eta, \lambda, \mu, \nu \in \mathbb{R}^2$ (all different),

$$\lim_{N\to\infty} \mathrm{Cov}\left(\zeta(T,\eta;N) - \zeta(T,\lambda;N),\, \zeta(S,\mu;N) - \zeta(S,\nu;N)\right)$$
$$= \frac{S}{\pi d\sqrt{a(1-a)}}\Big(G_\tau(|\eta-\mu|) - G_\tau(|\eta-\nu|) - G_\tau(|\lambda-\mu|) + G_\tau(|\lambda-\nu|)\Big)$$

where $\tau = (T-S)/T$ and $G_\tau(r) = -\ln(r^2) - \Gamma(0, r^2/2\tau)$

## Relation with Edwards-Wilkinson and GFF

The space-time covariance at $\mathcal{O}(\sqrt{N})$ away from the characteristic is the ones of the covariance of the Edwards-Wilkinson equation

$$\partial_t u(x,t) = \frac{1}{2}\Delta u(x,t) + \xi(x,t)$$

with $\xi$ space-time white noise and GFF initial conditions

$$\mathrm{Cov}\Big(\zeta_{aN}^{N}(1), \zeta_{bcN}^{cN}(1)\Big) = \frac{N}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dW \oint_{\Gamma_{0,W}} dZ \frac{1}{Z-W}$$
$$\times \left( \int_0^\infty dX \frac{e^{NF(c,b,W,X)}}{X-W} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dU \frac{e^{NG(c,b,W,U)}}{W+U} \right)$$
$$\times \left( \int_0^\infty dY \frac{e^{NF(1,a,Z,Y)}}{Y-Z} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dV \frac{e^{NG(1,a,Z,V)}}{Z+V} \right),$$

where

$$F(c,b,W,X) = bc\ln(X-W) + (1-b)c\ln(X) - X,$$
$$G(c,b,W,U) = -bc\ln(U) - (1-b)c\ln(W+U) + U + W.$$

$$\mathrm{Cov}\Big(\zeta_{aN}^{N}(1), \zeta_{bcN}^{cN}(1)\Big) = \frac{N}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dW \oint_{\Gamma_{0,W}} dZ \, \frac{1}{Z-W}$$
$$\times \left( \int_0^\infty dX \, \frac{e^{NF(c,b,W,X)}}{X-W} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dU \, \frac{e^{NG(c,b,W,U)}}{W+U} \right)$$
$$\times \left( \int_0^\infty dY \, \frac{e^{NF(1,a,Z,Y)}}{Y-Z} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dV \, \frac{e^{NG(1,a,Z,V)}}{Z+V} \right),$$

Key idea: Study $H(W,X,U) = F(c,b,W,X) + G(c,b,W,U)$.
$W, Z$ are slow varying variables, $X, U, Y, V$ are fast varying variables.
(1) Integrate first $X, U, Y, V$.
(2) Integrate after that $W, Z$.

Difficulty: After (1), there are mixed-terms which are bounded only along some non-explicitly known paths for $W, Z$. The mixed terms vanish only after (2).