LINEAR SPACES OF TILINGS

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Thm (Brooks, Smith, Stone, Tutte 1939): Given a rectangle tiling of a rectangle, there is a tiling of another rectangle with the same combinatorics and prescribed aspect ratios.
At a “degenerate” vertex, choose a resolution:
Proof idea: Associate to a rectangle tiling a harmonic function on a planar network.

\begin{align*}
\text{voltage} &= \text{y-coordinate} \\
\text{current} &= \text{width} \\
\text{conductance} &= \text{aspect ratio} \\
\text{energy} &= \text{area}
\end{align*}
**Thm:** Given a tiling of a convex polygon $P$ with convex polygons, and, fixing slopes, a new shape (up to scale in $\mathbb{R}$) for every tile. Then there is a combinatorially equivalent* tiling with these shapes, of a (new) convex polygon $P'$.

**Lemma:** When $P$ is a $k$-gon,

$$\# \text{ parameters} = \# \text{ internal lines} + k - 3 = \# \text{ tiles} - 1.$$
generic
generic
generic
generic
nongeneric
nongeneric
nongeneric
Associated to a convex polygon tiling is a bipartite network...
...which has dimer covers (when we remove all but one outer edge).
Why? Because it has a fractional dimer cover:

(Dimer covers are the vertices of the polytope of fractional dimer covers.)
There is an associated $|W| \times |B|$ matrix $K$ (a signed, weighted adjacency matrix) with $K_{bw} = \pm \ell$ if black segment $b$ is an edge of face $w$ of length $\ell$.

$K$ is a “Kasteleyn matrix”: product of signs around a face is $(-1)^{k/2+1}$.
What if we don’t fix slopes, just the bipartite graph?
Thm: The space of tilings with $n$ segments, fixed boundary and fixed combinatorics is homeomorphic to $\mathbb{R}^{2n}$. Global coordinates are *biratio coordinates* $\{X_i\}$. 

\[
X = \frac{ac}{bd}
\]
Thm: Given a tiling of a convex polygon $P$ with convex polygons, and, fixing slopes, a new shape (up to scale in $\mathbb{R}$) for every tile. Then there is a combinatorially equivalent* tiling with these shapes, of a (new) convex polygon $P'$.

Proof. Let $K$ be the signed adjacency matrix.

$$K = \begin{pmatrix}
    i & -1 - i & 1 & 1 & -1 + i & -i \\
    -1 & -1 & -1 & 0 & 0 & 0 \\
    f & 0 & -e & e & 0 & f \\
    a - b & a & 0 & a + b & b & 0 \\
    0 & -d & c - d & 0 & c & c + d
\end{pmatrix}$$
$\begin{align*}
(x_1, x_2, x_3, x_4) \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
f & 0 & -e & e & 0 & f \\
a - b & a & 0 & a + b & b & 0 \\
0 & -d & c - d & 0 & c & c + d
\end{pmatrix}
&= \begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
1
\end{pmatrix}
\end{align*}$

Find a left nullvector $W$ for $K_I$ ("interior" columns of $K$).
Scale tile $i$ by $x_i$; these fit together to tile.
Since $K_I$ has full rank, $\exists$ nonzero solution.
Areas  “Tile areas determine the tiling”

Polygons (or closed polygonal curves) with fixed edge slopes

Proof by picture:

Given an $n$-gon, the space of closed polygonal curves with the same edge slopes is $\cong \mathbb{R}^{n-2}$.

**Thm (Thurston):** If $P$ is convex, on this space the signed area is a quadratic form of signature $(1, n - 3)$.

$$A = C_3x_3^2 - C_1x_1^2 - C_2x_2^2$$
For fixed area, there are two components to the space, called orientations:

- triangle
- quadrilateral
- pentagon

Each component has a natural Riemannian metric.
The space of combinatorially equivalent tilings with fixed slopes is a linear cone. To add tilings, just add tile lengths, with sign.

**Thm:** For fixed slopes and orientations, the tile areas determine the tiling.

**Proof:** Suppose two tilings have same tile areas and orientations. Then their average tiling will have all tile areas greater than average. \(\square\)
Thm [Wimer, Koren, Cederbaum 1988]:
Given a rectangle tiling of a rectangle there is an isotopic tiling in which the rectangles have prescribed areas.

Question: What orientations and areas are achievable in general?
Let $\Psi : \{\text{Intercepts}\} \rightarrow \{\text{Areas}\}$. It is injective.

**Theorem:** $D\Psi = K$.

Therefore $K^{-1} : \{d\text{Areas}\} \rightarrow \{d\text{Intercepts}\}$, which gives dimer probabilities, has certain positivity properties...

**Conclusion:**

The (inverse) Kasteleyn matrix can be interpreted as a geometric object.
thank you for your attention!