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Chamber behaviour of mixed Hurwitz numbers

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Abstract

Hurwitz numbers are enumerative objects counting functions between Riemann surfaces with fixed ramification data. Classical and monotone double Hurwitz numbers are two classes of Hurwitz-type counts of specific interest. Mixed double Hurwitz numbers are a combinatorial interpolation between those invariants. They are known to be piecewise polynomial. We give an algorithm involving tropical covers computing these polynomials in each chamber. Moreover, we use this approach to give wall-crossing formulas in genus zero.

Classical Hurwitz numbers

- Count degree d covers

$$f : S \to \mathbb{P}^1(\mathbb{C})$$

of the projective line by a Riemann surface S with fixed genus g and fixed ramification data over fixed branch points in $\mathbb{P}^1(\mathbb{C})$.

- Equivalent description by monodromy representations
  - Single Hurwitz numbers:

$$H_g(\mu) = \frac{1}{d!}\#\{(\sigma, \tau_1, \dots \tau_r)\}, \text{ where}$$

1. each τ_i is a transposition in $\mathbb{S}_d$,
2. σ is of cycle type μ,
3. $\sigma \circ \tau_1 \circ \cdots \circ \tau_r = \mathrm{id}$.

  - Double Hurwitz numbers:

$$H_g(\mu_1, \mu_2) = \frac{1}{d!}\#\{(\sigma_1, \tau_1, \dots \tau_r, \sigma_2)\}, \text{ where}$$

1. each τ_i is a transposition in $\mathbb{S}_d$,
2. σ_i is of cycle type μ_i,
3. $\sigma_1 \circ \tau_1 \circ \cdots \circ \tau_r \circ \sigma_2 = \mathrm{id}$.

Properties of Hurwitz numbers

Single Hurwitz numbers

- Cut-and-Join relations
- Eynard-Orantin topological recursion
- ELSV formula (connection to intersection numbers)
- Polynomiality

Properties of Hurwitz numbers

Double Hurwitz numbers

- Cut-and-Join relations
- Piecewise polynomiality
- Wall-crossing formulas

Monotone/Mixed Hurwitz numbers

- Monotone Hurwitz numbers: Transposition satisfy monotonicity condition
  - For $\tau_i = (r_i\ s_i)$ $(r_i < s_i)$, we require $s_i \leq s_{i+1}$
- Mixed Hurwitz numbers: Combinatorial interpolation between monotone and classical Hurwitz numbers
  - Monotonicity condition required for first k transposition: $s_i \leq s_{i+1}$ for $i = 1, \dots, k-1$

Previous results

Monotone Hurwitz numbers

- Motivated by random matrices – coefficients in the expansion of the Harish-Chandra-Itzykson-Zuber integral
- Eynard-Orantin topological recursion
- ELSV-type formula for single monotone Hurwitz numbers

Mixed Hurwitz numbers

- Piecewise polynomiality for mixed double Hurwitz numbers

Open Questions

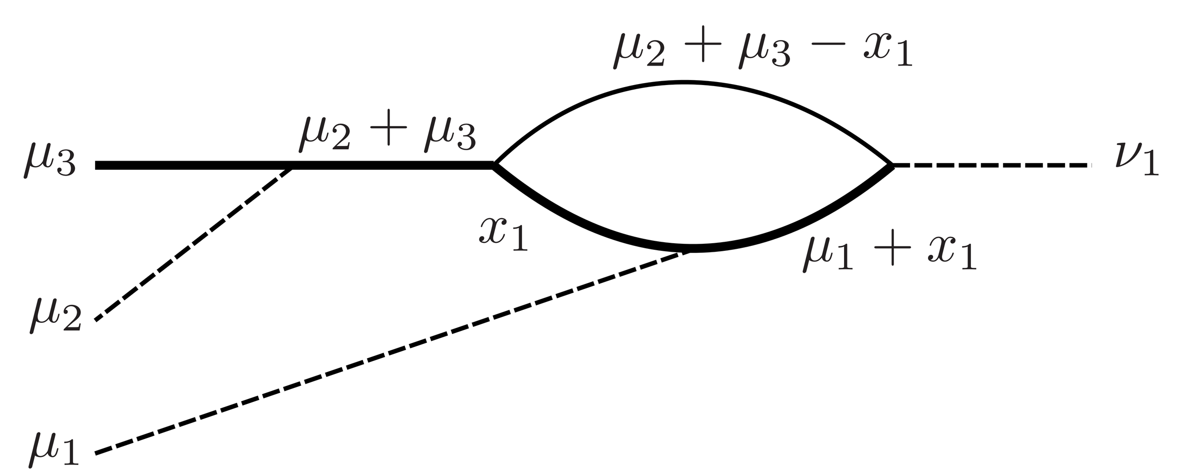
- Compute the polynomials for mixed double Hurwitz numbers in each chamber
- Give recursive wall-crossing formulas for mixed double Hurwitz numbers

Our Toolbox

- Correspondence theorem between classical and tropical double Hurwitz numbers [2] (wall-crossing formulas in [3])
- Correspondence theorem between classical and tropical monotone orbifold Hurwitz numbers [4]

Our strategy

- We generalised the tropical correspondence theorems for mixed double Hurwitz numbers



- We used the tropical approach and developed an algorithm to compute the polynomial in each chamber (using volumes of polytopes)
- We introduced a new counting problem generalising mixed Hurwitz numbers
- Our algorithm can also compute the polynomials for this new counting problem
- We proved recursive wall-crossing formulas for this new counting problem in genus 0

References

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