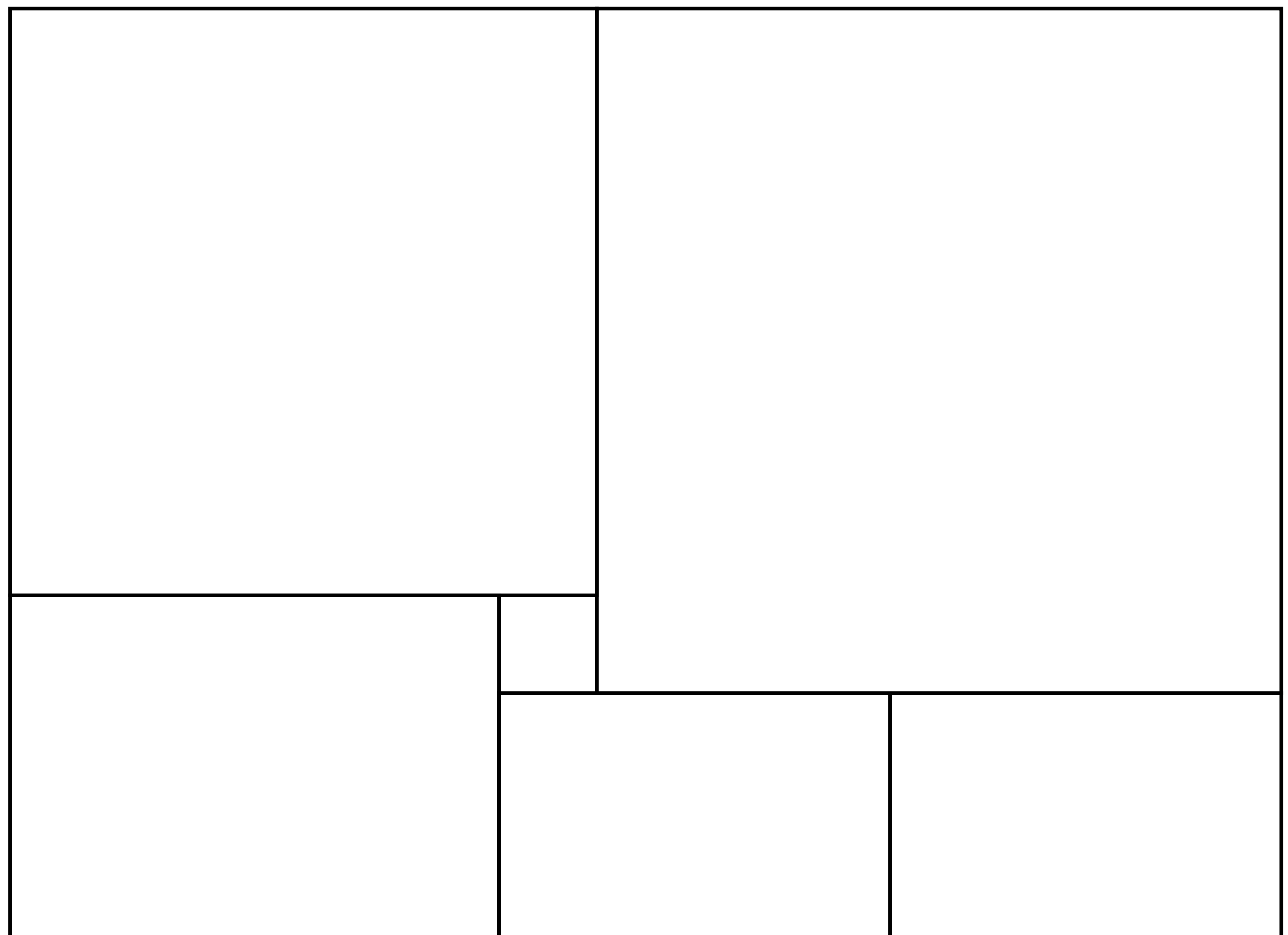
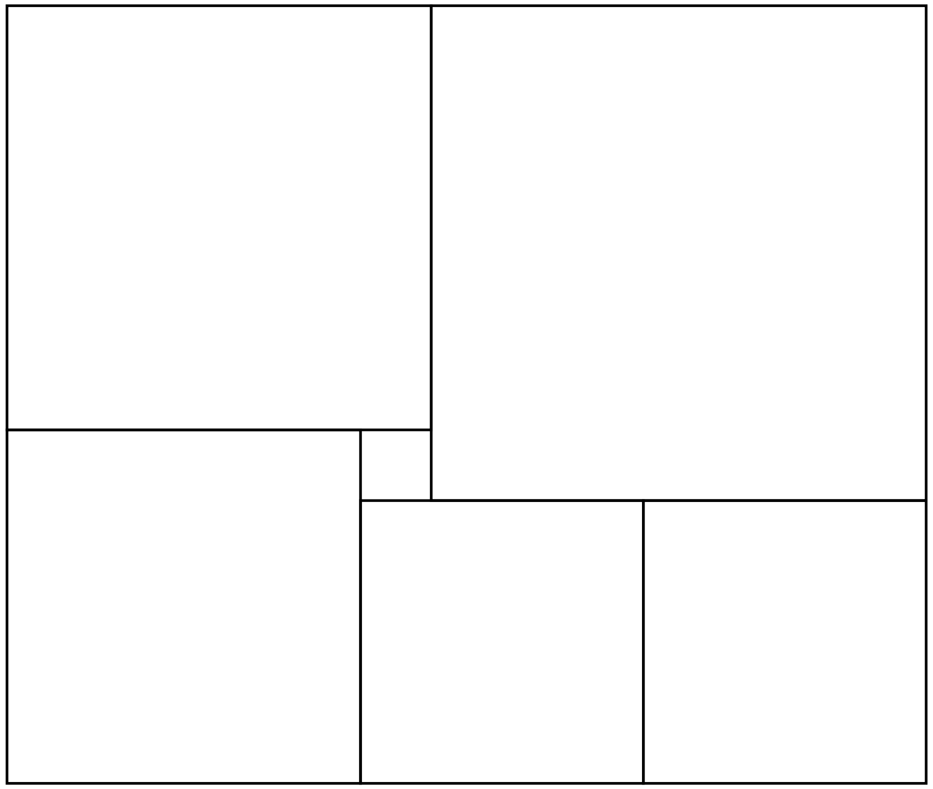
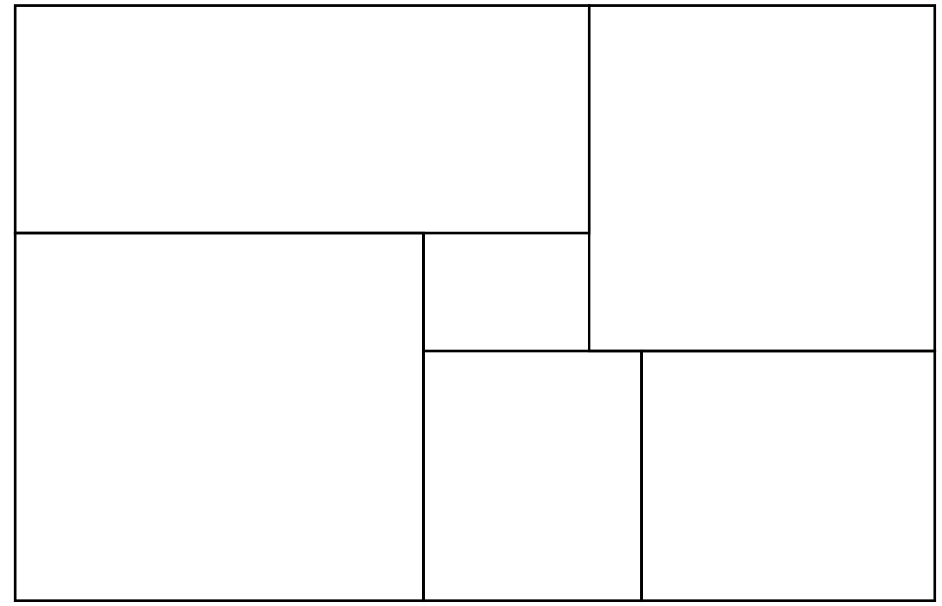
Linear spaces of tilings

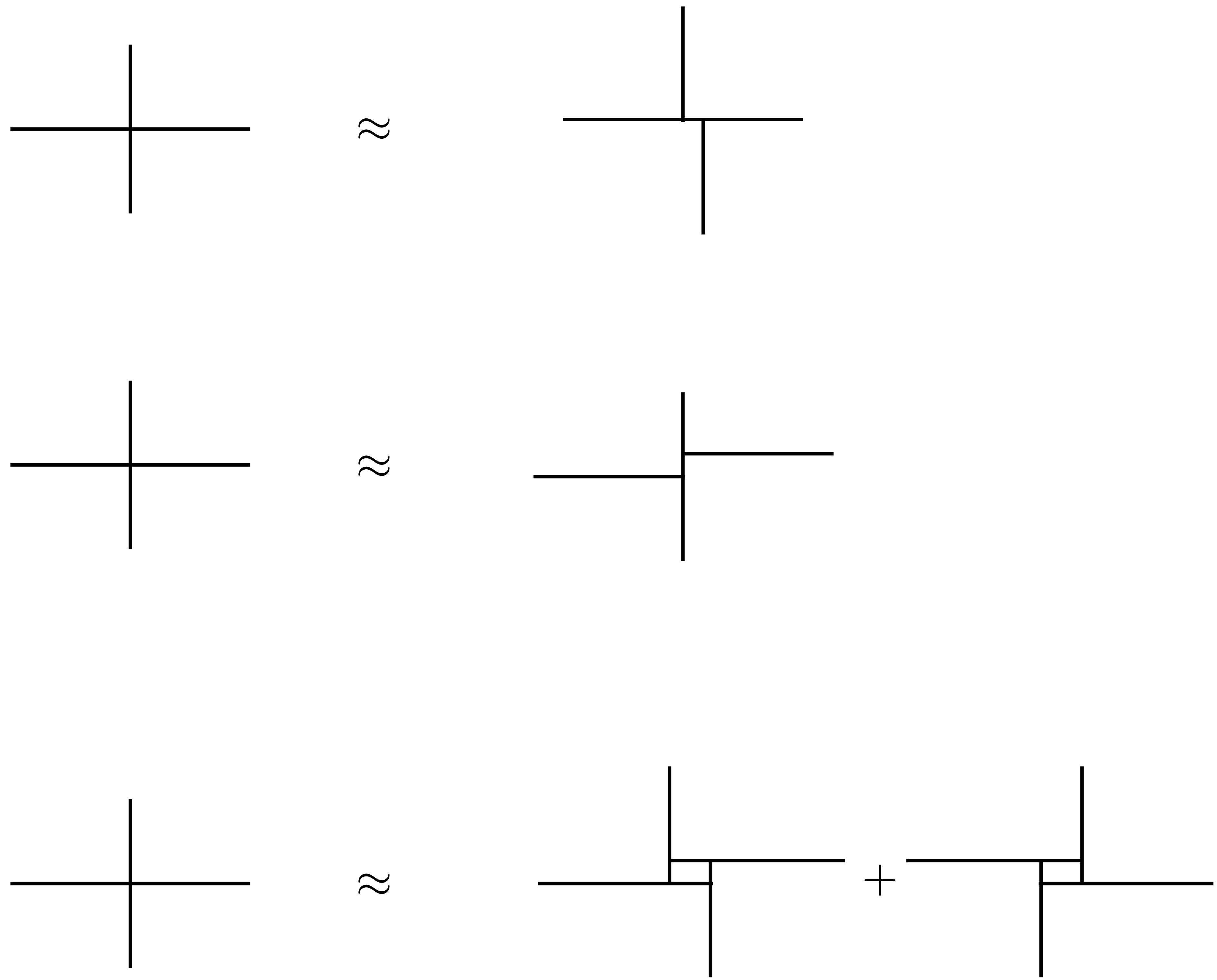
Richard Kenyon (Brown University)



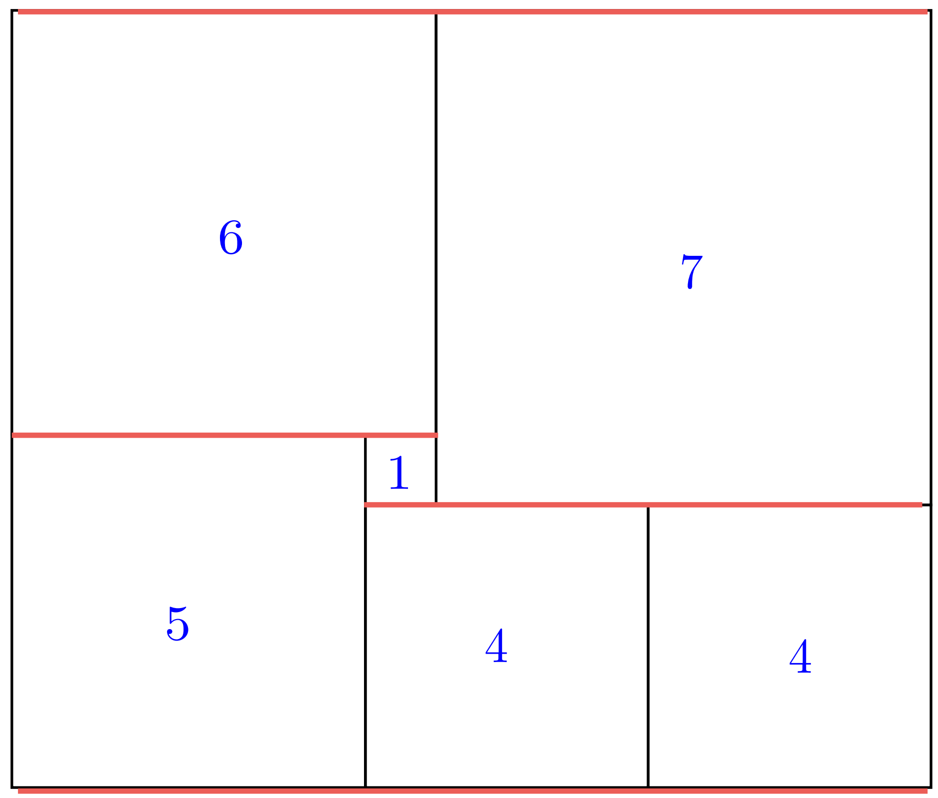
 

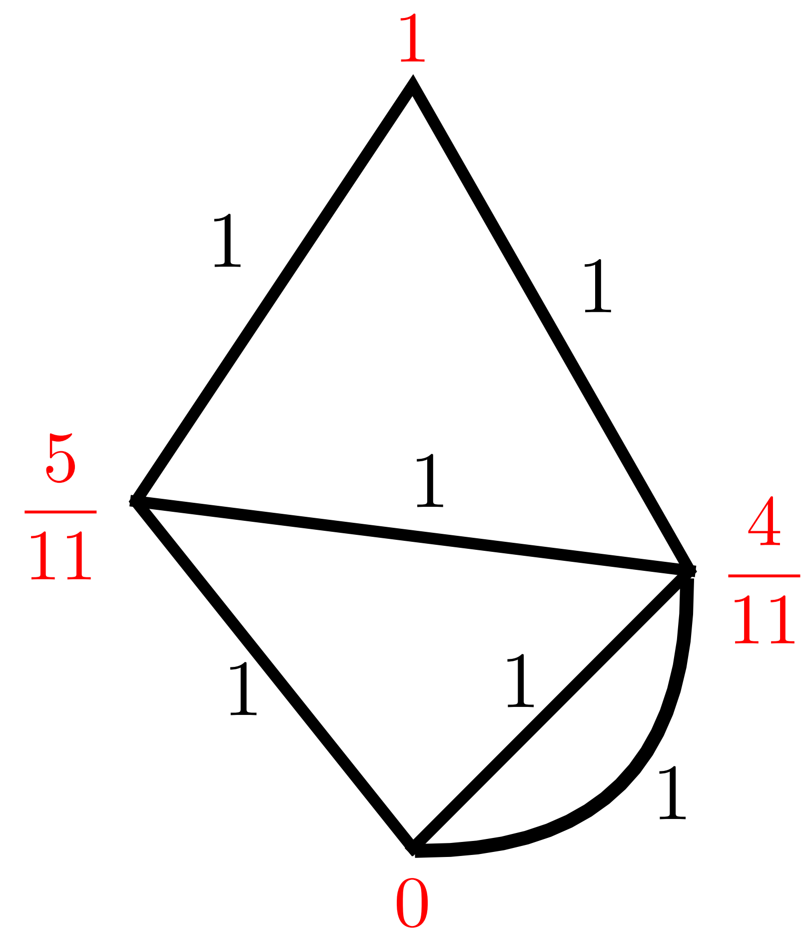
Thm (Brooks, Smith, Stone, Tutte 1939): Given a rectangle tiling of a rectangle, there is a tiling of another rectangle with the same combinatorics and prescribed aspect ratios.

At a “degenerate” vertex, choose a resolution:



Proof idea: Associate to a rectangle tiling a harmonic function on a planar network.

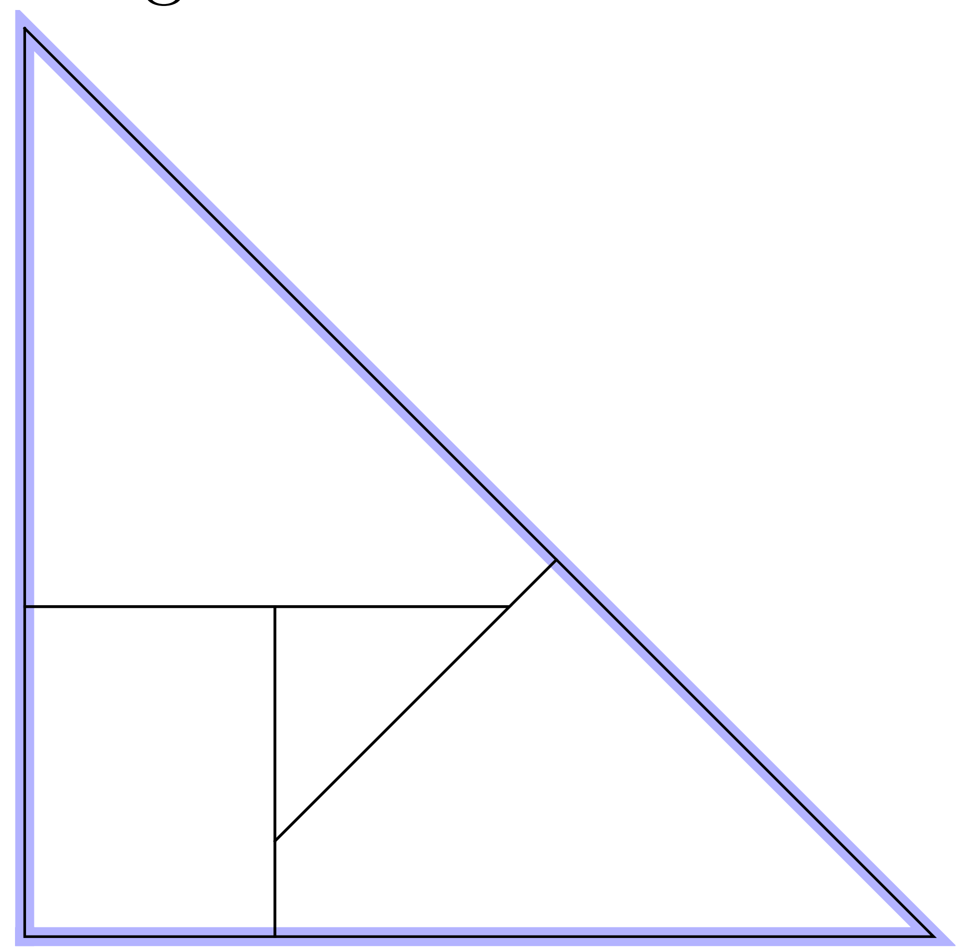




voltage = y-coordinate
current = width
conductance = aspect ratio
energy = area

$\square$

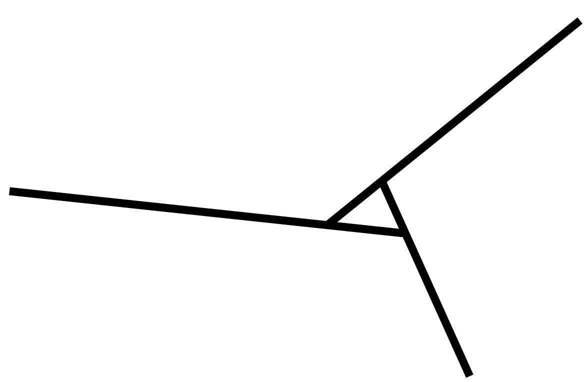
A generalization



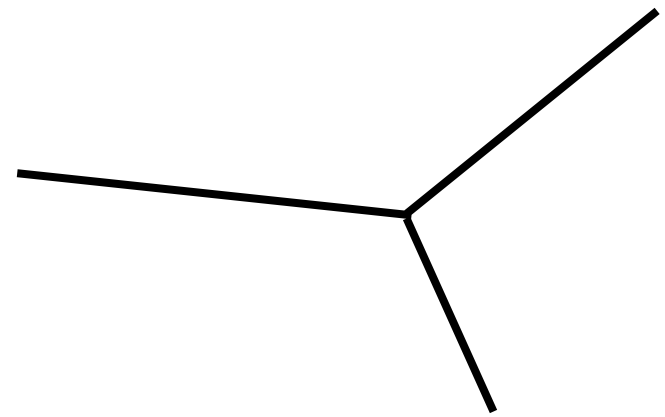
Thm: Given a tiling of a convex polygon P with convex polygons, and, fixing slopes, a new shape (up to scale in $\mathbb{R}$) for every tile. Then there is a combinatorially equivalent* tiling with these shapes, of a (new) convex polygon P'.

Lemma: When P is a k-gon,

$$\# \text{ parameters} = \# \text{ internal lines} + \text{k - 3} = \# \text{ tiles - 1}.$$

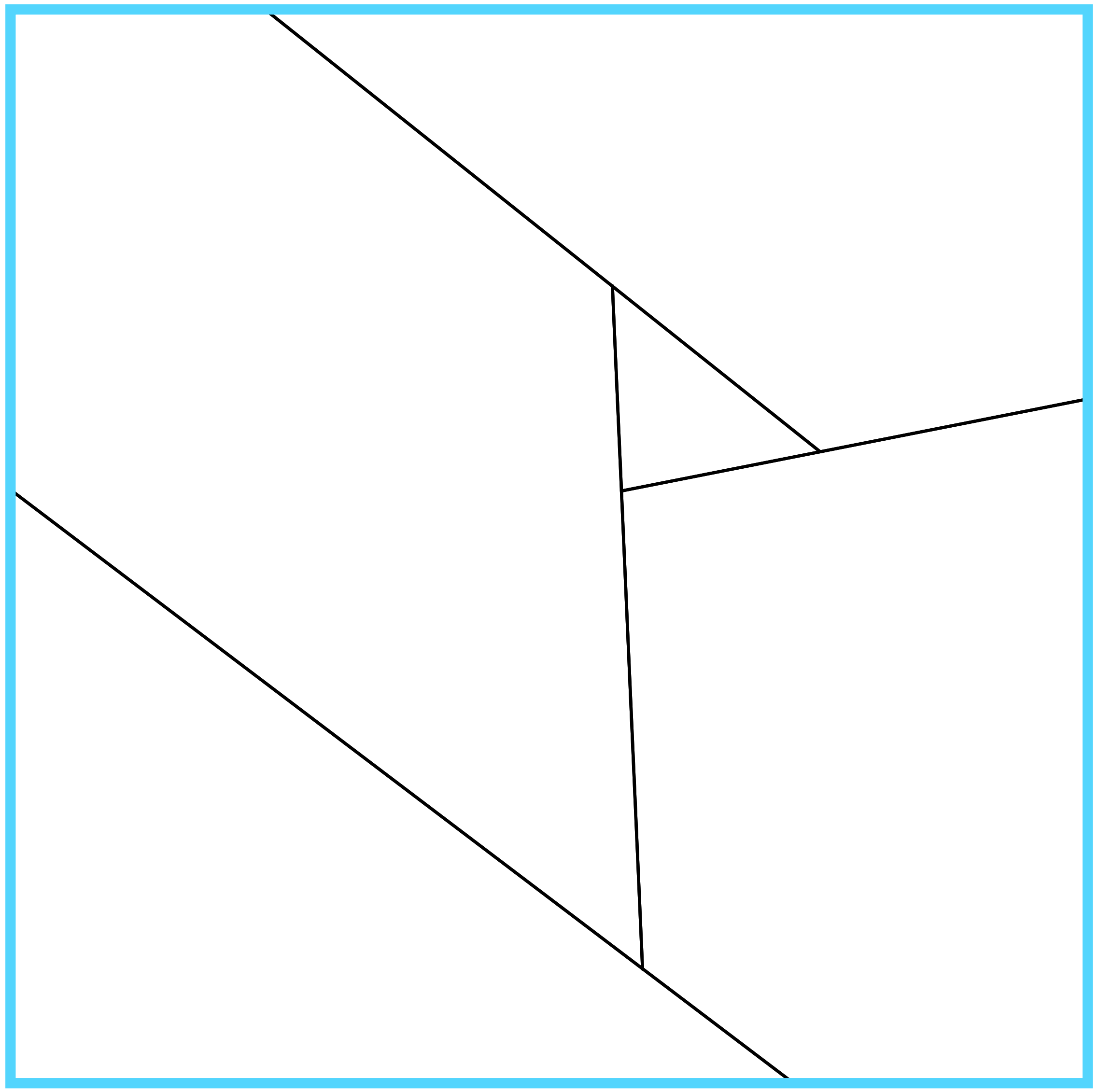


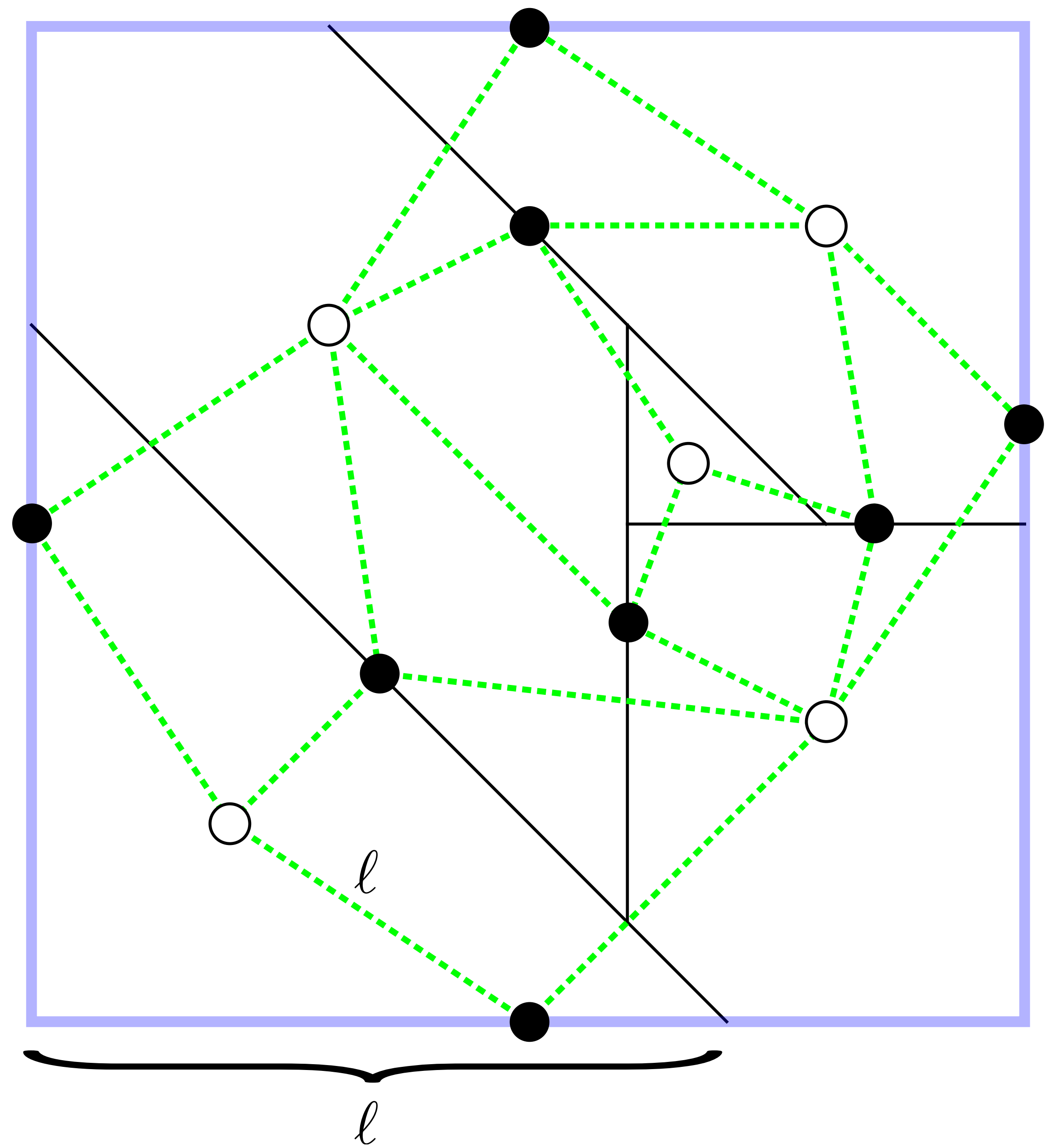
generic

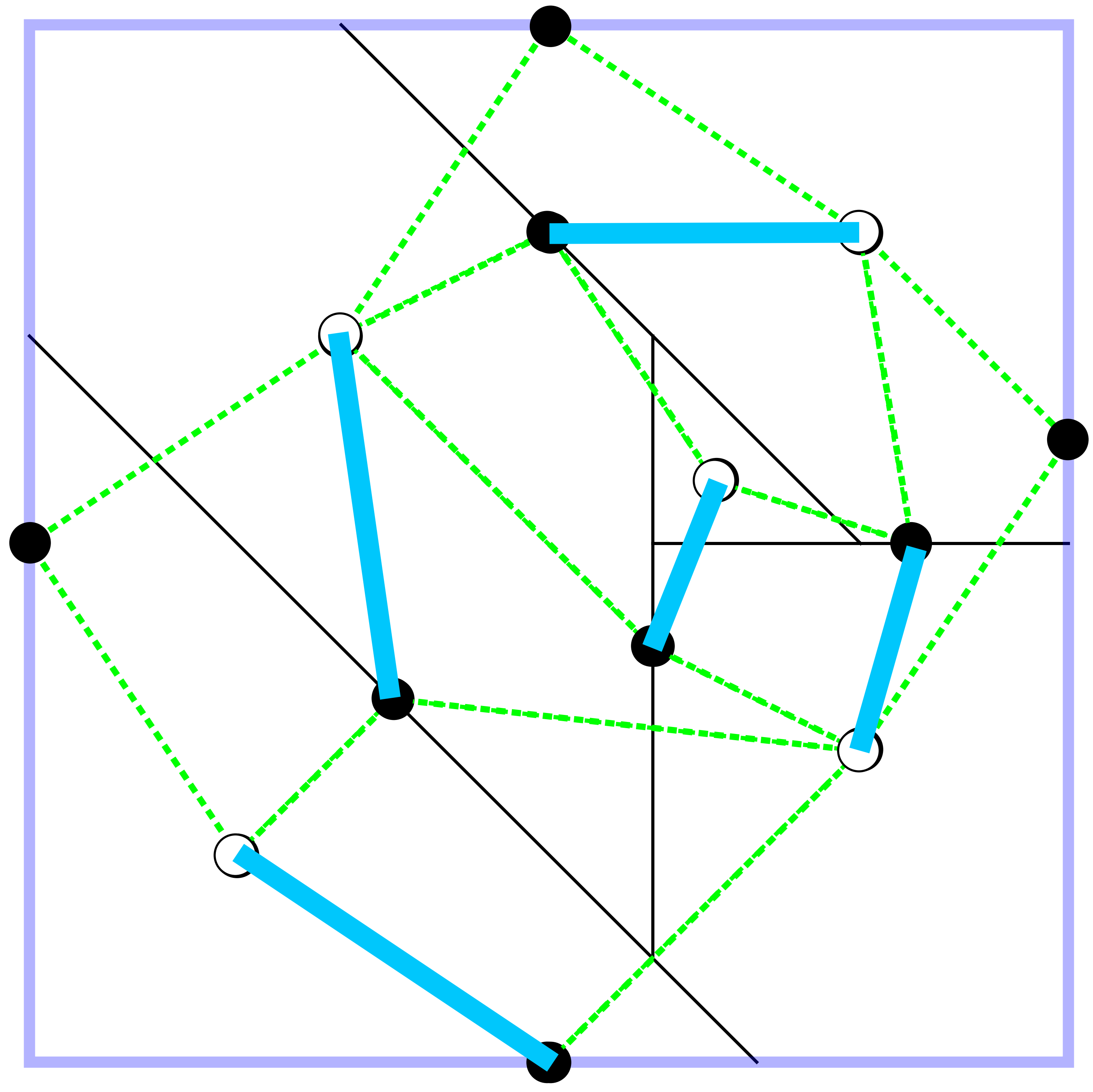


nongeneric

Associated to a convex polygon tiling is a bipartite network...



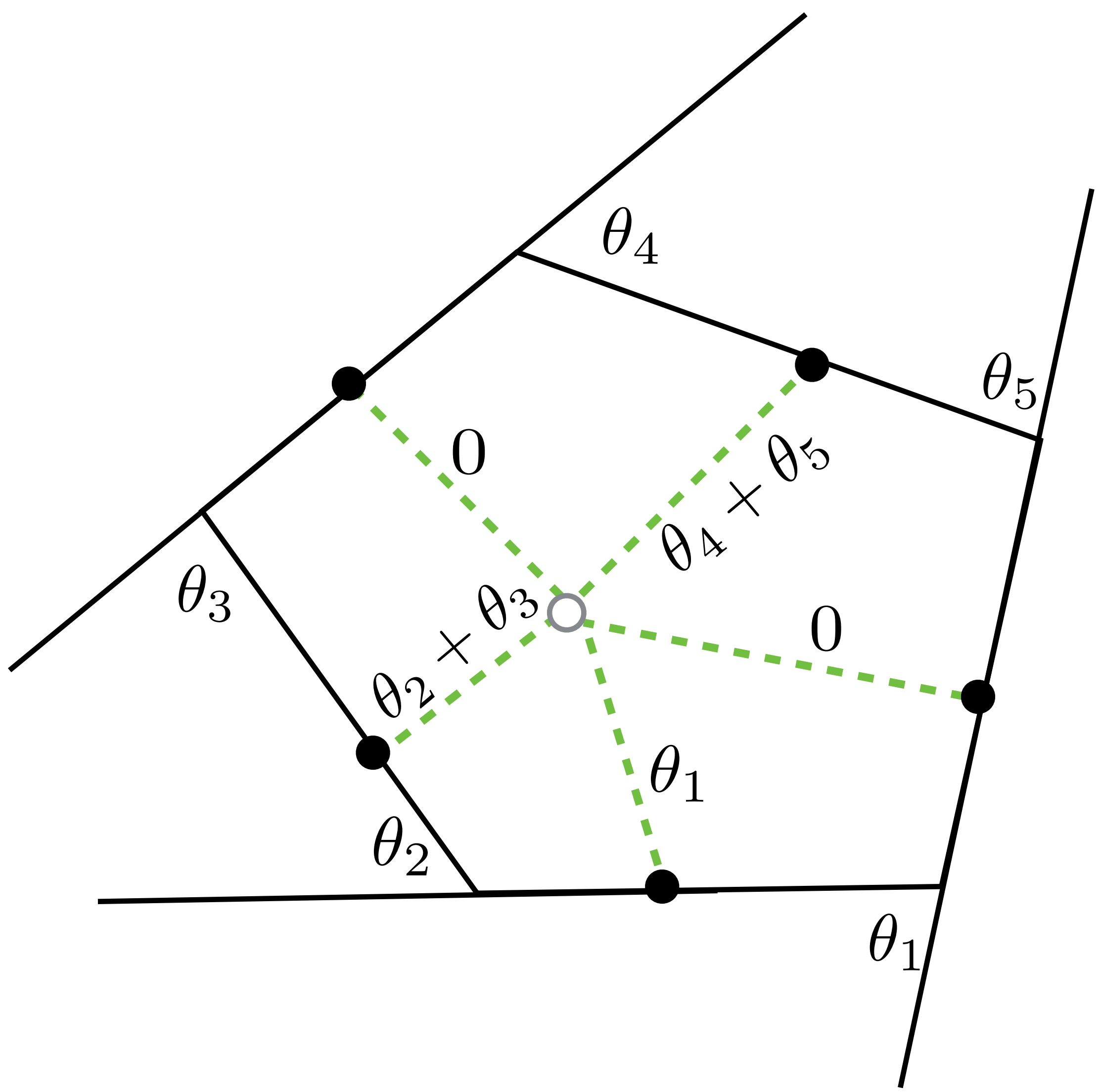


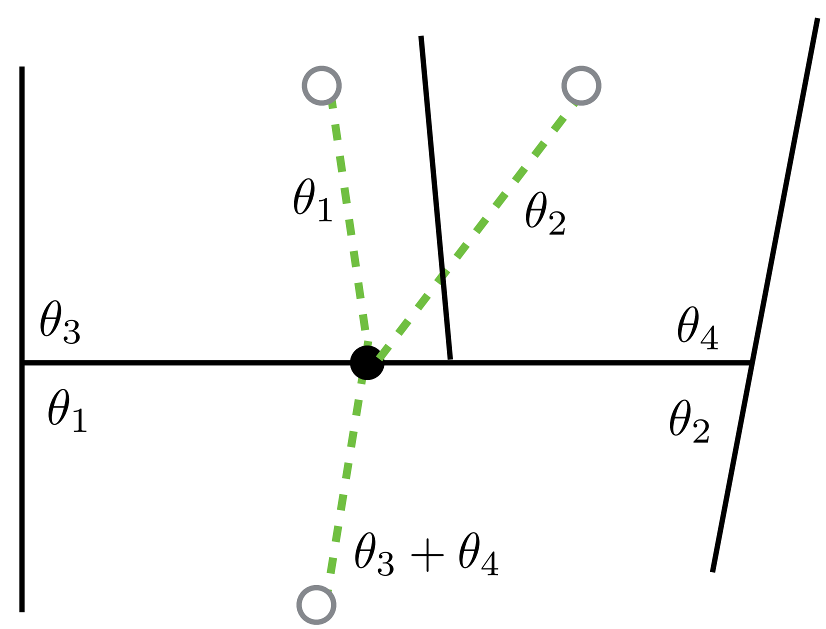


...which has dimer covers (when we remove all but one outer edge).

Why? Because it has a fractional dimer cover:

(Dimer covers are the vertices of the polytope of fractional dimer covers.)

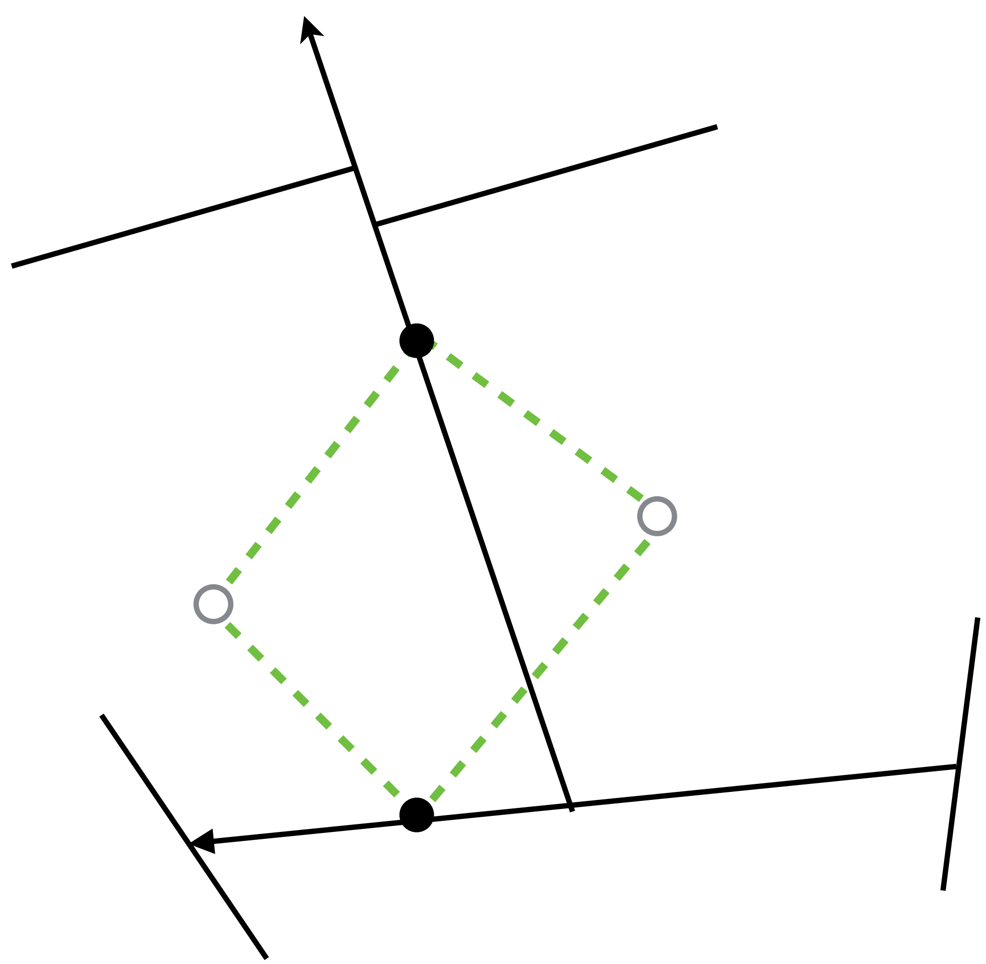




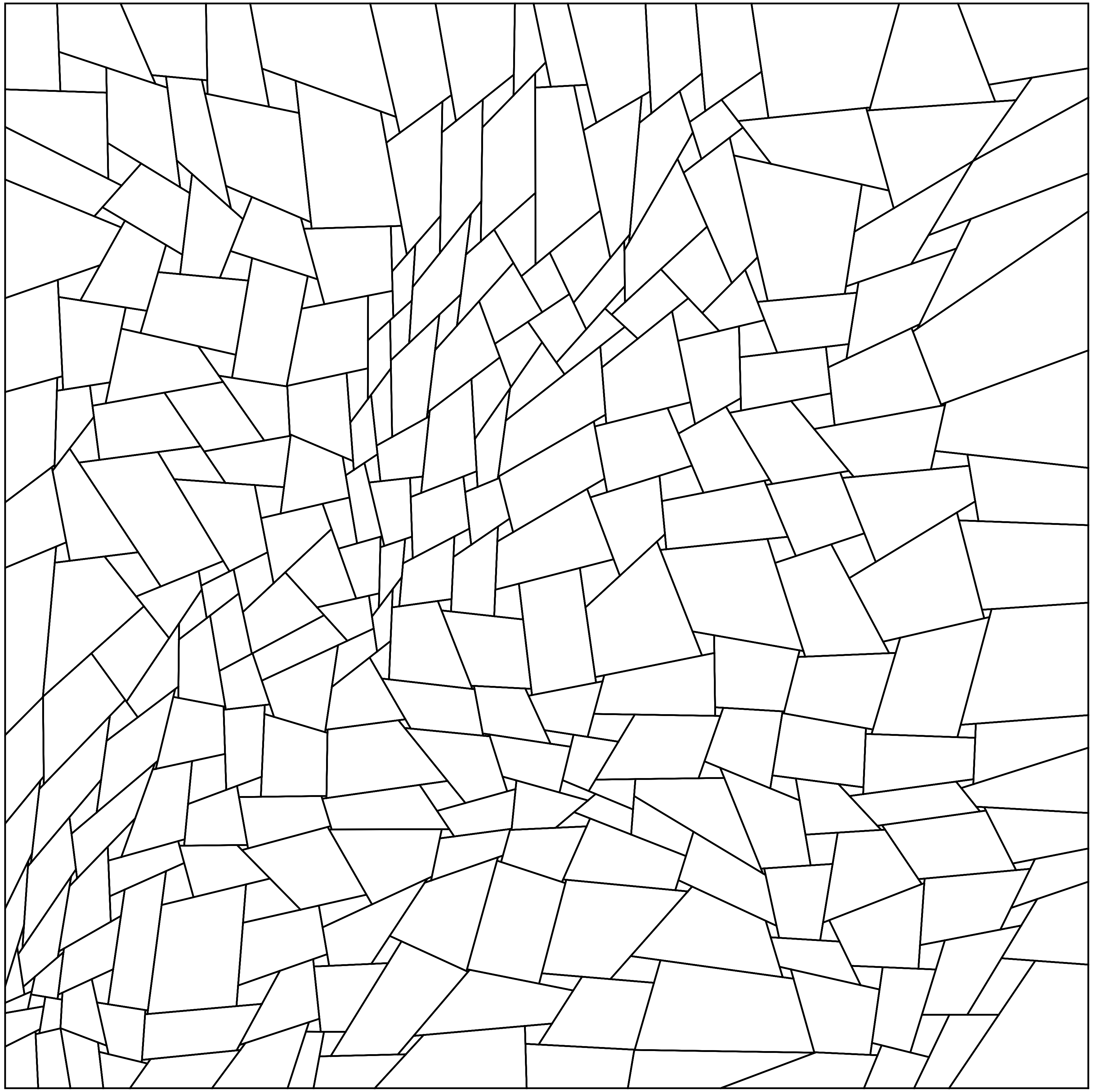
There is an associated $|W| \times |B|$ matrix K (a signed, weighted adjacency matrix) with $K_{bw} = \pm\ell$ if black segment b is an edge of face w of length ℓ.

sign depends on which side of edge face lies

K is a "Kasteleyn matrix": product of signs around a face is $(-1)^{k/2+1}$.

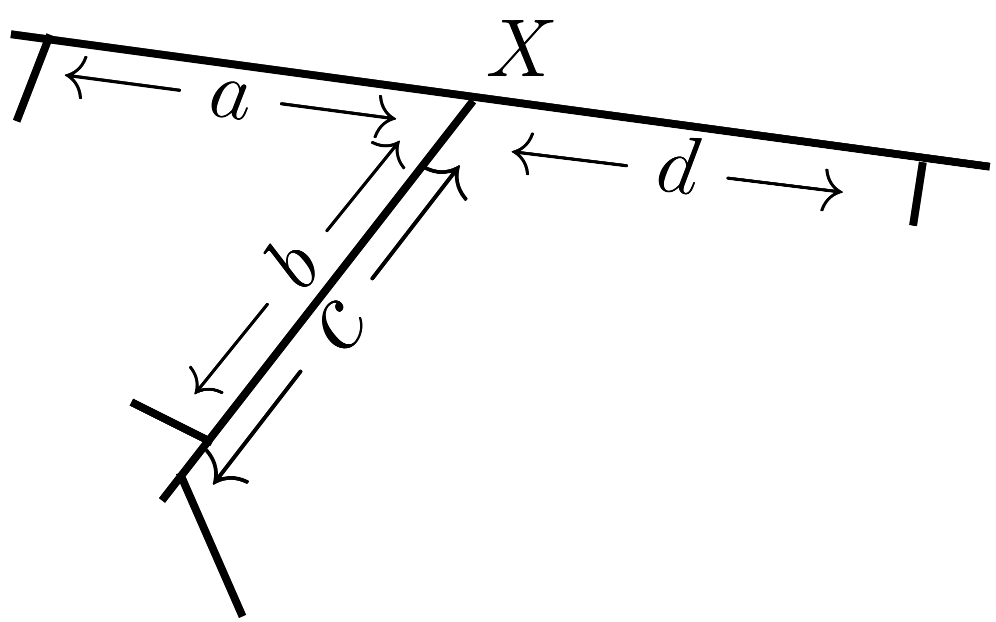


What if we don't fix slopes, just the bipartite graph?



(follows from [K-Sheffield 2003])

Thm: The space of tilings with n segments, fixed boundary and fixed combinatorics is homeomorphic to $\mathbb{R}_+^{2n}$. Global coordinates are *biratio coordinates* $\{X_i\}$.

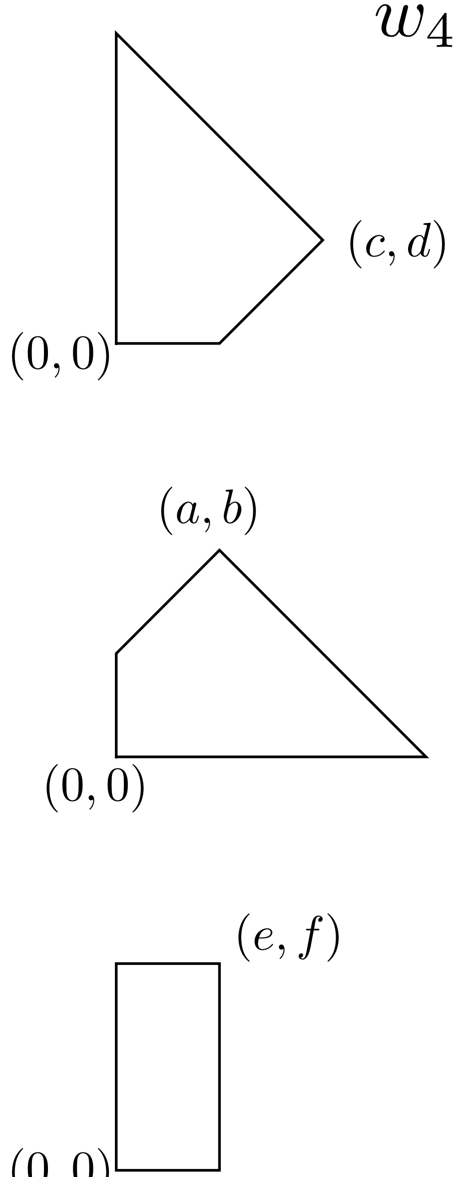


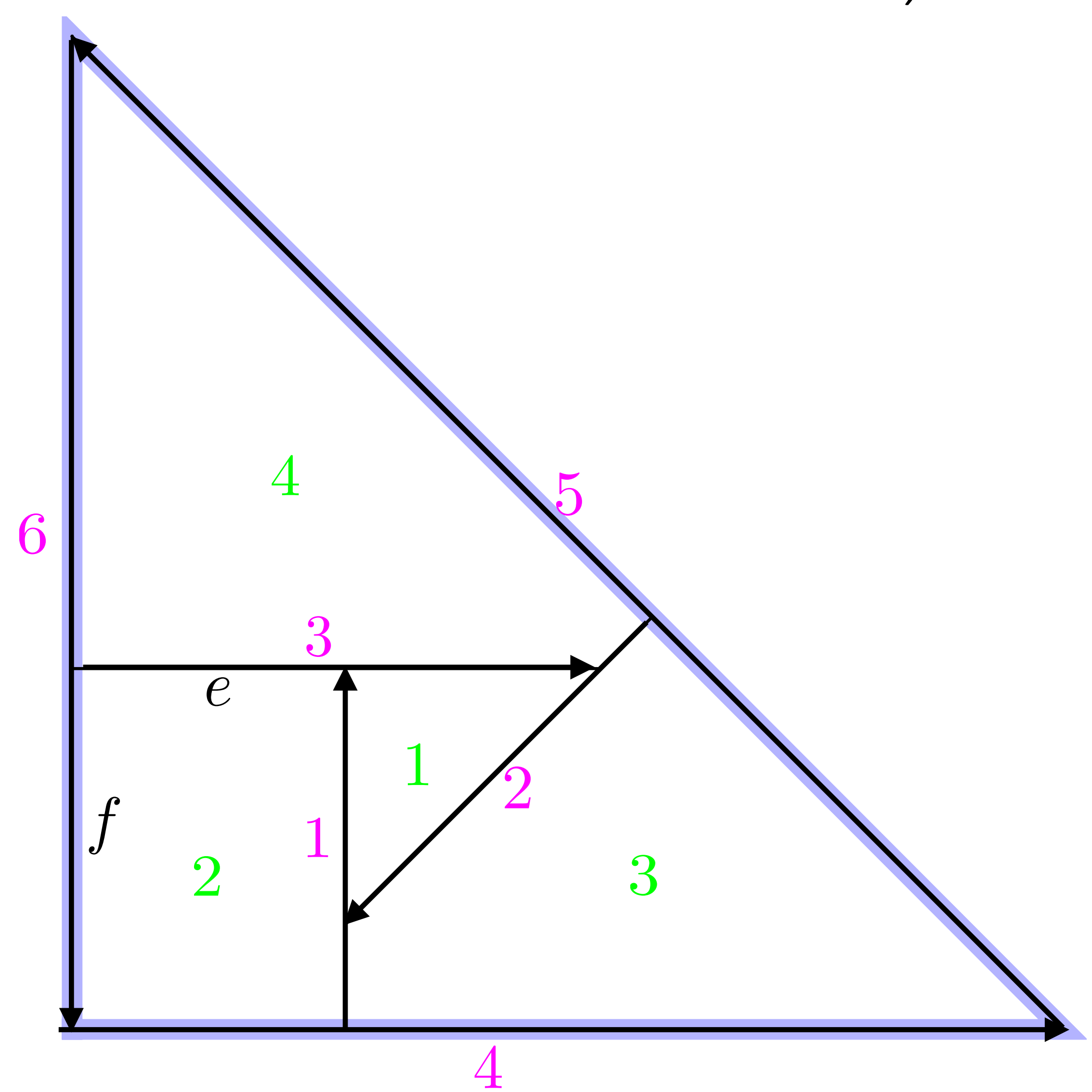
$$X = \frac{ac}{bd}$$

Thm: Given a tiling of a convex polygon P with convex polygons, and, fixing slopes, a new shape (up to scale in $\mathbb{R}$) for every tile. Then there is a combinatorially equivalent* tiling with these shapes, of a (new) convex polygon P'.

Proof. Let K be the signed adjacency matrix.

$$K = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{array} \begin{array}{c} \begin{array}{cccccc} i & -1-i & 1 & 1 & -1+i & -i \end{array} \\ \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ f & 0 & -e & e & 0 & f \\ a-b & a & 0 & a+b & b & 0 \\ 0 & -d & c-d & 0 & c & c+d \end{pmatrix} \end{array}$$





$$
(x_1, x_2, x_3, x_4) \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ f & 0 & -e & e & 0 & f \\ a-b & a & 0 & a+b & b & 0 \\ 0 & \underbrace{-d & c-d}_{K_I} & 0 & c & c+d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}^T
$$

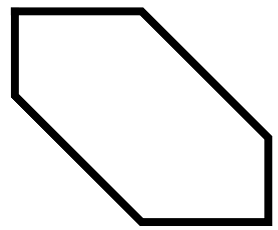
Find a left nullvector W for K_I ("interior" columns of K).

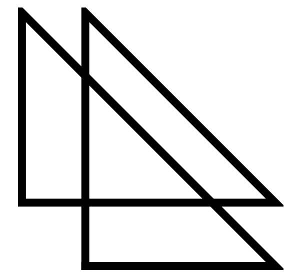
Scale tile i by x_i; these fit together to tile.

Since K_I has full rank, $\exists$ nonzero solution. $\square$

Areas "Tile areas determine the tiling"

Polygons (or closed polygonal curves) with fixed edge slopes

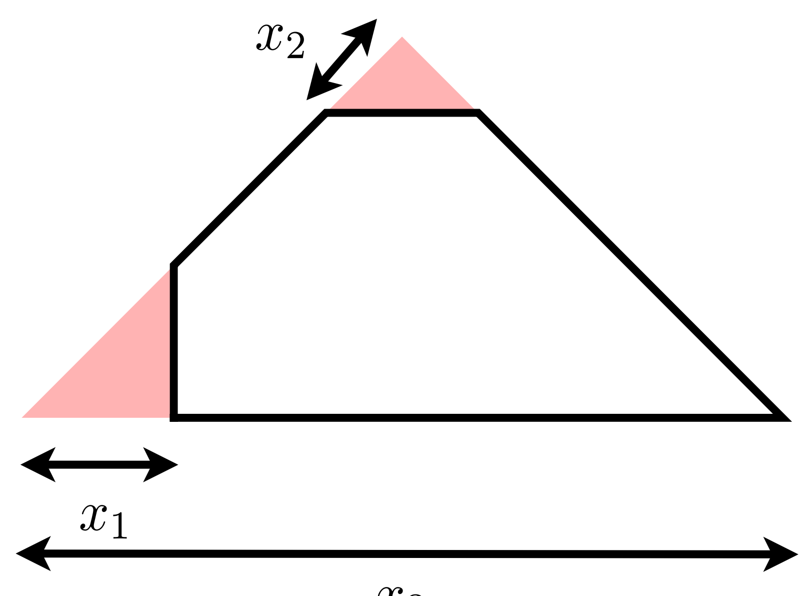




Given an n-gon, the space of closed polygonal curves with the same edge slopes is $\cong \mathbb{R}^{n-2}$.

Thm (Thurston): If P is convex, on this space the signed area is a quadratic form of signature $(1, n-3)$.

Proof by picture:



$$A = C_3 x_3^2 - C_1 x_1^2 - C_2 x_2^2$$

$\square$

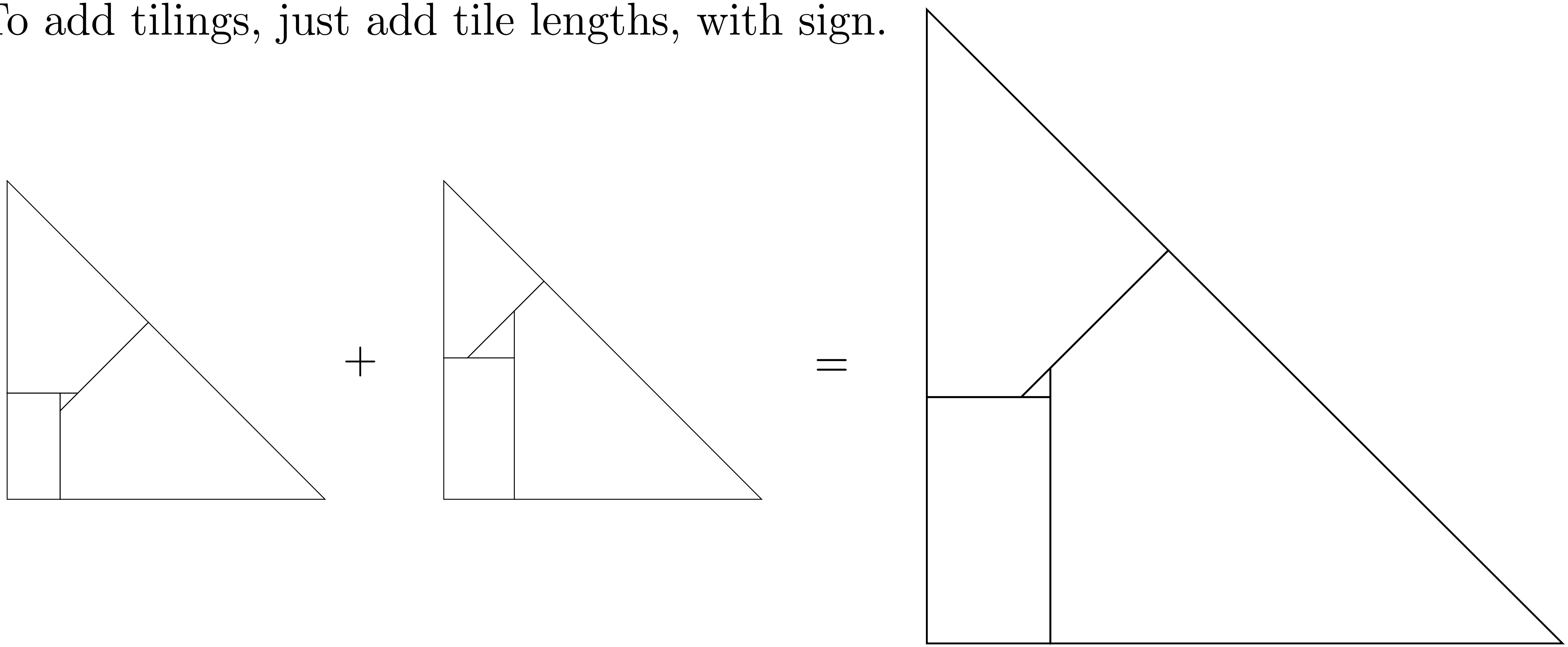
For fixed area, there are two components to the space, called orientations:

triangle quadrilateral pentagon

Each component has
a natural Riemannian metric.

The space of combinatorially equivalent tilings with fixed slopes is a linear cone.

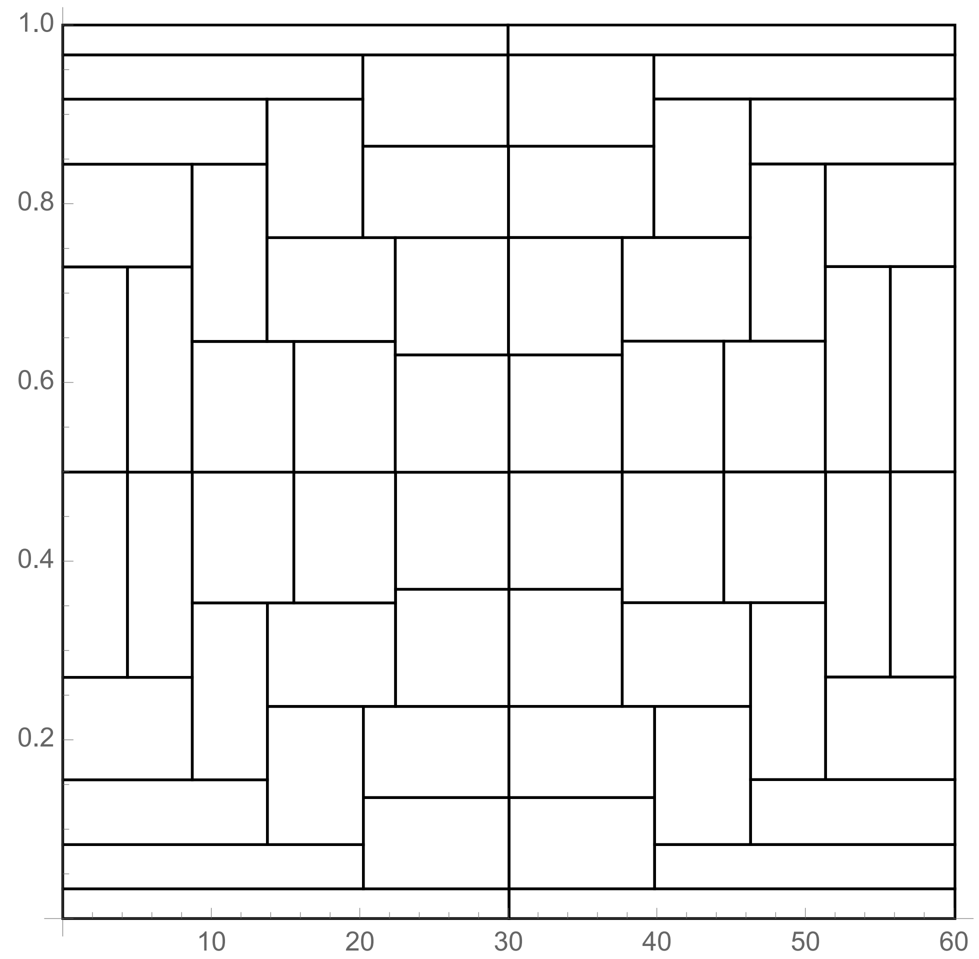
To add tilings, just add tile lengths, with sign.



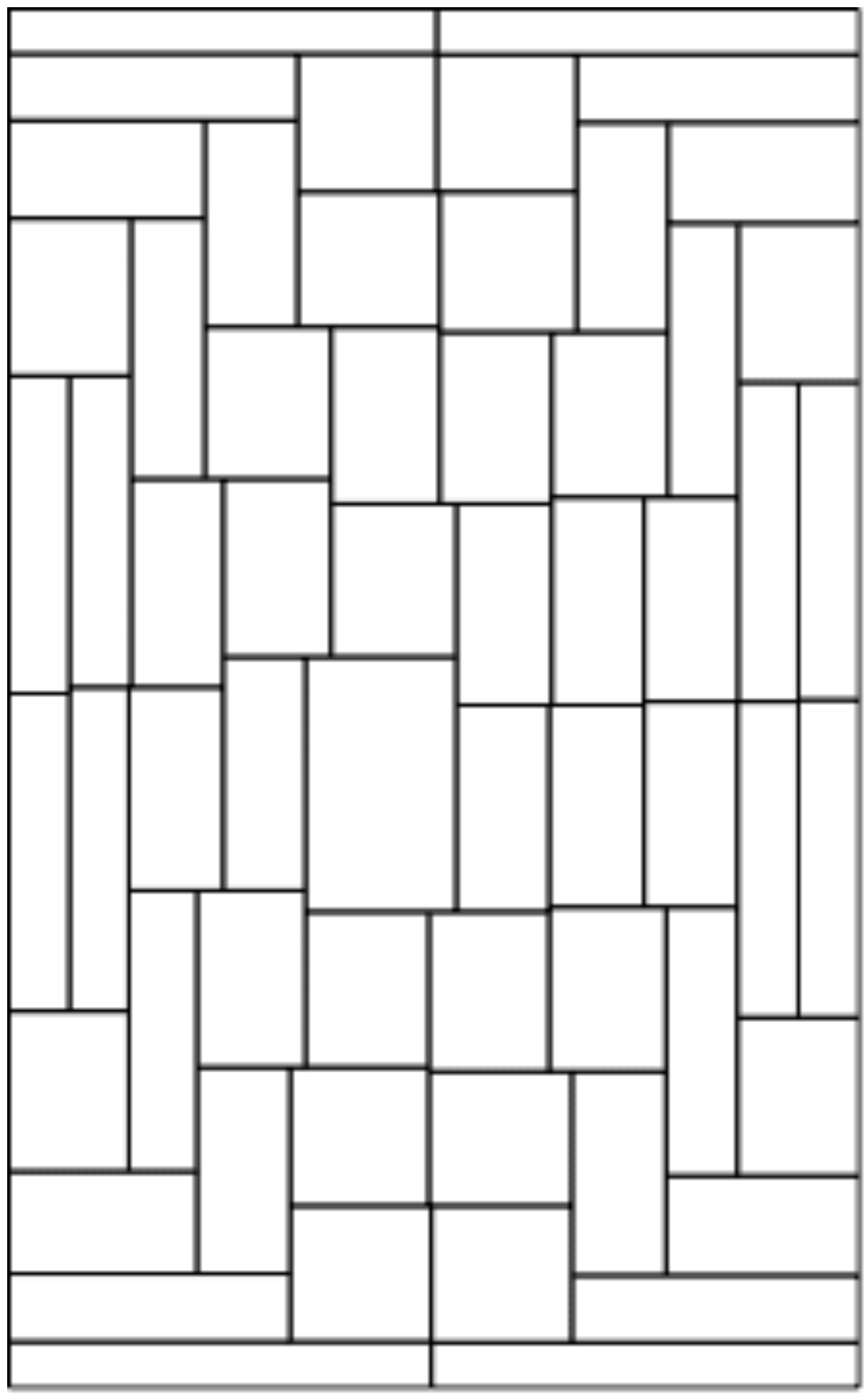
Thm: For fixed slopes and orientations, the tile areas determine the tiling.

Proof: Suppose two tilings have same tile areas and orientations. Then their average tiling will have all tile areas greater than average. $\square$

Thm [Wimer, Koren, Cederbaum 1988]:
Given a rectangle tiling of a rectangle there is an isotopic tiling in which the rectangles have prescribed areas.



Question: What orientations and areas are achievable in general?



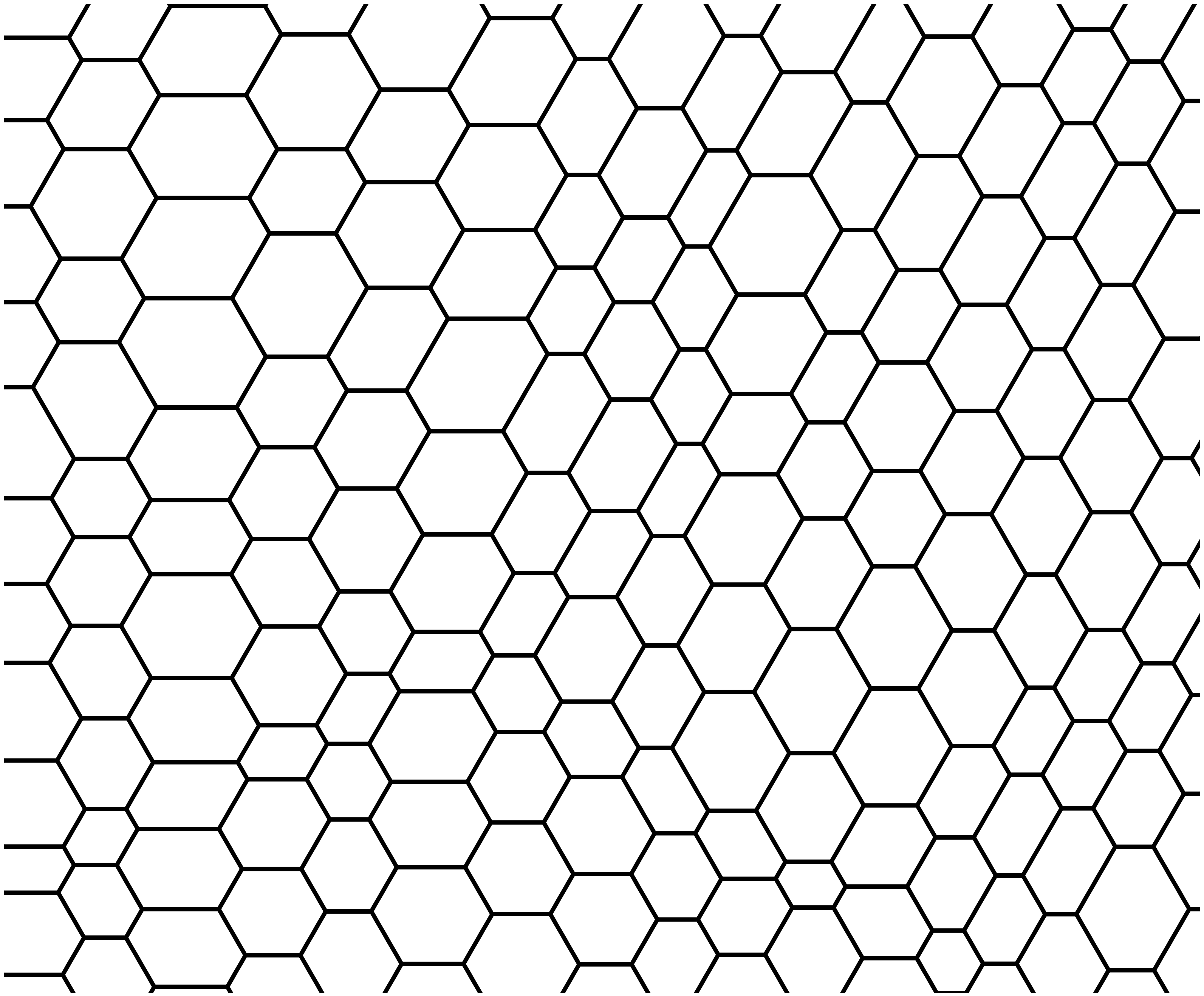
Let $\Psi : \{\text{Intercepts}\} \to \{\text{Areas}\}$. It is injective.

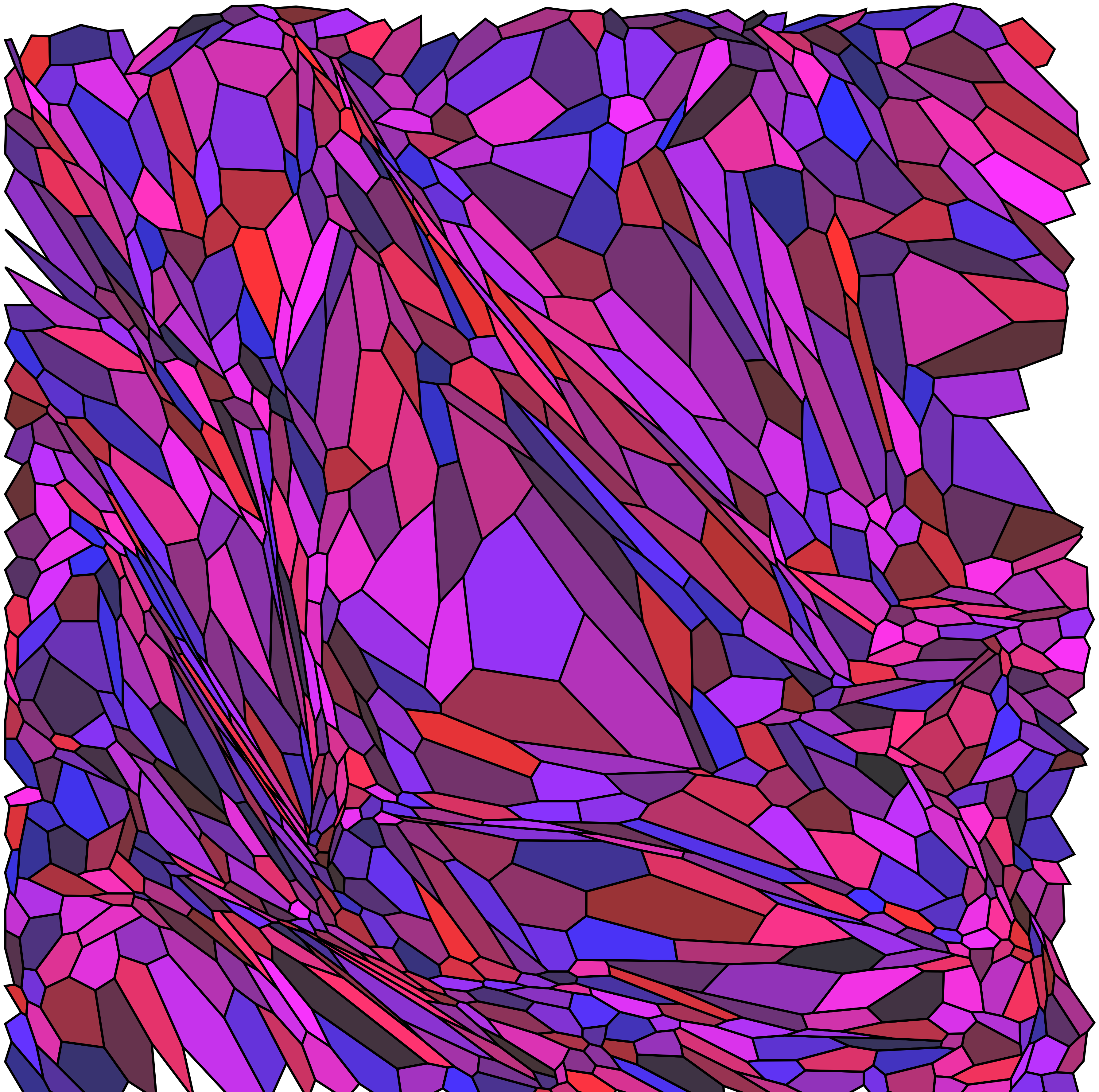
Theorem: $D\Psi = K$.

Therefore $K^{-1} : \{d\text{Areas}\} \to \{d\text{Intercepts}\}$, which gives dimer probabilities, has certain positivity properties...

Conclusion:

The (inverse) Kasteleyn matrix can be interpreted as a geometric object.





thank you for your attention!