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Recent Progress on the Diameter of Polyhedra and Simplicial Complexes

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We review several recent results on the diameter of polytopes, polyhedra and simplicial complexes, motivated by the (now disproved, but not quite solved) Hirsch Conjecture.

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Introduction

The Hirsch Conjecture, understood in a broad sense, asked what is the maximum possible combinatorial diameter of a convex polyhedron of dimension d and with n facets. Let us denote this number $H(n, d)$. Although the original conjecture $H(n, d) \leq n - d$ has been disproved [7, 9], the underlying problem is still wide open:

- The known counter-examples violate the conjecture only by a constant and *small* factor (25% in the case of unbounded polyhedra, 5% for bounded polytopes).
- No polynomial upper bound is known for $H(n, d)$. All we know is $H(n, d) \leq n^{\log d+2}$ (*quasi-polynomial* bound of Kalai and Kleitman [6]) and $H(n, d) \leq 2^{d-3}n$ (linear bound *in fixed dimension* by Larman [8]).

Some recent attempts of settling this question go by looking at the problem in the more general context of *pure simplicial complexes*: *What is the maximum diameter of the dual graph of a simplicial* $(d-1)$*-sphere or* $(d-1)$*-ball with* n *vertices?*

Here a simplicial $(d-1)$-ball or sphere is a simplicial complex homeomorphic to the $(d-1)$-ball or sphere. These complexes are necessarily *pure* (all the maximal simplices have the same dimension). The dual graph of a pure simplicial complex is the graph whose vertices are the maximal simplices (a. k. a. *facets*) and whose edges correspond to adjacent facets. We can also remove the sphere/ball condition and ask the same for all pure simplicial complexes. Some recent results in this direction are:

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- For arbitrary pure simplicial complexes the diameter can be exponential, in the order of $n^{2d/3}$ [5].
- For complexes in which every star is strongly connected (that is, the dual graph of every star is connected) the Kalai-Kleitman and the Larman bounds stated above hold, essentially with the same proofs. These complexes have been called *normal* or *locally strongly connected* in the literature.
- For complexes which are not only normal but also *flag* (meaning that the complex is the *clique complex* of its 1-skeleton), the original Hirsch bound holds [1].

Going back to polytopes, there is also a recent bound in terms of n, d and the maximum determinant of the system defining the polytope [2] and a recent construction of polytopes which fail to have the k-decomposability property, for arbitrarily large k [4].

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