Quantisation in Chern-Simons theory via geometric constructions

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Geometric Quantisation

Geometric quantisation produces a quantum Hilbert space $\mathcal{H}$ given the following data:

- A symplectic manifold (M, ω);
- A pre-quantum line bundle for (M, ω), i.e. a triple $(\mathscr{L}, h, \nabla)$ with $\mathscr{L}$ a complex line bundle over M, h a Hermitian metric on it and ∇ a connection on it so that $\nabla h = 0$ and:
$$F_\nabla = -i\omega \qquad (1)$$
- A polarisation on M, e.g. a Kähler structure *or* a Lagrangian foliation.

$\mathcal{H}$ is then the space of smooth polarised sections of $\mathscr{L}$, i.e. holomorphic or covariantly constant along the leaves as the case may be.

Moduli spaces of flat connections

Let Σ be a closed, oriented smooth surface of genus g, and for $G = \mathrm{SU}(n)$ or $\mathrm{SL}(n, \mathbb{C})$ call $\mathcal{M}_G$ the moduli spaces of reductive flat G-connections. The space $\mathrm{T}_{[A]}\mathcal{M}_G$ is isomorphic to the first A-twisted cohomology of Σ:

$$\mathrm{T}_{[A]}(\mathcal{M}_G) \simeq H^1(\Sigma, d_A) \qquad (2)$$

$\mathcal{M}_G$ has a symplectic structure given by the Atiyah-Bott form:

$$\omega_G([\eta_1], [\eta_2]) = 4\pi \int_\Sigma \langle \eta_1 \wedge \eta_2 \rangle \qquad (3)$$

For brevity we drop $\mathrm{SU}(n)$ and replace $\mathrm{SL}(n, \mathbb{C})$ with $\mathbb{C}$ when they appear as subscripts. Notice that $\mathcal{M}$ sits inside $\mathcal{M}_\mathbb{C}$ as a symplectic subspace.

Chern-Simons theory

Classical theory The Chern-Simons (CS) theory is a 3-dimensional gauge field theory whose classical solutions are the flat G-connections up to gauge transformations. One can think of $\mathcal{M}_G$ as the space of boundary conditions for the theory on 3-manifolds bounded by Σ.

The CS line bundle Let $t = k + is$ be a complex parameter, k a positive integer, and fix a flat connection A on Σ. The (exponentiated) action of the $\mathrm{SL}(n, \mathbb{C})$-theory at the level t with boundary condition A is well defined, but it takes values in an abstract Hermitian line $\mathscr{L}^{(t)}_{[A]}$ which depends functorially on $[A]$. As $[A]$ varies in $\mathcal{M}_\mathbb{C}$, these lines form a line bundle $\mathscr{L}^{(t)}$ with Hermitian structure h. Furthermore, a geometric construction based on the action defines a parallel transport operator associated to any curve in $\mathcal{M}_\mathbb{C}$, thus giving a connection ∇ on $\mathscr{L}^{(t)}$ which preserves h.

Fact 1: $(\mathscr{L}^{(t)}, h, \nabla)$ is a pre-quantum line bundle for the symplectic structure
$$\omega_t := \frac{1}{2}(t\omega_\mathbb{C} + \bar{t}\bar{\omega}_\mathbb{C}) \qquad (4)$$

Definition 1: $(\mathscr{L}^{(t)}, h, \nabla)$ is called the Chern-Simons line bundle at the level t.

The restriction of $\mathscr{L}^{(t)}$ to $\mathcal{M}$ has the topological type of $\mathscr{L}^k$, the k-th tensor power of $\mathscr{L}^{(1)}|_\mathcal{M}$.

Polarisations and metrics

Polarisations A Riemann surface structure σ on Σ induces a splitting

$$H^1(\Sigma, d_A) = H^{1,0}(\Sigma, d_A) \oplus H^{0,1}(\Sigma, d_A) \qquad (5)$$

The collection of the $H^{1,0}(\Sigma, d_A)$'s gives an integrable distribution on $\mathcal{M}_\mathbb{C}$, hence a real polarisation P_σ. Each leaf of P_σ intersects $\mathcal{M}$ at exactly one point, transversally, which implies the following:

Fact 2: The Hilbert space induced by σ is
$$\mathcal{H}_\sigma \simeq \mathcal{C}^\infty(\mathcal{M}, \mathscr{L}^k) \qquad (6)$$

Besides the polarisation, σ also induces a complex structure on $\mathcal{M}_\mathbb{C}$, hence a Kähler metric $g = g_\sigma$ with a Ricci potential F.

The Hitchin-Witten connection

Although $\mathcal{H}_\sigma$ can be seen as a σ-independent vector space, Witten argues in [2] that the identification is not natural from the viewpoint of gauge field theories. However, the various spaces can be arranged to form a bundle over the Teichmüller space:

$$\mathcal{H} = \mathcal{T} \times \mathcal{C}^\infty(\mathcal{M}, \mathscr{L}^k) \to \mathcal{T} \qquad (7)$$

On this bundle Witten defines a connection:

$$\tilde{\boldsymbol{\nabla}} = \boldsymbol{\nabla}^T + \frac{1}{2t}b - \frac{1}{2\bar{t}}\bar{b} + dF \qquad (8)$$

Here $\boldsymbol{\nabla}^T$ is the trivial connection on $\mathcal{H}$ and dF denotes the differential of $F \in \mathcal{C}^\infty(\mathcal{M} \times \mathcal{T})$ along $\mathcal{T}$. b is instead a 1-form on $\mathcal{T}$ with values in differential operators on $\mathscr{L}^k$, constructed from the metric g and its variation along $\mathcal{T}$.

Theorem 3: $\tilde{\boldsymbol{\nabla}}$ is projectively flat, i.e. its curvature is a complex-valued 2-form on $\mathcal{T}$ times the identity of $\mathcal{C}^\infty(\mathcal{M}, \mathscr{L}^k)$.

Due to contractibility of $\mathcal{T}$, the holonomy of $\tilde{\boldsymbol{\nabla}}$ identifies the quantum Hilbert spaces up to projective factors.

Embedding of $\mathrm{T}^*\mathcal{M}$ into $\mathcal{M}_\mathbb{C}$

Let σ be fixed, $g \geq 2$ and $[A] \in \mathcal{M}$. By the Narasimhan-Seshadri correspondence, $[A]$ defines a semi-stable holomorphic bundle E over Σ, while Serre duality gives an isomorphism:

$$H^1(\Sigma, E)^* \simeq H^0(\Sigma, K \otimes E) \qquad (9)$$

The space on the left-hand side is identified with the holomorphic co-tangent space of $\mathcal{M}$ at Σ That on the right is the space of Higgs fields on E, and non-Abelian Hodge theory gives an equivalence between $\mathcal{M}_\mathbb{C}$ and the moduli space of semi-stable Higgs bundles on Σ.

Theorem 4: The above identifications combine into an embedding $\mathrm{T}^*\mathcal{M} \to \mathcal{M}_\mathbb{C}$ whose image is an open dense.

Although the correspondences above require $g \geq 2$, the construction can be adjusted to obtain an embedding also in the case $g = 1$.

Functions and differential operators

Definition 2: Call a function $f \in \mathcal{C}^\infty(\mathcal{M}_\mathbb{C})$ of polynomial type if its pull-back to $\mathrm{T}^*\mathcal{M}$ is polynomial in the co-tangent variables.

To every function f of polynomial type corresponds a formal, totally symmetric tensor field T_f on $\mathcal{M}$ whose coordinates at each point are the coefficients of the relevant polynomial. Given $\psi \in \mathcal{H}_\sigma$, subsequent application of a combination of ∇ and the Levi-Civita connection of g gives a section $\nabla^r\psi$ of $(\mathrm{T}^*\mathcal{M})^{\otimes r} \otimes \mathscr{L}^k$ for every positive integer r.

Definition 3: If T is a tensor field on $\mathcal{M}$, call $\nabla^r_T\psi$ its full contraction with $\nabla^r\psi$.

These correspondences associate to every classical observable f of polynomial type a quantum operator. Up to weighting the r-th order part with $(-i\hbar)^r$ for some quantum parameter $\hbar$, this is a good candidate for the quantisation of f. Viceversa, every differential operator D of finite order r defines totally symmetric tensor fields $\sigma_j(D)$ so that:

$$\mathrm{rk}(\sigma_j(D)) = j \quad \text{and} \quad D = \sum_{j=0}^{r} \nabla^j_{\sigma_j(D)} \qquad (10)$$

Definition 4: The tensor field $\sigma_j(D)$ is called the j-th symbol of D.

This might be used for defining a star product.

Genus 1 and the A polynomial

If $g = 1$ and the Serre pairing is normalised properly, the embedding $\mathrm{T}^*\mathcal{M} \to \mathcal{M}_\mathbb{C}$ and the Levi-Civita connection are independent on σ.

Theorem 5: For fixed k, the above construction gives a star product for $\hbar = s/|t|^2$.

Moreover, the Hitchin-Witten connection is *flat* and is trivialised by the gauge transformation:

$$\exp(-r\Delta) \quad \text{where } e^{4kr} = -\frac{k - is}{k + is} \qquad (11)$$

The star-product can be made $\tilde{\boldsymbol{\nabla}}$-invariant by conjugating the operators by this.

An important class of observables is that of the A-polynomials of knots, which in this theory correspond to constraint equations.

Question: What does this construction give for the A-polynomial of a knot?

References

[1] J. E. Andersen, N. L. Gammelgaard *The Hitchin-Witten Connection and Complex Quantum Chern-Simons Theory.* arXiv:1409.103

[2] E. Witten *Quantization of Chern Simons Gauge Theory with Complex Gauge Group.* Comm. Math. Phys., 1991.

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