The GUE corners process for volume measures on lozenge tilings

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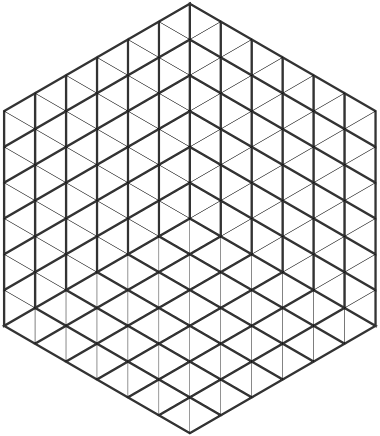
(joint with L. Petrov)

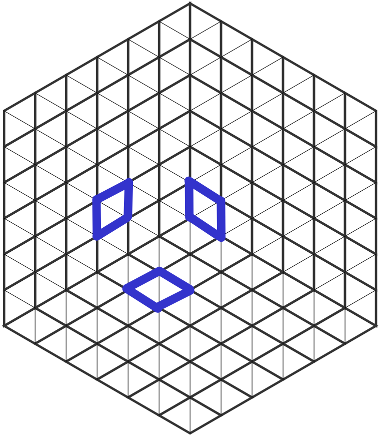
Lozenge tilings

Consider the triangular lattice.



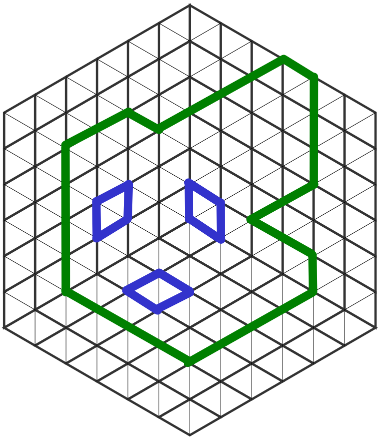
Lozenge tilings

Connecting two neighboring faces get rhombi of 3 different orientations. These are called lozenges.



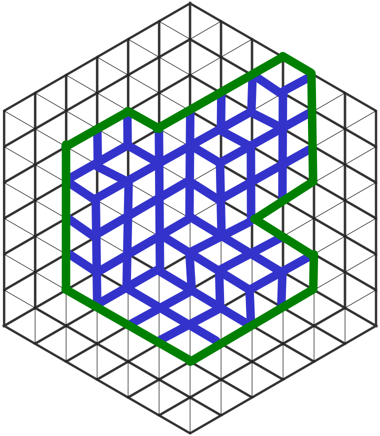
Lozenge tilings

Take a portion of the lattice which is tilable by lozenges.



Lozenge tilings

There are many different tilings.
The lozenge tiling model $\leftrightarrow$ study random tilings of a region.



Lozenge tilings as dimer configurations

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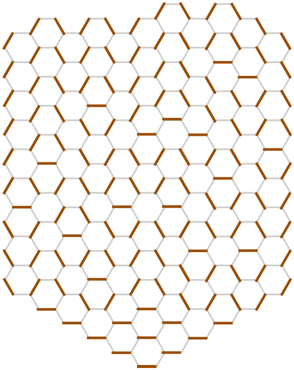
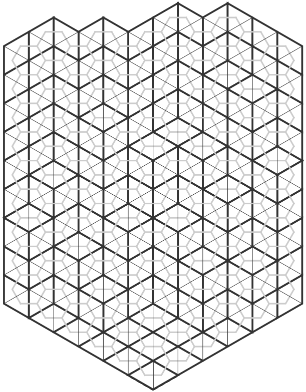
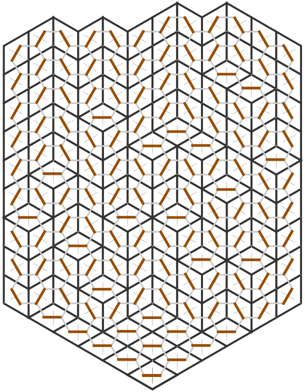


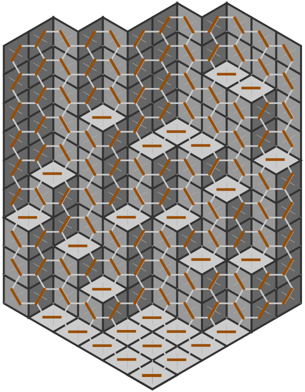
Lozenge tilings as dimer configurations



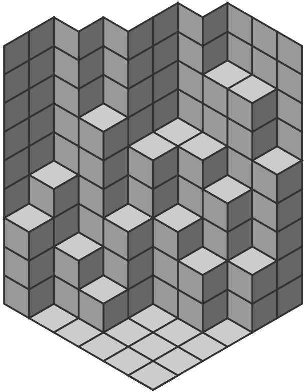
Lozenge tilings as dimer configurations



Lozenge tilings as dimer configurations

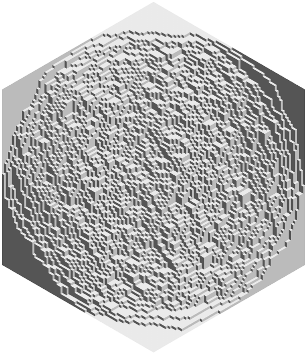


Lozenge tilings as dimer configurations



Limit shapes

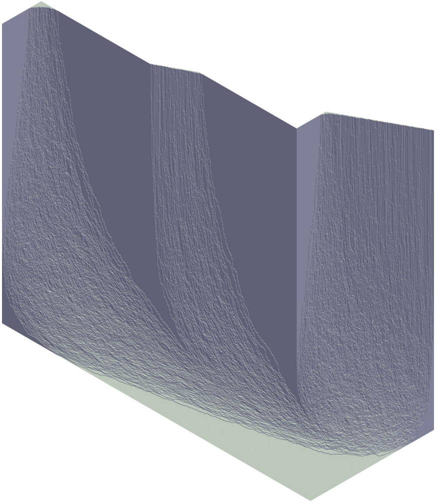
Take a region, (in this picture a hexagon of size $n \times n \times n$), and consider uniformly random tilings of it by lozenges in the limit when $n \to \infty$.



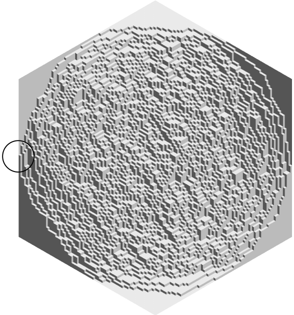
- Limit shape results via a variational principle - Cohn Larsen Propp 1998, Cohn Kenyon Propp 2001
- Frozen boundaries for polygonal regions and limiting processes - Kenyon Okounkov 2006, Kenyon Okounkov Sheffield 2006

Limit shapes - semi-infinite regions

Semi-infinite regions - Okounkov Reshetikhin 2003, 2007, Boutillier, M., Reshetikhin, Tingley 2012, M. 2011.



The birth of a random matrix



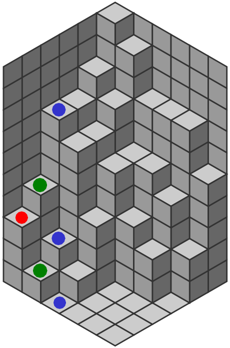
Conjecture (Okounkov-Reshetikhin 2006)

The point process at turning points converges to the GUE corners process.

Turning points

Consider the process near turning points:

- On a slice at distance k from the edge we have k horizontal lozenges. Let the heights be
$$x_1^k \leq x_2^k \leq \cdots \leq x_k^k.$$
- Slices interlace:



$$x_1^2 \leq x_1^1 \leq x_2^2$$
$$x_1^3 \leq x_1^2 \leq x_2^3 \leq x_2^2 \leq x_3^3$$

The GUE corners process

The Gaussian Unitary Ensemble is an ensemble of $N \times N$ hermitian random matrices H with independent Gaussian entries:

$$H_{i,i} \sim N(0,1)$$
$$\Re H_{i,j}, \Im H_{i,j} \sim N(0,1/2)$$
$$H_{j,i} = \overline{H_{i,j}}.$$

A different way to describe the law is to say H is an $N \times N$ hermitian random matrix with density

$$\frac{1}{Z_N} e^{-tr(H^2)/2},$$

where Z_N is a normalization constant.
Let $\lambda_1^k \le \lambda_2^k \le \cdots \le \lambda_k^k$ be the eigenvalues of the $k \times k$ corner of H.
Two neighboring rows interlace:

$$\begin{array}{ccccccccccc}
\lambda_1^k & & & & \lambda_2^k & & & \cdots & \lambda_{k-1}^k & & & & \lambda_k^k. \\
& \leq & & \geq & & \leq & & & & \leq & & \geq & \\
& & \lambda_1^{k-1} & & & & \lambda_2^{k-1} & \cdots & & & \lambda_{k-1}^{k-1}. & &
\end{array}$$

The GUE corners process

- The joint distribution of

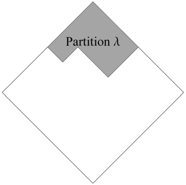
$$\begin{array}{c}
\lambda_1^k, \lambda_2^k, \quad \dots, \quad \lambda_{k-1}^k, \lambda_k^k \\
\lambda_1^{k-1}, \lambda_2^{k-1}, \dots, \lambda_{k-1}^{k-1} \\
\dots \\
\lambda_1^3, \lambda_2^3, \lambda_3^3 \\
\lambda_1^2, \lambda_2^2 \\
\lambda_1^1
\end{array}$$

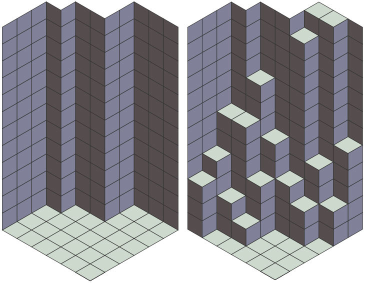
  is called the GUE corners process. Denote by $\mathbb{GUE}_k$.
- Conditioned on the top row and the interlacing condition, the lower entries are unifromly distributed.

The Okounkov-Reshetikhin conjecture

Conjecture (Okounkov-Reshetikhin 2006)

After appropriate centering and scaling the joint law of the heights x_i^k *of the horizontal lozenges in the first* k *slices converges to the joint law of the eigenvalues* λ_i^k *of the first* k *corners of a GUE random matrix.*





Consider the stacks of boxes with boundary λ, confined to the $aN \times bN$ box, with the distribution

$$Prob(\pi) \propto q^{|\pi|} = q^{volume},$$

for some $q \in (0,1)$, where $|\pi| = \sum \pi_{i,j}$ is the total volume.
Let $x_i^k, i \leq k$ be the heights of the first k slices as defined above.

Theorem (Okounkov-Reshetikhin 2006)

Let $q = e^{-1/N}$. There exist constants C_0 and C_1 such that in the limit $q \to 1$ we have $\frac{x_i^k - NC_0}{\sqrt{NC_1}} \to \mathbb{GUE}_k$ in distribution.

Uniform measure

- Johansson-Nordenstam proved convergence to the GUE corners process for the uniform measure in the case of a hexagon.
- Gorin-Panova: Let $x(1), x(2), \ldots$ be a sequence of signatures, where $x(N)$ has N parts. Consider the uniform distribution on lozenge tilings whose N'th slice is $x(N)$. Let $x^k(N)$ be the k'th slice.

Theorem (Gorin-Panova 2013)

Suppose there exists a nonconstant piecewise-differentiable weakly decreasing function $f(t)$ such that

$$\sum_{i=1}^{N} \left| \frac{x_i(N)}{N} - f(i/N) \right| = o(\sqrt{N})$$

as $N \to \infty$. Then for every k, as $N \to \infty$, we have

$$\frac{x^k(N) - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k,$$

in the sense of weak convergence, for some explicit constants $E(f)$ and $S(f)$.

Volume measure

Let $x(1), x(2), \ldots$ be a sequence of signatures, where $x(N)$ has N parts. Consider the uniform distribution on lozenge tilings whose N'th slice is $x(N)$. Let $x^k(N)$ be the k'th slice.

Theorem (Main Theorem)

Suppose there exists a nonconstant weakly decreasing function $f(t)$ such that $x_i(N)/N$ converges pointwise and uniformly to f. E.g.

$$\left|\frac{x_i(N)}{N} - f(i/N)\right| = o(1/\sqrt{N})$$

as $N \to \infty$ will suffice. Then for every k, as $N \to \infty$ and $q \to 1$ as $q = e^{-\gamma/N}$ for some constant $\gamma \geq 0$, we have

$$\frac{x^k(N) - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k,$$

in the sense of weak convergence, for some explicit constants $E(f)$ and $S(f)$.

Remarks

- f only needs to be weakly decreasing.
- $\gamma = 0$ recovers the Gorin-Panova results.
- If we let f to be piecewise constant, we get the result for certain polygonal regions, incuding the hexagon.
- Here q must converge to 1 at the rate $q = e^{-\gamma/N}$. Work in progress with Greta Panova to study what happens when $q = e^{-\gamma/g(N)}$ with $g(N) \to \infty$, $g(N) << N$.

Proof via saddle point analysis

Theorem (Petrov 2012)

For any $1 \le K < N$, $\kappa \in \operatorname{sign} K$, *and* $\nu \in \operatorname{sign} N$, *we have the following formula for the distribution of the signature* κ *in the* K*th row arising as a projection of the measure* q^{vol} *on signatures with top row* ν*:*

$$P_K(\kappa) = s_\kappa(1, q, \ldots, q^{K-1}) \cdot (-1)^{K(N-K)} q^{(N-K)|\kappa|} q^{-K(N-K)(N+2)/2} \times \det[A_i(\kappa_j - j)]_{i,j=1}^{K},$$

where s_κ *is the Schur polynomial, and*

$$A_i(x) = A_i(x \mid K, N, \nu) = \frac{1 - q^{N-K}}{2\pi \mathbf{i}} \oint_{\mathfrak{C}(x)} dz \, \frac{(zq^{1-x}; q)_{N-K-1}}{\prod_{r=i}^{N-K+i}(z - q^{-r})} \prod_{r=1}^{N} \frac{z - q^{-r}}{z - q^{\nu_r - r}}.$$

Here the positively (counter-clockwise) oriented simple contour $\mathfrak{C}(x)$ *encircles points* $q^x, q^{x+1}, \ldots, q^{\nu_1 - 1}$, *and not the possible poles* $q^{x-1}, q^{x-2}, \ldots, q^{\nu_N - N}$.

Do a saddle point analysis of the correlation kernel. First do the piecewise constant case, in which case the asymptotically leading term is a rational function.

Breaking away from the GUE corners process

Given $\{q_i\}_{i\in\mathbb{Z}}$, $q_i > 0$, consider with the distribution

$$Prob(\pi) \propto \prod_{i\in\mathbb{Z}} q_i^{|\pi^i|},$$

where $|\pi^i|$ is the total volume of the i-th slice of π.

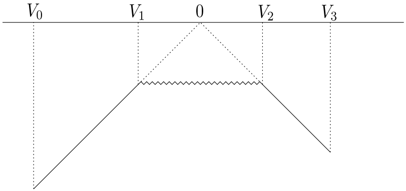
Periodic weights

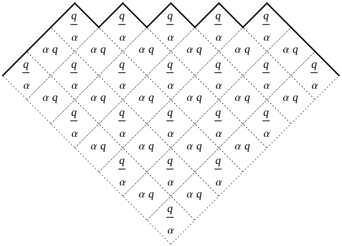
- Consider weights with

$$q_0 = q_{2k} \text{ and } q_1 = q_{2k+1} \ \forall k \in \mathbb{Z}.$$

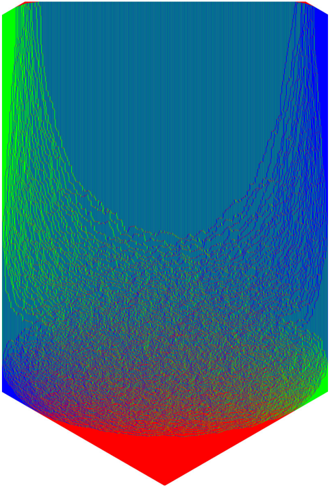
- What scaling limit should we study?
- Nothing new, if you take $q_0 \to 1^-$ and $q_1 \to 1^-$.
- More interesting: $\alpha \geq 1$, $q_0 = \alpha q$, $q_1 = \alpha^{-1} q$ and $q \to 1^-$.

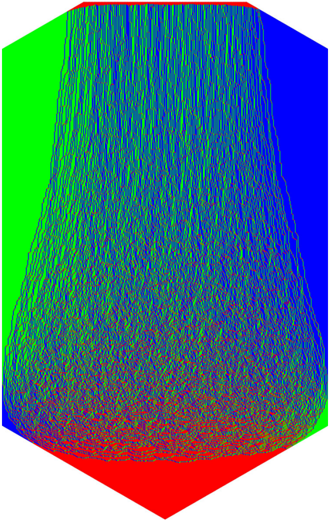
Periodic weights





Bounded floor: A sample

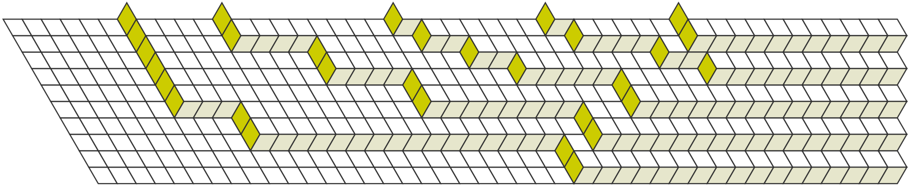


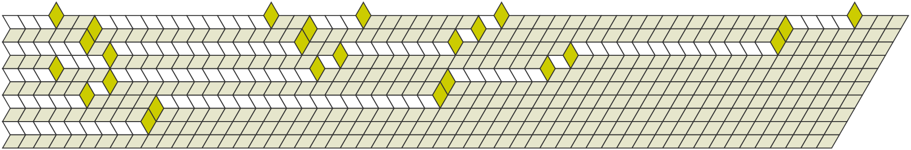


Turning points

- There are two turning points near each vertical boundary section.
- The fact that there are two turning points implies that locally you do not have the interlacing property from slice to slice.
- The distance between the turning points converges to zero when α converges to 1.

Turning points and the GUE Corners process





Turning point correlations

Theorem (M.)

Let χ be the expected hight of a turning point and let $h_i = \lfloor \frac{\chi}{r} \rfloor + \frac{\tilde{h}_i}{r^{\frac{1}{2}}}$, where $q = e^{-r}$. The correlation functions near a turning point of the system with periodic weights are given by

$$\lim_{r \to 0} r^{-\frac{1}{2}} K_{\lambda,\bar{q}}((t_1, h_1), (t_2, h_2)) = \frac{1}{(2\pi \mathrm{i})^2} \iint e^{\frac{\sigma^2}{2}(\zeta^2 - \omega^2)} \frac{e^{\tilde{h}_2 \omega}}{e^{\tilde{h}_1 \zeta}} \frac{\omega^{\lfloor \frac{t_2+e}{2} \rfloor}}{\zeta^{\lfloor \frac{t_1+e}{2} \rfloor}} \frac{d\zeta \ d\omega}{\zeta - \omega},$$

where e is 1 or 2 depending on whether $\chi = \chi_{bottom}$ or $\chi = \chi_{top}$.

Remark: Not surprising that we don't get the GUE-corners process. Conditioned on the k'th slice, the previous slices are not uniform.
Remark: If we restrict the process to horizontal lozenges of only even or only odd distances from the edge, then the correlation kernel coincides with the correlation kernel of the GUE-corners process, so we have two GUE-corners processes non-trivially correlated.
Remark: Interlacing is not a geometric constraint anymore.

Intermediate regime

- Question: What happens when $\alpha \to 1$?
- Consider two-periodic weights q_t given by

$$q_t = \begin{cases} e^{-r+\gamma r^{1/2}}, & t \text{ is even} \\ e^{-r-\gamma r^{1/2}}, & t \text{ is odd} \end{cases}, \tag{1}$$

  where $\gamma > 0$ is an arbitrary constant. This is an intermediate regime between the homogeneous weights and the inhomogeneous weights considered earlier.
- The macroscopic limit shape and correlations in the bulk are the same as in the homogeneous case.
- Periodicity disappears in the limit and we have a $\mathbb{Z} \times \mathbb{Z}$ translation invariant ergodic Gibbs measure in the bulk. However, the local point process at turning points is different from the homogeneous one. In particular, while we only have one turning point near each edge, we still do not have the GUE corners process, but rather a one-parameter deformation of it.

Turning point correlations in the intermediate regime

Theorem (M.)

Let χ be the expected hight of a turning point and let $h_i = \lfloor \frac{\chi}{r} \rfloor + \frac{\tilde{h}_i}{r^{\frac{1}{2}}}$, where $q = e^{-r}$. The correlation functions near a turning point of the system with periodic weights(1) *are given by*

$$\lim_{r \to 0} r^{-\frac{1}{2}} K_{\lambda,\bar{q}}((t_1, h_1),(t_2, h_2))$$

$$= \frac{1}{(2\pi \mathrm{i})^2} \iint e^{C_{cr}(\zeta^2-\omega^2)} \frac{e^{\tilde{h}_2\omega}}{e^{\tilde{h}_1\zeta}} \frac{\omega^{\lfloor \frac{t_2+1}{2} \rfloor}}{\zeta^{\lfloor \frac{t_1+1}{2} \rfloor}} \frac{(\omega-\gamma)^{\lfloor \frac{t_2+2}{2} \rfloor}}{(\zeta-\gamma)^{\lfloor \frac{t_1+2}{2} \rfloor}} \frac{d\zeta \; d\omega}{\zeta - \omega}.$$

Thank you for your attention.