Lefschetz for non-Archimedean Jacobians

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Introduction

Let $Y$ be a smooth projective variety of dimension $n$ over a non-Archimedean field $K$, and let $D$ be an ample divisor on $Y$. As a consequence of comparison theorems of Berkovich and of the $\ell$-adic Lefschetz hyperplane theorem, the inclusion $D^{an} \hookrightarrow Y^{an}$ induces isomorphisms between the $\mathbb{Q}$-cohomology groups of dimensions $< n - 1$, and an injection in dimension $n - 1$.

However, unlike the case of complex analytic spaces, this does not hold in general if we replace $\mathbb{Q}$ by $\mathbb{Z}$, or if we replace cohomology groups with homotopy groups.

**Question.** For the analytifications of specific families of projective varieties, is it possible to establish a Lefschetz hyperplane theorem for $\mathbb{Z}$-cohomology and homotopy groups?

Let $X$ be a smooth projective curve over $K$ of genus $g$, and let $J$ be its Jacobian. Suppose $K$ is algebraically closed, and choose an identification $\text{Pic}^d(X) \cong J$. Let $W_d$ denote the locus in $J$ of effective divisor classes of degree $d$.

**Theorem 1** (S, 2016). For $1 \leq d \leq g - 1$, the inclusion $W_d^{an} \hookrightarrow J^{an}$ induces isomorphisms between $\mathbb{Z}$-cohomology (resp. homotopy) groups of dimensions $< d$, and an injection (resp. surjection) in dimension $d$.

Recall that $W_{g-1}$ is equal, up to translation, to the theta divisor $\Theta$ of $J$. We thus obtain, as a special case of the above theorem, a Lefschetz hyperplane theorem for $\mathbb{Z}$-cohomology and homotopy groups for the pair $(J^{an}, G^{an})$.

**Tropicalization**

To prove Theorem 1, we tropicalize $W_d^{an}$ and show that this tropicalization is a homotopy equivalence. Recall that $X^{an}$ has a tropicalization $\Gamma$ (see Figure 1), which is a metric graph. There is a natural tropicalization map from $J^{an}$ to the Jacobian $J(\Gamma)$ of $\Gamma$, which sends $W_d^{an}$ to a subset $W_d(\Gamma)$ of $J(\Gamma)$.

**Theorem 2** (S, 2016). The map $W_d^{an} \rightarrow W_d(\Gamma)$ is a homotopy equivalence for all $d$.

The tropicalization map $\text{trop} : J^{an} \rightarrow J(\Gamma)$ is naturally identified with the map from $\text{Pic}^d(X)$ to $\text{Pic}(\Gamma)$ given by tropicalizing the divisors on $X$. In particular, this allows us to identify $W_d(\Gamma)$ with the locus in $\text{Pic}(\Gamma)$ of effective tropical divisor classes of degree $d$.

**Morphisms with Projective Fibers**

The natural map $\text{Sym}^d(\Gamma) \rightarrow W_d(\Gamma)$ has contractible fibers, and therefore is a homotopy equivalence. Theorem 2 can then be reduced to showing that the map $\text{Sym}^{d,an}(X) \rightarrow W_d^{an}$ is a homotopy equivalence. This is derived as a special case of the following.

**Theorem 3** (S, 2016). Let $f : \tilde{Z} \rightarrow Y$ be a morphism of quasi-projective $K$-varieties. Suppose that there is a finite stratification $Y = \bigsqcup Y_i$ such that $f : Z \times_Y Y_i \rightarrow Y_i$ is a projective bundle of rank $r_i$ over $Y_i$.

Then, there is a finite extension $K \subset L$ such that $f_L^{an} : (Z_L)^{an} \rightarrow (Y_L)^{an}$ is a homotopy equivalence.

Brown and Foster have shown, following the MMP approach developed by Mustaţă – Nicaise and Nicaise – Xu, that over $K = \mathbb{C}((t))$, if $f : Z \rightarrow Y$ is a projective bundle with $Y$ smooth, then $f^{an} : Z^{an} \rightarrow Y^{an}$ is a homotopy equivalence.

Our proof of Theorem 3 follows a different approach, which works over $K$ of arbitrary characteristic and allows arbitrarily bad singularities for $Y$. A crucial component of our argument is the theorem of Hrushovski – Loeser – Poonen stating that over $K$ with a countable dense subset, the analytification $Y^{an}$ of a quasi-projective $K$-scheme of dimension $d$ embeds into $\mathbb{R}^{2d+1}$. In particular, $Y^{an}$ is metrizable and has a countable dense subset, which allows us to apply the Vietoris-Begle-Smale mapping theorem for maps with contractible fibers.

Figure 1: The tropicalization of an analytic curve $X^{an}$.

We thank D. Ranganathan for permission to use the picture on the left.