FPSAC 2013 Paris, France DMTCS proc. **AS**, 2013, 9–20

Quiver mutation and combinatorial DT-invariants

Bernhard Keller[†]

Université Paris Diderot – Paris 7, Institut Universitaire de France, Institut de Mathématiques de Jussieu – Paris Rive Gauche, UMR 7586 du CNRS, Paris, France

received 2013-05-26,

A quiver is an oriented graph. Quiver mutation is an elementary operation on quivers. It appeared in physics in Seiberg duality in the nineties and in mathematics in the definition of cluster algebras by Fomin-Zelevinsky in 2002. We show how, for large classes of quivers Q, using quiver mutation and quantum dilogarithms, one can construct the combinatorial DT-invariant, a formal power series intrinsically associated with Q. When defined, it coincides with the 'total' Donaldson-Thomas invariant of Q (with a generic potential) provided by algebraic geometry (work of Joyce, Kontsevich-Soibelman, Szendroi and many others). We illustrate combinatorial DT-invariants on many examples and point out their links to quantum cluster algebras and to (infinite) generalized associahedra.

Un carquois est un graphe orienté. La mutation des carquois est une opération élémentaire sur les carquois. Elle est apparue en physique dans la dualité de Seiberg dans les années 90 et en mathématiques dans la définition des algèbres amassées par Fomin–Zelevinsky en 2002. Nous montrons comment, pour de grandes classes de carquois Q, à l'aide de la mutation des carquois et des dilogarithmes quantiques, on peut construire l'invariant DT combinatoire, une série formelle associée intrinsèquement à Q. Quand cet invariant est défini, il est égal à l'invariant de Donaldson–Thomas 'total' associé à Q (avec un potentiel générique) qui est fourni par la géométrie algébrique (travaux de Joyce, Kontsevich–Soibelman, Szendroi et beaucoup d'autres). Nous illustrons les invariants DT combinatoires sur beaucoup d'exemple et évoquons leurs liens avec les algèbres amassées quantiques et des associaèdres généralisés (infinis).

Keywords: Cluster algebra, quiver mutation, Donaldson-Thomas invariants

1 Introduction

A quiver is an oriented graph. Quiver mutation is an elementary operation on quivers. It appeared in physics already in the nineties in Seiberg duality, cf. Seiberg (1995). In mathematics, quiver mutation was introduced by Fomin and Zelevinsky (2002) as the basic combinatorial ingredient of their definition of cluster algebras. Thus, quiver mutation is linked to the large array of subjects where cluster algebras have subsequently turned out to be relevant, cf. for example the cluster algebras portal maintained by

[†]Email: `keller@math.jussieu.fr`

Fomin (2002) and the survey articles by Fomin (2010), Leclerc (2010), Reiten (2010), Williams (2012). Among these links, the one to representation theory and algebraic geometry has been particularly fruitful. It has allowed to 'categorify' cluster algebras and thereby to prove conjectures about them which seem beyond the scope of the purely combinatorial methods, cf. for example the articles of Derksen et al. (2010), Geißet al. (2011), Plamondon (2011), Cerulli Irelli et al. (2012), ...

The constructions and results we present in this talk are another manifestation of this fruitful interaction. They are inspired by the theory of Donaldson–Thomas invariants as it has been developped by Bridgeland, Joyce and Song (2009), Kontsevich and Soibelman (2008, 2010), Nagao (2010), Reineke (2011), Szendrői (2008) and many others. In this theory, one assigns Donaldson–Thomas invariants to three-dimensional, possibly non commutative, Calabi–Yau varieties. These invariants exist in many different versions. Here we use the 'total' Donaldson–Thomas invariant, which is a certain power series in (slighly) non commutative variables. One important construction of non commutative 3-Calabi–Yau varieties takes as its input a quiver (with a generic potential). Thus, there is a Donaldson–Thomas invariant associated with 'each' quiver (some technical problems remain to be solved for a completely general definition). It turns out that for a suprisingly large class of quivers, it is possible to construct this invariant in a combinatorial way using products of quantum dilogarithm series associated with so-called reddening sequences of quiver mutations. This construction yields the definition of the combinatorial DT-invariant, which is the main point of this talk (section 4). It is an important fact that a given quiver may admit many distinct reddening sequences. Each of them yields a product decomposition for the combinatorial DT-invariant and in this way, one obtains many interesting quantum dilogarithm identities (section 5).

Let us emphasize that the use of (products of) quantum dilogarithm series in the study of (quantum) cluster algebras goes back to the insight of Fock and Goncharov (2009a,b). They also pioneered their application in the study of (quantum) dilogarithm identities, which was subsequently developped by many authors. We refer to Nakanishi (2012) for a survey. Geometric as well as combinatorial constructions of DT-invariants also appear in physics, cf. for example Cecotti et al. (2010), Alim et al. (2011a,b), Cecotti et al. (2011), Cecotti and Vafa (2011), Gaiotto et al. (2009, 2010a,b), Xie (2012), ...

2 Quiver mutation

A *quiver* is an oriented graph, i.e. a quadruple $Q = (Q_0, Q_1, s, t)$ formed by a set of vertices Q_0, a set of arrows Q_1 and two maps s and t from Q_1 to Q_0 which send an arrow α respectively to its source $s(\alpha)$ and its target $t(\alpha)$. In practice, a quiver is given by a picture as in the following example

$$
\begin{tikzcd}
Q: & & 3 \arrow[dl, "\lambda"'] & & 5 \arrow[loop left, "\alpha"] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & 6 \\
& 1 \arrow[rr, "\nu"'] & & 2 \arrow[ul, "\mu"'] \arrow[r, shift left, "\beta"] & 4. \arrow[l, shift left, "\gamma"]
\end{tikzcd}
$$

An arrow α whose source and target coincide is a *loop*; a 2-*cycle* is a pair of distinct arrows β and γ such that $s(\beta) = t(\gamma)$ and $t(\beta) = s(\gamma)$. Similarly, one defines n-*cycles* for any positive integer n. A vertex i of a quiver is a *source* (respectively a *sink*) if there is no arrow with target i (respectively with source i). A *Dynkin quiver* is a quiver whose underlying graph is a Dynkin diagram of type A_n, $n \geq 1$, D_n, $n \geq 4$, or E_6, E_7, E_8.

By convention, in the sequel, by a quiver we always mean a finite quiver without loops nor 2-cycles whose set of vertices is the set of integers from 1 to n for some $n \geq 1$. Up to an isomorphism fixing the vertices, such a quiver Q is given by the *skew-symmetric matrix* $B = B_Q$ whose coefficient b_{ij} is the difference between the number of arrows from i to j and the number of arrows from j to i for all $1 \leq i, j \leq n$. Conversely, each skew-symmetric matrix B with integer coefficients comes from a quiver.

Let Q be a quiver and k a vertex of Q. The *mutation* $\mu_k(Q)$ is the quiver obtained from Q as follows:

1) for each subquiver $i \xrightarrow{\beta} k \xrightarrow{\alpha} j$, we add a new arrow $[\alpha\beta] : i \to j$;
2) we reverse all arrows with source or target k;
3) we remove the arrows in a maximal set of pairwise disjoint 2-cycles.

For example, if k is a source or a sink of Q, then the mutation at k simply reverses all the arrows incident with k. In general, if B is the skew-symmetric matrix associated with Q and B' the one associated with $\mu_k(Q)$, we have

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \,; \\ b_{ij} + \operatorname{sign}(b_{ik}) \max(0, b_{ik}b_{kj}) & \text{else.} \end{cases} \tag{2.1}$$

This is the *matrix mutation rule* for skew-symmetric (more generally: skew-symmetrizable) matrices introduced by Fomin and Zelevinsky (2002), cf. also Fomin and Zelevinsky (2007).

One checks easily that μ_k is an involution. For example, the quivers

$$\begin{tikzcd}[column sep=small] & 1 \arrow[dr] & \\ 2 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & 1 \arrow[dl] & \\ 2 & & 3 \arrow[ul] \end{tikzcd} \tag{2.2}$$

are linked by a mutation at the vertex 1. Notice that these quivers are drastically different: The first one is a cycle, the second one the Hasse diagram of a linearly ordered set.

Two quivers are *mutation equivalent* if they are linked by a finite sequence of mutations. For example, it is an easy exercise to check that any two orientations of a tree are mutation equivalent. Using the quiver mutation applet by Keller (2006) or the Sage package by Musiker and Stump (2011) one can check that the following three quivers are mutation equivalent

$$\begin{tikzcd}[column sep=tiny] & & & 1 \arrow[dr] & & & \\ & & 2 \arrow[ur] \arrow[dr] & & 3 \arrow[ll] \arrow[dr] & & \\ & 4 \arrow[ur] \arrow[dr] & & 5 \arrow[ll] \arrow[ur] \arrow[dr] & & 6 \arrow[ll] \arrow[dr] & \\ 7 \arrow[ur] & & 8 \arrow[ll] \arrow[ur] & & 9 \arrow[ll] \arrow[ur] & & 10 \arrow[ll] \end{tikzcd} \qquad \begin{tikzcd}[column sep=small, row sep=small] & 10 \arrow[r] \arrow[dl] & 5 \arrow[dd] & & \\ 1 & & & & \\ & & 7 \arrow[d] & 6 \arrow[l] & \\ & & 8 \arrow[dr] & & 4 \arrow[ul] \\ & & & 9 \arrow[ur] \arrow[d] & \\ & & & 2 & 3 \arrow[uu] \end{tikzcd} \qquad \begin{tikzcd}[column sep=small, row sep=small] & 4 \arrow[dr] & & 5 \arrow[dl] & 3 \arrow[l] \\ & & 6 \arrow[dl] & & \\ & 7 \arrow[dd] & & & 2 \arrow[uu] \\ & & & 10 \arrow[ur] \arrow[dr] & \\ & 8 \arrow[r] & 9 \arrow[ur] & & 1 \end{tikzcd} \;. \tag{2.3}$$

The common *mutation class* of these quivers contains 5739 quivers (up to isomorphism). The mutation class of 'most' quivers is infinite. The classification of the quivers having a finite mutation class was achieved by by Felikson et al. (2012a) (and by Felikson et al. (2012b) in the skew-symmetric case): in addition to the quivers associated with triangulations of surfaces (with boundary and marked points, cf. Fomin et al. (2008)), the list contains 11 exceptional quivers, the largest of which is in the mutation class of the quivers (2.3).

3 Green quiver mutation

Let Q be a quiver without loops nor 2-cycles. The *framed quiver* $\tilde{Q}$ is obtained from Q by adding, for each vertex i, a new vertex i' and a new arrow $i \to i'$. Here is an example:

$$Q: \; 1 \to 2 \qquad \tilde{Q}: \begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 1' & 2' \end{tikzcd} .$$

The vertices i' are called *frozen vertices* because we never mutate at them. Now suppose that we have transformed $\tilde{Q}$ into $\tilde{Q}\,'$ by a finite sequence of mutations (at non frozen vertices). A vertex i of Q is *green* in $\tilde{Q}\,'$ if there are no arrows $j\,' \to i$ in $\tilde{Q}\,'$. Otherwise, it is *red*. A sequence $\mathbf{i} = (i_1, \ldots, i_N)$ is *green* if for each $1 \leq t \leq N$, the vertex i_t is green in the partially mutated quiver $\tilde{Q}(\mathbf{i}, t)$ defined by

$$\tilde{Q}(\mathbf{i}, t) = \mu_{i_{t-1}} \ldots \mu_{i_2} \mu_{i_1}(\tilde{Q}),$$

where for $t = 1$, we have the empty mutation sequence and obtain the initial quiver $\tilde{Q}$. It is *maximal green* if it is green and all the vertices of the final quiver $\mu_{\mathbf{i}}(\tilde{Q})$ are red (so that indeed, the sequence $\mathbf{i}$ cannot be extended to any strictly longer green sequence).

In Figure 1, we have encircled the green vertices. We see that in this example, we have two maximal green sequences 12 and 212 and that the final quivers associated with these two sequences are isomorphic by an isomorphism which fixes the frozen vertices. We call such an isomorphism a *frozen isomorphism*. A sequence $\mathbf{i} = (i_1, \ldots, i_N)$ is *reddening* if all vertices of the final quiver

$$\mu_{\mathbf{i}}(\tilde{Q}) = \tilde{Q}(\mathbf{i}, N)$$

are red. Of course, maximal green sequences are reddening. In the example of Figure 1, the sequence $1, 2, 1, 2, 1, 2, 1$ is reddening but not green.

Theorem 3.1 *If* $\mathbf{i}$ *and* $\mathbf{i}'$ *are reddening sequences, there is a frozen isomorphism between the final quivers*

$$\mu_{\mathbf{i}}(\tilde{Q}) \xrightarrow{\sim} \mu_{\mathbf{i}'}(\tilde{Q}).$$

The statement of the theorem is purely combinatorial but the known proofs (cf. section 7 of Keller (2012) and the references given there) are based on representation theory and geometry. For an *arbitrary* sequence $\mathbf{i}$ of non frozen vertices, we define the c-matrix $C(\mathbf{i})$ as the $n \times n$-matrix occuring in the right upper corner of the skew-symmetric matrix associated with the final quiver $\mu_{\mathbf{i}}(\tilde{Q})$, so that we have

$$B_{\mu_{\mathbf{i}}(\tilde{Q})} = \begin{bmatrix} * & C(\mathbf{i}) \\ * & * \end{bmatrix}.$$

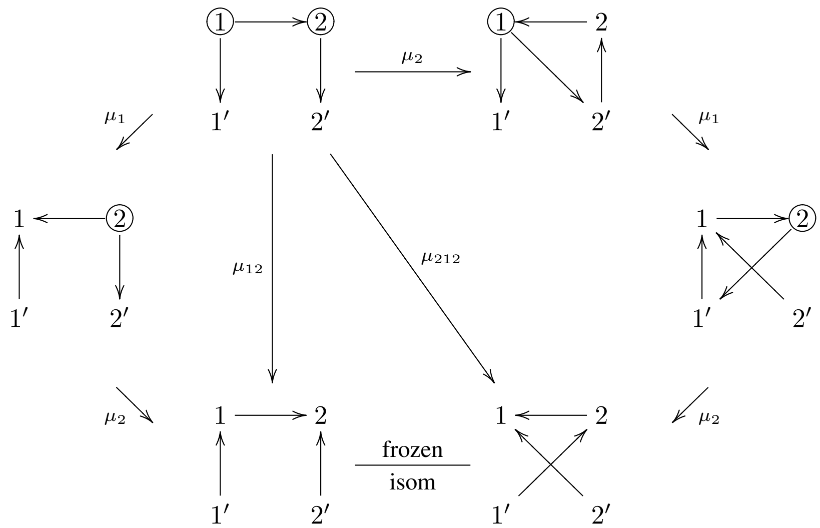


Fig. 1: The two maximal green sequences for A_2

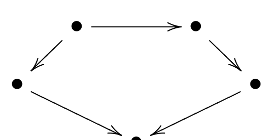
Thus, the (i, j)-coefficient of the matrix $C(\mathbf{i})$ is the difference between the number of arrows $i \to j'$ and $j' \to i$. The *c-vectors* associated with the sequence $\mathbf{i}$ are by definition the columns of the matrix $C(\mathbf{i})$. The following statement is known as the *sign coherence of c-vectors*.

Theorem 3.2 (Derksen et al. (2010)) *Each c-vector lies in* $\mathbb{N}^n$ *or* $(-\mathbb{N})^n$.

Again, the known proof of this combinatorial statement uses representation theory and geometry. The *oriented exchange graph* of the quiver Q is defined to be the quiver $\mathcal{E}_Q$ whose vertices are the frozen isomorphism classes of the quivers $\mu_{\mathbf{i}}(\tilde{Q})$, where $\mathbf{i}$ is an arbitrary sequence of vertices of Q, and where we have an arrow

$$\tilde{Q}' \to \mu_j(\tilde{Q}')$$

whenever j is a green vertex of $\tilde{Q}'$. For example, if Q is the quiver $1 \to 2$, then we see from Figure 1 that the oriented exchange graph is the oriented pentagon



By Theorem 3.1, the quiver $\mathcal{E}_Q$ has at most one sink. One can also show that it always has a unique source. An arbitrary sequence $\mathbf{i}$ of non frozen vertices corresponds to a *walk* in $\mathcal{E}_Q$ (a sequence of arrows and formal inverses of arrows) and a reddening sequence to a *path* (a formal composition of arrows) from the source to the sink. If Q is mutation equivalent to a quiver whose underlying graph is a Dynkin diagram of type A_n (resp. Δ), then $\mathcal{E}_Q$ is an orientation of the 1-skeleton of the nth Stasheff associahedron (resp. of the generalized associahedron of type Δ, cf. Chapoton et al. (2002)).

One can show that $\mathcal{E}_Q$ is the Hasse graph of a poset (a subposet of the set of torsion subcategories, cf. section 7.7 of Keller (2012)). If Q is the linear orientation of A_n, this poset the Tamari lattice. For certain classes of quivers, this poset is studied from the viewpoint of representation theory for example by Adachi et al. (2012), Brüstle et al. (2012), King and Qiu (2011), Koenig and Yang (2012) and Ladkani (2007).

Not all quivers admit reddening sequences. For example the quiver

$$\begin{tikzcd} & \bullet \arrow[dl, shift left] \arrow[dl, shift right] & \\ \bullet \arrow[rr, shift left] \arrow[rr, shift right] & & \bullet \arrow[ul, shift left] \arrow[ul, shift right] \end{tikzcd}$$

does not admit a reddening sequence. On the other hand, reddening sequences do exist for large classes of quivers, cf. section 5 below. In particular, each acyclic quiver (=quiver without oriented cycles) admits a maximal green sequence corresponding to an increasing enumeration of the vertices for the order defined by the existence of a path.

4 Combinatorial DT-invariants

Our aim is to associate an intrinsic formal power series $\mathbb{E}_Q$ with each quiver Q admitting a reddening sequence. For quiver with a unique vertex and no arrows, this series will be the *quantum dilogarithm series*

$$\begin{aligned} \mathbb{E}(y) &= 1 + \frac{q^{1/2}}{q-1} \cdot y + \cdots + \frac{q^{n^2/2} y^n}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})} + \cdots \\ &\in \mathbb{Q}(q^{1/2})[[y]], \end{aligned}$$

where $q^{1/2}$ is an indeterminate whose square is denoted by q and y is an indeterminate. This series is a classical object with many remarkable properties, cf. for example Zagier (1991). We will focus on one of them, namely the *pentagon identity*: If y_1 and y_2 are two indeterminates which q-commute, i.e. $y_1 y_2 = q y_2 y_1$, then we have

$$\mathbb{E}(y_1)\mathbb{E}(y_2) = \mathbb{E}(y_2)\mathbb{E}(q^{-1/2} y_1 y_2)\mathbb{E}(y_1). \tag{4.1}$$

It is due to Faddeev and Volkov (1993) and Faddeev and Kashaev (1994); a recent account can be found in Volkov (2012). Notice a striking structural similarity between this identity and the diagram in Figure 1: The two factors $\mathbb{E}(y_1)\mathbb{E}(y_2)$ on the left correspond to the two mutations in the path on the left, the three factors $\mathbb{E}(y_2)\mathbb{E}(q^{-1/2}y_1y_2)\mathbb{E}(y_1)$ on the right correspond to the three mutations in the path on the right and the equality corresponds to the frozen isomorphism between the final quivers. The common value of the two products will be defined as the combinatorial DT-invariant $\mathbb{E}_Q$ associated with the quiver $Q : 1 \to 2$.

Let Q be a quiver. For *any* sequence $\mathbf{i} = (i_1, \ldots, i_N)$ of vertices of Q, we will define a *product* $\mathbb{E}_{Q,\mathbf{i}}$ of quantum dilogarithm series. For this, let $\lambda_Q : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be the bilinear antisymmetric form associated with the matrix B_Q. Define the *complete quantum affine space* as the algebra

$$\hat{\mathbb{A}}_Q = \mathbb{Q}(q^{1/2})\langle\langle y^\alpha, \alpha \in \mathbb{N}^n \mid y^\alpha y^\beta = q^{1/2\,\lambda(\alpha,\beta)} y^{\alpha+\beta} \rangle\rangle.$$

This is a slightly non commutative deformation of an ordinary commutative power series algebra in n indeterminates. We define the *product*

$$\mathbb{E}_{Q,\mathbf{i}} = \mathbb{E}(y^{\varepsilon_1 \beta_1})^{\varepsilon_1} \cdots \mathbb{E}(y^{\varepsilon_N \beta_N})^{\varepsilon_N},$$

where the product is taken in $\hat{\mathbb{A}}_Q$, the vector β_t is the t-th column of the c-matrix $C(i_1, \ldots, i_{t-1})$ and ε_t is the common sign of the entries of this column (Theorem 3.2), $1 \leq t \leq N$.

Theorem 4.1 *If* $\mathbf{i}$ *and* $\mathbf{i}'$ *are two sequences of vertices of* Q *such that there is a frozen isomorphism between* $\mu_{\mathbf{i}}(\tilde{Q})$ *and* $\mu_{\mathbf{i}'}(\tilde{Q})$*, then we have the equality*

$$\mathbb{E}_{Q,\mathbf{i}} = \mathbb{E}_{Q,\mathbf{i}'}.$$

The theorem is proved in section 7.11 of Keller (2012) and independently in Nagao (2011). In particular, if $\mathbf{i}$ and $\mathbf{i}'$ are two reddening sequences, then by Theorem 4.1, the above equality holds. More generally, the theorem shows that to each vertex of the oriented exchange graph $\mathcal{E}_Q$, a canonical power series in $\hat{\mathbb{A}}_Q$ is associated. The one associated to the unique sink (if it exists) is $\mathbb{E}_Q$ and any reddening sequence gives a product expansion for $\mathbb{E}_Q$.

Definition 4.2 *If* Q *admits a reddening sequence* $\mathbf{i}$*, the* combinatorial DT-invariant of Q *is defined as*

$$\mathbb{E}_Q = \mathbb{E}_{Q,\mathbf{i}} \in \hat{\mathbb{A}}_Q.$$

The adjoint combinatorial DT-invariant of Q *is* $DT_Q = \Sigma \circ Ad(\mathbb{E}_Q) : Frac(\hat{\mathbb{A}}_Q) \to Frac(\hat{\mathbb{A}}_Q)$, *where* $Frac(\hat{\mathbb{A}}_Q)$ *is the non commutative field of fractions of* $\hat{\mathbb{A}}_Q$ *(cf. the Appendix of Berenstein and Zelevinsky (2005)) and* Σ *its automorphism determined by* $\Sigma(y^\alpha) = y^{-\alpha}$ *for all* $\alpha \in \mathbb{N}^n$.

For the agreement with the geometrically defined DT-invariant, we refer to section 7 of Keller (2012). In physics, an equivalent procedure has been discovered independently, cf. Xie (2012) and the references given there. It is easy to check that for $Q : 1 \to 2$, the above definition yields the left and right hand sides of the pentagon identity (4.1) associated with the two maximal green sequences of Figure 1 so that indeed, the combinatorial DT-invariant $\mathbb{E}_Q$ equals these two products. One can show that in this case, the adjoint combinatorial DT-invariant satisfies $(DT_Q)^5 = \mathrm{Id}$. Below, we will explore some remarkable generalizations of this example.

5 Examples

5.1 Dynkin quivers

Let Q be an *alternating* Dynkin quiver, i.e. a simply laced Dynkin diagram endowed with an orientation such that each vertex is a source or a sink, for example

$$Q = \vec{A}_5 : \bullet \longleftarrow \circ \longrightarrow \bullet \longleftarrow \circ \longrightarrow \bullet \, .$$

Let i_+ be the sequence of all sources $\circ$ and i_- the sequence of all sinks $\bullet$ (in any order). One can show that in this case, the sequence $\mathbf{i} = i_+ i_-$ is maximal green and so is

$$\mathbf{i}' = \underbrace{i_- i_+ i_- \ldots}_{h \text{ factors}} \quad ,$$

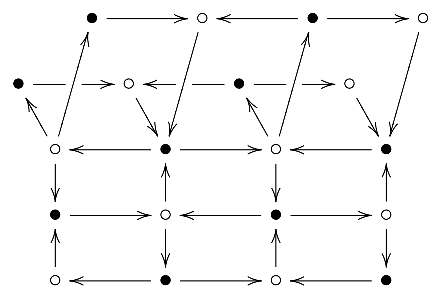


Fig. 2: The quiver $\vec{A}_4 \square \vec{D}_5$

where h is the Coxeter number of the underlying graph of Q. Thus, we have $\mathbb{E}(\mathbf{i}) = \mathbb{E}(\mathbf{i}')$ and $\mathbb{E}_Q$ is the common value. The identity $\mathbb{E}(\mathbf{i}) = \mathbb{E}(\mathbf{i}')$ is due to Reineke (2010). Using the geometry of the generalized associahedra of Chapoton et al. (2002), one can show that it is a consequence of the pentagon identity, cf. Qiu (2011) for another approach. For the adjoint combinatorial DT-invariant, we have

$$DT_Q^{h+2} = \text{Id}.$$

This is closely related to the original form of the periodicity conjecture of Zamolodchikov (1991) proved by Fomin and Zelevinsky (2003). We refer to Brüstle et al. (2012) for the study of maximal green sequences for more general acyclic quivers.

5.2 Square products of Dynkin quivers

Let Q_1 and Q_2 be alternating Dynkin quivers and $Q = Q_1 \square Q_2$ their square product as defined in section 8 of Keller (2010). For example, for suitable orientations of A_4 and D_5, the square product is depicted in Figure 2. There are no longer sources or sinks in the square product. However, we can consider the sequence i_+ of all even vertices $\circ$ (corresponding to a pair of sources or a pair of sinks) and the sequence i_- of all odd vertices $\bullet$ (corresponding to a mixed pair). Let

$$\begin{aligned} \mathbf{i} \; &= i_+ i_- i_+ \ldots \quad \text{with } h \text{ factors,} \\ \mathbf{i}' &= i_- i_+ i_- \ldots \quad \text{with } h' \text{ factors,} \end{aligned}$$

where h and h' are the Coxeter numbers of the Dynkin diagrams underlying the two quivers. One can check that both of these sequences are maximal green. In particular, the combinatorial DT-invariant is well-defined and we have $\mathbb{E}_Q = \mathbb{E}(\mathbf{i}) = \mathbb{E}(\mathbf{i}')$. It is an open question whether these identities are consequences of the pentagon identity. One can show (cf. section 5.7 of Keller (2011) or section 8.3.2 of Cecotti et al. (2010)) that in this case, the adjoint combinatorial DT-invariant satisfies

$$(DT_Q)^m = \text{Id}, \quad \text{where} \quad m = \frac{2(h+h')}{\gcd(h,h')}.$$

5.3 Quivers from reduced expressions in Coxeter groups

If R is an acyclic quiver and $\tilde{w}$ a reduced expression for an element of the Coxeter group associated with the underlying graph of R, there is a canonical quiver Q associated with the pair $(R, \tilde{w})$, cf. Berenstein et al. (2005). For example, if Q is A_4 with the linear orientation and $\tilde{w}$ a suitable expression for the longest element, one obtains the first quiver in (2.3). As shown by Geißet al. (2011), such quivers always admit maximal green sequences. In the A_4-example, such a sequence is given by

$$7, 8, 9, 10, 4, 5, 6, 2, 3, 1, 7, 8, 9, 2, 3, 1, 7, 8, 4, 7.$$

Thus, the combinatorial DT-invariant is well-defined. In the above example (and for all members of the 'triangular' family it belongs to), the adjoint combinatorial DT-invariant satisfies $(DT_Q)^6 = \mathrm{Id}$. It is an open question for which pairs $(R, \tilde{w})$ the invariant DT_Q is of finite order.

5.4 Another product construction

The following quiver is obtained as the triangle product (cf. section 8 of Keller (2010)) of a quiver of type A_3 with the quiver appearing in the first column (the arrows marked by 2 are double arrows).

$$\begin{tikzcd}
7 \arrow[r] & 8 \arrow[r] \arrow[dl] & 9 \arrow[dl] \\
4 \arrow[u] \arrow[r] & 5 \arrow[u] \arrow[r] \arrow[dl, "2"] & 6 \arrow[u] \arrow[dl, "2"] \\
1 \arrow[u, "2"] \arrow[r] & 2 \arrow[u, "2"] \arrow[r] & 3 \arrow[u, "2"]
\end{tikzcd}$$

It admits the maximal green sequence $3, 6, 9, 2, 5, 8, 1, 4, 7, 3, 6, 9, 2, 5, 8, 3, 6, 9$. In this case, the invariant DT_Q is of infinite order.

References

T. Adachi, O. Iyama, and I. Reiten. τ-tilting theory. arXiv:1210.1036 [math.RT], 2012.

M. Alim, S. Cecotti, C. Córdova, S. Espahbodi, A. Rastogi, and C. Vafa. $\mathcal{N} = 2$ quantum field theories and their BPS quivers. arXiv:1112.3984 [hep-th], 2011a.

M. Alim, S. Cecotti, C. Córdova, S. Espahbodi, A. Rastogi, and C. Vafa. BPS quivers and spectra of complete $\mathcal{N} = 2$ quantum field theories. arXiv:1109.4941 [hep-th], 2011b.

A. Berenstein and A. Zelevinsky. Quantum cluster algebras. *Adv. Math.*, 195(2):405–455, 2005.

A. Berenstein, S. Fomin, and A. Zelevinsky. Cluster algebras. III. Upper bounds and double Bruhat cells. *Duke Math. J.*, 126(1):1–52, 2005.

T. Brüstle, G. Dupont, and M. Pérotin. On maximal green sequences. arXiv:1205.2050 [math.RT], 2012.

S. Cecotti and C. Vafa. Classification of complete $N = 2$ supersymmetric field theories in 4 dimensions. arXiv: 1103.5832 [hep-th], 2011.

S. Cecotti, A. Neitzke, and C. Vafa. R-twisting and $4d$/$2d$-Correspondences. arXiv:10063435 [physics.hep-th], 2010.

S. Cecotti, C. Córdova, and C. Vafa. Braids, walls and mirrors. arXiv:1110.2115 [hep-th], 2011.

G. Cerulli Irelli, B. Keller, D. Labardini-Fragoso, and P.-G. Plamondon. Linear independence of cluster monomials for skew-symmetric cluster algebras. arXiv:1203.1307 [math.RT], to appear in Compositio, 2012.

F. Chapoton, S. Fomin, and A. Zelevinsky. Polytopal realizations of generalized associahedra. *Canad. Math. Bull.*, 45(4):537–566, 2002. Dedicated to Robert V. Moody.

H. Derksen, J. Weyman, and A. Zelevinsky. Quivers with potentials and their representations I: Mutations. *Selecta Mathematica*, 14:59–119, 2008.

H. Derksen, J. Weyman, and A. Zelevinsky. Quivers with potentials and their representations II: Applications to cluster algebras. *J. Amer. Math. Soc.*, 23:749–790, 2010.

L. Faddeev and A. Y. Volkov. Abelian current algebra and the Virasoro algebra on the lattice. *Phys. Lett. B*, 315(3-4):311–318, 1993.

L. D. Faddeev and R. M. Kashaev. Quantum dilogarithm. *Modern Phys. Lett. A*, 9(5):427–434, 1994.

A. Felikson, M. Shapiro, and P. Tumarkin. Skew-symmetric cluster algebras of finite mutation type. *J. Eur. Math. Soc. (JEMS)*, 14(4):1135–1180, 2012a. ISSN 1435-9855. doi: 10.4171/JEMS/329. URL `http://dx.doi.org/10.4171/JEMS/329`.

A. Felikson, M. Shapiro, and P. Tumarkin. Cluster algebras of finite mutation type via unfoldings. *Int. Math. Res. Not. IMRN*, (8):1768–1804, 2012b. ISSN 1073-7928. doi: 10.1093/imrn/rnr072. URL `http://dx.doi.org/10.1093/imrn/rnr072`.

V. V. Fock and A. B. Goncharov. Cluster ensembles, quantization and the dilogarithm. *Annales scientifiques de l'ENS*, 42(6):865–930, 2009a.

V. V. Fock and A. B. Goncharov. The quantum dilogarithm and representations of quantum cluster varieties. *Invent. Math.*, 175(2):223–286, 2009b.

S. Fomin. Cluster algebras portal. `www.math.lsa.umich.edu/˜fomin/cluster.html`, 2002.

S. Fomin. Total positivity and cluster algebras. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 125–145, New Delhi, 2010. Hindustan Book Agency.

S. Fomin and A. Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

S. Fomin and A. Zelevinsky. Y-systems and generalized associahedra. *Ann. of Math. (2)*, 158(3):977–1018, 2003.

S. Fomin and A. Zelevinsky. Cluster algebras IV: Coefficients. *Compositio Mathematica*, 143:112–164, 2007.

S. Fomin, M. Shapiro, and D. Thurston. Cluster algebras and triangulated surfaces. I. Cluster complexes. *Acta Math.*, 201(1):83–146, 2008.

D. Gaiotto, G. W. Moore, and A. Neitzke. Four-dimensional wall-crossing via three-dimensional field theory. arXiv:0807.4723 [physics.hep-th], 2008.

D. Gaiotto, G. W. Moore, and A. Neitzke. Wall-crossing, Hitchin systems, and the WKB approximation. arXiv:0907.3987 [physics.hep-th], 2009.

D. Gaiotto, G. W. Moore, and A. Neitzke. Framed BPS States. arXiv:1006.0146 [physics.hep-th], 2010a.

D. Gaiotto, G. W. Moore, and A. Neitzke. Four-dimensional wall-crossing via three-dimensional field theory. *Comm. Math. Phys.*, 299(1):163–224, 2010b.

C. Geiß, B. Leclerc, and J. Schröer. Kac-Moody groups and cluster algebras. *Adv. Math.*, 228(1):329–433, 2011. ISSN 0001-8708. doi: 10.1016/j.aim.2011.05.011. URL `http://dx.doi.org/10.1016/j.aim.2011.05.011`.

D. Joyce and Y. Song. A theory of generalized Donaldson-Thomas invariants. II. Multiplicative identities for Behrend functions. arXiv:0901.2872 [math.AG], 2009.

B. Keller. Quiver mutation in Java. Java applet available at the author's home page, 2006.

B. Keller. Cluster algebras, quiver representations and triangulated categories. In T. Holm, P. Jørgensen, and R. Rouquier, editors, *Triangulated categories*, volume 375 of *London Mathematical Society Lecture Note Series*, pages 76–160. Cambridge University Press, 2010.

B. Keller. On cluster theory and quantum dilogarithm identities. In *Representations of algebras and related topics*, EMS Ser. Congr. Rep., pages 85–116. Eur. Math. Soc., Zürich, 2011. doi: 10.4171/101-1/3. URL `http://dx.doi.org/10.4171/101-1/3`.

B. Keller. Cluster algebras and derived categories. arXiv:1202.4161 [math.RT], 2012.

A. King and Y. Qiu. Exchange graphs of acyclic Calabi–Yau categories. arXiv:1109.2924 [math.RT], 2011.

S. Koenig and D. Yang. Silting objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras. arXiv:1203.5657 [math.RT], 2012.

M. Kontsevich and Y. Soibelman. Stability structures, Donaldson-Thomas invariants and cluster transformations. arXiv:0811.2435 [math.AG], 2008.

M. Kontsevich and Y. Soibelman. Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants. arXiv:1006.2706 [math.AG], 2010.

S. Ladkani. Universal derived equivalences of posets of tilting modules. arXiv:0708.1287 [math.RT], 2007.

B. Leclerc. Cluster algebras and representation theory. In *Proceedings of the International Congress of Mathematicians. Volume IV*, pages 2471–2488, New Delhi, 2010. Hindustan Book Agency.

G. Musiker and C. Stump. A compendium on the cluster algebra and quiver package in Sage. arXiv:1102.4844 [math.CO], 2011.

K. Nagao. Donaldson-Thomas theory and cluster algebras. arXiv:1002.4884 [math.AG], 2010.

K. Nagao. Quantum dilogarithm identities. *RIMS Kokyuroku Bessatsu*, B28:164–170, 2011.

T. Nakanishi. Tropicalization method in cluster algebras. *Contemp. Math.*, 580:95–115, 2012.

P.-G. Plamondon. Cluster algebras via cluster categories with infinite-dimensional morphism spaces. *Compositio Mathematica*, 147:1921–1954, 2011.

Y. Qiu. Stability conditions and quantum dilogarithm identities for Dynkin quivers. arXiv:1111.1010 [math.AG], 2011.

M. Reineke. Poisson automorphisms and quiver moduli. *J. Inst. Math. Jussieu*, 9(3):653–667, 2010.

M. Reineke. Cohomology of quiver moduli, functional equations, and integrality of Donaldson-Thomas type invariants. *Compos. Math.*, 147(3):943–964, 2011. ISSN 0010-437X. doi: 10.1112/S0010437X1000521X. URL `http://dx.doi.org/10.1112/S0010437X1000521X`.

I. Reiten. Cluster categories. In *Proceedings of the International Congress of Mathematicians. Volume I*, pages 558–594, New Delhi, 2010. Hindustan Book Agency.

N. Seiberg. Electric-magnetic duality in supersymmetric non-abelian gauge theories. *Nuclear Phys. B*, 435(1-2):129–146, 1995. arXiv:hep-th/9411149.

B. Szendrői. Non-commutative Donaldson-Thomas invariants and the conifold. *Geom. Topol.*, 12(2): 1171–1202, 2008. doi: 10.2140/gt.2008.12.1171. URL `http://dx.doi.org/10.2140/gt.2008.12.1171`.

A. Y. Volkov. Pentagon identity revisited. *Int. Math. Res. Not. IMRN*, (20):4619–4624, 2012. ISSN 1073-7928.

L. Williams. Cluster algebras: an introduction. arXiv:1212.6263 [math.RA], 2012.

D. Xie. BPS spectrum, wall crossing and quantum dilogarithm identity. arXiv:1211.7071 [physics.hep-th], 2012.

D. Zagier. Polylogarithms, Dedekind zeta functions and the algebraic K-theory of fields. In *Arithmetic algebraic geometry (Texel, 1989)*, volume 89 of *Progr. Math.*, pages 391–430. Birkhäuser Boston, Boston, MA, 1991.

A. B. Zamolodchikov. On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories. *Phys. Lett. B*, 253(3-4):391–394, 1991.