Counting curls and hunting monomials: Tropical mirror symmetry for elliptic curves

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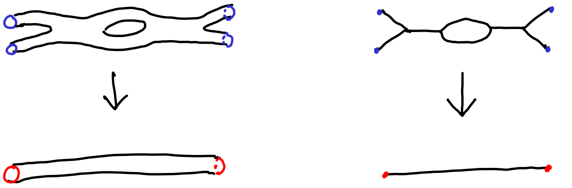
Tropical geometry is...

- ... geometry over the max-plus semifield

  ($\rightarrow$ optimization, complexity theory)
- ... a finite shadow of non-archimedean analytic geometry
- ... an efficient combinatorial tool to study degenerations in algebraic geometry



map of Riemann surfaces   "combinatorial shadow"

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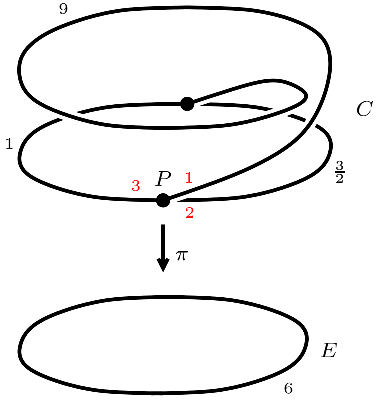
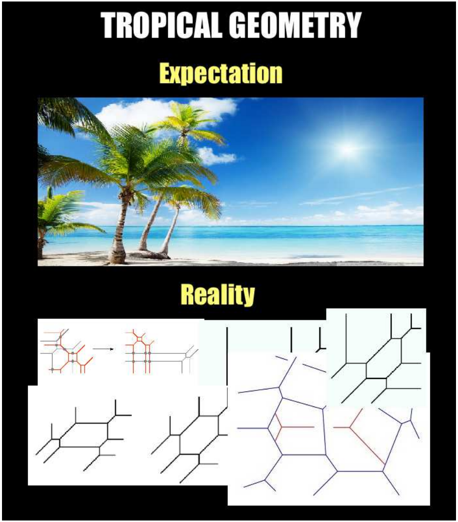
Tropical covers of an elliptic curve

Fix a tropical elliptic curve E, i.e. a circle, e.g. of length 6.



No contrast



Tropical geometry in enumerative geometry

Theorem (Correspondence Theorem (Cavalieri-Johnson-M, Bertrand-Brugallé-Mikhalkin))

$$N_{d,g} = N_{d,g}^{\text{trop}}$$

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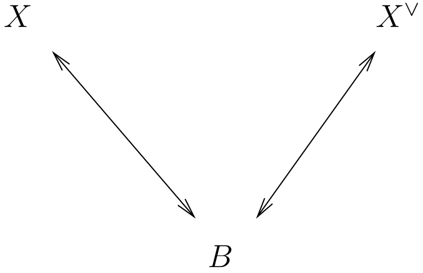
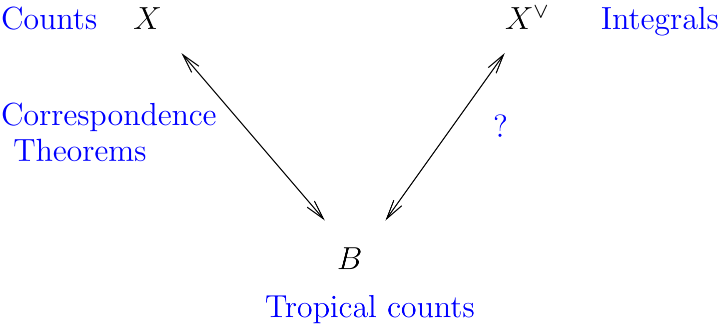
Gross-Siebert mirror symmetry

X $\qquad\qquad\qquad$ $X^\vee$

Gross-Siebert mirror symmetry



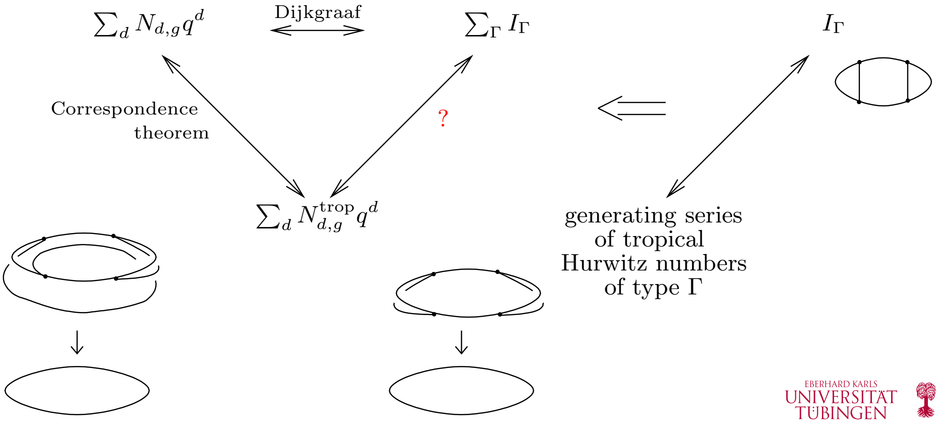
Gross-Siebert mirror symmetry



Tropical mirror symmetry of elliptic curves

2013: Böhm, Bringmann, Buchholz, M.

I_Γ= Feynmanintegral for the graph Γ



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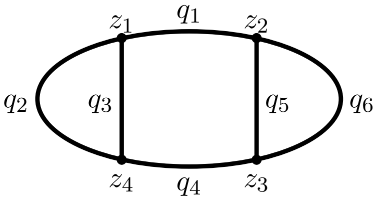
Feynman graphs

Feynman graphs: Let Γ be a 3-valent connected graph of genus g. Γ has $2g-2$ vertices and $3g-3$ edges.

Fix a labeling $z_1, \ldots, z_{2g-2}$ for the vertices and $q_1, \ldots, q_{3g-3}$ for the edges.



Notation

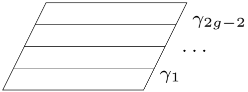
For the edge q_k, we say z_{k_1} and z_{k_2} are its adjacent vertices (no matter which order).

Feynman integrals

Definition (The Propagator)

$$P(z,q) := \frac{1}{4\pi^2}\wp(z,q) + \frac{1}{12}E_2(q^2).$$

Integration paths:



Definition

$$I_\Gamma(q) = \int_{\gamma_{2g-2}} \dots \int_{\gamma_1} \left(\prod_{k=1}^{3g-3} P(z_{k_1} - z_{k_2}, q) \right) dz_1 \dots dz_{2g-2}.$$

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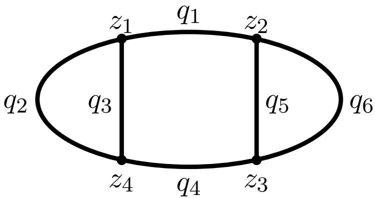
Example



Definition

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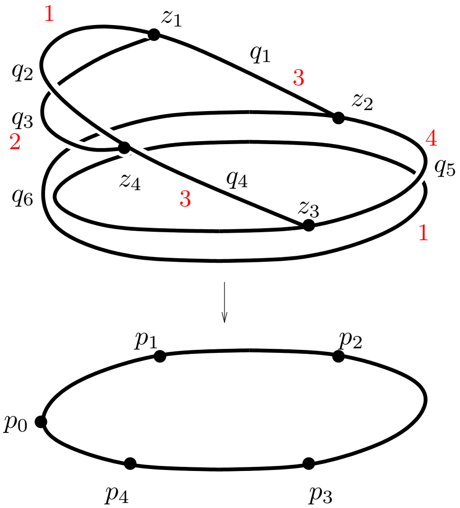
Labeled tropical covers



Numbers of labeled tropical covers

Let $\underline{a} = (a_1, \ldots, a_{3g-3})$ be a tuple of natural numbers.

Fix a Feynman graph Γ. Fix a base point $p_0 \in E$.

Definition

$N^{\text{trop}}_{\underline{a},\Gamma}$ = weighted number of labeled tropical covers $\pi : C \to E$

- where C is of combinatorial type Γ,
- where the weighted sum of points in $\pi^{-1}(p_0) \cap q_k$ is a_k
- with ramifications (3-valent vertices) at the fixed points.

Each such cover is counted with $\prod$ weights.

Tropical mirror symmetry

Definition (Refined integrals)

$$I_\Gamma(q_1,\ldots,q_{3g-3}) = \int_{\gamma_{2g-2}} \ldots \int_{\gamma_1} \left(\prod_{k=1}^{3g-3} P(z_{k_1}-z_{k_2}, q_k) \right) dz_1 \ldots dz_{2g-2}.$$

Theorem (BBBM, 13)

$N^{\mathrm{trop}}_{\underline{a},\Gamma}$ = *the coefficient of* $\underline{q}^{\underline{a}}$ *in* $I_\Gamma(q_1,\ldots,q_{3g-3})$.

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Tropical mirror symmetry

Theorem (BBBM, 13)

$N^{\mathrm{trop}}_{\underline{a},\Gamma} =$ *the coefficient of* $\underline{q}^{\underline{a}}$ *in* $I_\Gamma(q_1, \ldots, q_{3g-3})$.

Corollary (Right arrow in the triangle)

$$\sum_d N^{\mathrm{trop}}_{d,g} q^d = \sum_\Gamma I_\Gamma.$$

Proof: Set $q_k = q$ for all k and count correctly.

Corollary (Top arrow)

Mirror symmetry for elliptic curves.

Theorem (Goujard-Möller, 16)

New quasimodularity results for generating function for fixed graph.

Computing the integrals

- Coordinate change $x_k = e^{i\pi z_k}$.
- Produces circles from integration paths.
- Yields a factor of $\frac{1}{x_k}$ in the integral for every k (derivative of the inverse function).
- Computing the integral= computing residues, resp. the constant term of the product of propagators after coordinate change.

Theorem (Propagator after coord. change)

$$P(x,q) = \sum_{w=1}^{\infty} w x^w + \sum_{a=1}^{\infty} \sum_{w|a} w(x^w + x^{-w}) q^a.$$

For the integral, plug in $\frac{x_{k_1}}{x_{k_2}} = e^{i\pi(z_{k_1} - z_{k_2})}$.

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Hunting monomials

For $\underline{a}$, the coefficient of $\underline{q}^{\underline{a}}$ in $I_\Gamma(q_i)$ is the in all x_i constant coefficient of

$$\prod_{k|a_k=0}\left(\sum_{w_k=1}^{\infty} w_k\cdot\left(\frac{x_{k_1}}{x_{k_2}}\right)^{w_k}\right)\cdot$$
$$\prod_{k|a_k\neq 0}\left(\sum_{w_k|a_k} w_k\left(\left(\frac{x_{k_1}}{x_{k_2}}\right)^{w_k}+\left(\frac{x_{k_2}}{x_{k_1}}\right)^{w_k}\right)\right),$$

i.e. $\sum\prod_k w_k$, where the sum goes over all tuples

$$\left(\underline{a},\left(w_k\left(\frac{x_{k_1}}{x_{k_2}}\right)^{w_k}\right)_k\right)$$

such that the product of the $w_k\left(\frac{x_{k_1}}{x_{k_2}}\right)^{w_k}$ is constant in all x_i, and $w_k|a_k$ if $a_k>0$.

Curls from monomials

- Draw "cut" E with the p_i,
- draw x_i above p_i,
- if $a_k = 0$ draw edge from x_{k_1} to x_{k_2} of weight w_k directly,
- if $a_k > 0$, draw edge from x_{k_1} to x_{k_2}, but "curled" $\frac{a_k}{w_k}$ times