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A unified bijective framework for planar maps

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Abstract. Planar maps are connected planar graphs embedded in the plane. In the last fifteen years, a bijective approach for studying planar maps has been developped; and there are now dozens of bijections between classes of maps and classes of trees. We present a unified way to think about these bijections. Roughly speaking, we show that all the bijective results for maps can be recovered by specializing a unique "master bijection". The subtlety is that the master bijection acts on *oriented maps*. Thus, the known bijections are recovered by choosing a suitable orientation for the maps in the class considered, and then applying the master bijection. The suitable orientations implicitly used in the known bijections are in fact part of an infinite family $(\Omega_d)_{d\geq 0}$ of orientations that we characterize. The parameter d is related to the girth of the maps.

Keywords: Bijection, Planar maps, Trees, Girth

Planar maps are connected planar graphs embedded in the plane, considered up continuous deformations. Planar maps have been actively studied in combinatorics ever since the seminal work of William Tutte in the sixties. Along the years, deep connections have been fruitfully exploited between planar maps and subjects as diverse as the combinatorics of the symmetric group, graph drawing algorithms, random matrix theory, statistical mechanics, and 2D quantum gravity.

In the last decade, following the seminal work of Cori and Vauquelin Cori and Vauquelin (1981), Arquès Arquès (1986) and Schaeffer Schaeffer (1998), many bijections have been discovered between classes of maps (e.g. triangulations, bipartite maps) and classes of trees Schaeffer (1998). These bijections provide the "proofs from the Book" for the many simple-looking counting formulas discovered by Tutte and his followers. Moreover they proved to be invaluable tools in order to study the metric properties of maps, finding algorithms for maps, and solving statistical mechanics models on maps.

There are now dozens of bijections between classes of planar maps and classes of trees; see for instance Schaeffer (1998, 1997); Bouttier et al. (2002); Poulalhon and Schaeffer (2003); Bouttier et al. (2004); Fusy et al. (2008); Fusy (2009); Bousquet-Mélou and Schaeffer (2000, 2002); Bouttier et al. (2007); Bernardi (2007); Fusy et al. (2009). We will present a bijective framework, developed jointly with Eric Fusy, which unifies and extend these bijections Bernardi and Fusy (2012a,b, 2013b,a). There are two ingredients to our approach:

- A master bijection between a class of oriented maps and a class of trees. The master bijection Φ is illustrated in Figures 1 and 2.
- The existence of certain canonical orientations for planar maps of given girth.

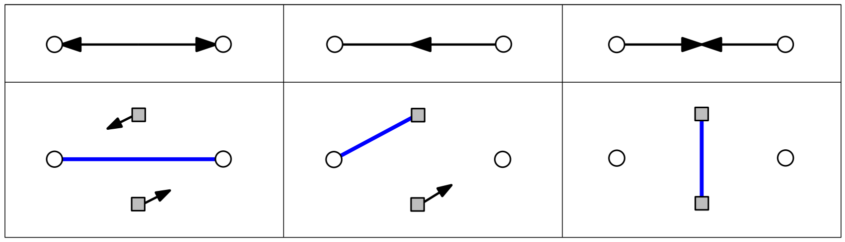

Fig. 1: The three types of edges in a bi-oriented map and the local rule of the master bijection Φ.

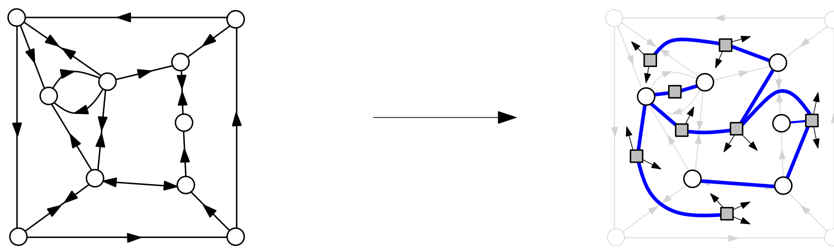

Fig. 2: The master bijection Φ applied to a bi-oriented map: the image is a tree with black and white vertices with some decorations on black vertices.

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