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A uniform model for Kirillov–Reshetikhin crystals

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Abstract. We present a uniform construction of tensor products of one-column Kirillov–Reshetikhin (KR) crystals in all untwisted affine types, which uses a generalization of the Lakshmibai–Seshadri paths (in the theory of the Littelmann path model). This generalization is based on the graph on parabolic cosets of a Weyl group known as the parabolic quantum Bruhat graph. A related model is the so-called quantum alcove model. The proof is based on two lifts of the parabolic quantum Bruhat graph: to the Bruhat order on the affine Weyl group and to Littelmann's poset on level-zero weights. Our construction leads to a simple calculation of the energy function. It also implies the equality between a Macdonald polynomial specialized at $t = 0$ and the graded character of a tensor product of KR modules.

Résumé. Nous présentons une construction uniforme pour les produits tensoriels des cristaux de Kirillov–Reshetikhin de type colonne, pour tous les types affines symétriques, qui utilise une généralisation des chemins de Lakshmibai–Seshadri (dans la théorie des chemins de Littelmann). Cette généralisation est basée sur un graphe sur les classes parabolique d'un groupe de Weyl appelé le graphe de Bruhat parabolique quantique. Un modèle lié est le modèle des alcôves quantique. La preuve est basée sur deux relèvements du graphe de Bruhat parabolique quantique: dans l'ordre de Bruhat affine et dans un ensemble ordonné des poids de niveau zero. Notre construction donne une formule simple pour la fonction d'énergie. Elle donne aussi l'égalité d'un polynôme de Macdonald spécialisé à $t = 0$ avec le caractère gradué d'un produit tensoriel des modules de Kirillov–Reshetikhin.

Keywords: Parabolic quantum Bruhat graph, Kirillov–Reshetikhin crystals, energy function, Lakshmibai–Seshadri paths, Macdonald polynomials.

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1 Introduction

Our goal in this series of papers (see [LNSSS1, LNSSS2]) is to obtain a uniform construction of tensor products of one-column Kirillov–Reshetikhin (KR) crystals. As a consequence we shall prove the equality $P_\lambda(q) = X_\lambda(q)$, where $P_\lambda(q)$ is the Macdonald polynomial $P_\lambda(q,t)$ specialized at $t = 0$ and $X_\lambda(q)$ is the graded character of a simple Lie algebra coming from tensor products of KR modules. Both the Macdonald polynomials and KR modules are of arbitrary untwisted affine type. The index λ is a dominant weight for the simple Lie subalgebra obtained by removing the affine node. Macdonald polynomials and characters of KR modules have been studied extensively in connection with various fields such as statistical mechanics and integrable systems, representation theory of Coxeter groups and Lie algebras (and their quantized analogues given by Hecke algebras and quantized universal enveloping algebras), geometry of singularities of Schubert varieties, and combinatorics.

Our point of departure is a theorem of Ion [Ion], which asserts that the nonsymmetric Macdonald polynomials at $t = 0$ are characters of Demazure submodules of highest weight modules over affine algebras. This holds for the Langlands duals of untwisted affine root systems (and type $A_{2n}^{(2)}$ in the case of nonsymmetric Koornwinder polynomials). Our results apply to the untwisted affine root systems. The overlapping cases are the simply-laced affine root systems $A_n^{(1)}$, $D_n^{(1)}$ and $E_{6,7,8}^{(1)}$.

It is known [FL, FSS, KMOU, KMOTU, ST, Na] that certain affine Demazure characters (including those for the simply-laced affine root systems) can be expressed in terms of KR crystals, which motivates the relation between P and X. For types $A_n^{(1)}$ and $C_n^{(1)}$, the equality $P = X$ was achieved in [Le2, LeS] by establishing a combinatorial formula for the Macdonald polynomials at $t = 0$ from the Ram–Yip formula [RY], and by using explicit models for the one-column KR crystals [FOS]. It should be noted that, in types $A_n^{(1)}$ and $C_n^{(1)}$, the one-column KR modules are irreducible when restricted to the canonical simple Lie subalgebra, while in general this is not the case. For the cases considered by Ion [Ion], the corresponding KR crystals are perfect. This is not necessarily true for the untwisted affine root systems considered in this work, especially for the untwisted non-simply-laced affine root systems.

In this work we provide a type-free approach to the equality $P = X$ for untwisted affine root systems. Lenart's specialization [Le2] of the Ram–Yip formula for Macdonald polynomials uses the quantum alcove model [LeL1], whose objects are paths in the quantum Bruhat graph (QBG), which was defined and studied in [BFP] in relation to the quantum cohomology of the flag variety. On the other hand, Naito and Sagaki [NS1, NS2, NS4, NS5] gave models for tensor products of KR crystals of one-column type in terms of projections of level-zero Lakshmibai–Seshadri (LS) paths to the classical weight lattice. Hence we need to establish a bijection between the quantum alcove model and projected level-zero LS paths.

In analogy with [BFP] and inspired by the quantum Schubert calculus of homogeneous spaces [Mi, P] we define the parabolic quantum Bruhat graph (PQBG), which is a directed graph structure on parabolic quotients of the Weyl group with respect to a parabolic subgroup. We construct two lifts of the PQBG. The *first lift* is from the PQBG to the Bruhat order of the affine Weyl group. This is a parabolic analogue of the lift of the QBG to the affine Bruhat order [LS], which is the combinatorial structure underlying Peterson's theorem [P]; the latter equates the Gromov-Witten invariants of finite-dimensional homogeneous spaces with the Pontryagin homology structure constants of Schubert varieties in the affine Grassmannian. We obtain Diamond Lemmas for the PQBG via projection of the standard Diamond Lemmas for the affine

Weyl group. We find a *second lift* of the PQBG into a poset of Littelmann [Li] for level-zero weights and characterize its local structure (such as cover relations) in terms of the PQBG. Littelmann's poset was defined in connection with LS paths for arbitrary (not necessarily dominant) weights, but the local structure was not previously known. The weight poset precisely controls the combinatorics of the level-zero LS paths and therefore their classical projections, which we formulate directly as quantum LS paths. Finally, we describe a bijection between the quantum alcove model and the quantum LS paths.

The paper is organized as follows. In Section 2 we prepare the background and define the PQBG. Section 3 is reserved for the two lifts of the PQBG and the statement of the Diamond Lemmas. In Section 4 we describe KR crystals in terms of the quantum LS paths and the quantum alcove model in [LeL1]; we also give simple combinatorial formulas for the energy function. Finally, we conclude in Section 5 with the results on (nonsymmetric) Macdonald polynomials at $t = 0$.

Acknowledgments

We would like to thank Thomas Lam for helpful discussions during FPSAC 2012, and Daniel Orr for his discussions about Ion's work [Ion]. We used SAGE [Sa] and SAGE-COMBINAT [Sa-comb] to discover properties about the level-zero weight poset and to obtain some of the pictures in [LNSSS1].

2 Background

2.1 Untwisted affine root datum

Let $I_{\mathrm{af}} = I \sqcup \{0\}$ (resp. I) be the Dynkin node set of an untwisted affine algebra $\mathfrak{g}_{\mathrm{af}}$ (resp. its canonical subalgebra $\mathfrak{g}$), W_{af} (resp. W) the affine (resp. finite) Weyl group with simple reflections r_i for $i \in I_{\mathrm{af}}$ (resp. $i \in I$), and $X_{\mathrm{af}} = \mathbb{Z}\delta \oplus \bigoplus_{i\in I_{\mathrm{af}}} \mathbb{Z}\Lambda_i$ (resp. $X = \bigoplus_{i\in I} \mathbb{Z}\omega_i$) the affine (resp. finite) weight lattice. Let $\{\alpha_i \mid i \in I_{\mathrm{af}}\}$ be the simple roots, $\Phi^{\mathrm{af}} = W_{\mathrm{af}}\{\alpha_i \mid i \in I_{\mathrm{af}}\}$ (resp. $\Phi = W\{\alpha_i \mid i \in I\}$) the set of affine real roots (resp. roots), and $\Phi^{\mathrm{af}+} = \Phi^{\mathrm{af}} \cap \bigoplus_{i\in I_{\mathrm{af}}} \mathbb{Z}_{\geq 0}\alpha_i$ (resp. $\Phi^+ = \Phi \cap \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$) the set of positive affine real (resp. positive) roots. Furthermore, $\Phi^{\mathrm{af}-} = -\Phi^{\mathrm{af}+}$ (resp. $\Phi^- = -\Phi^+$) are the negative affine real (resp. negative) roots. Let $X_{\mathrm{af}}^\vee = \mathrm{Hom}_{\mathbb{Z}}(X_{\mathrm{af}}, \mathbb{Z})$ be the dual lattice, $\langle \cdot\,,\,\cdot\rangle : X_{\mathrm{af}}^\vee \times X_{\mathrm{af}} \to \mathbb{Z}$ the evaluation pairing, and $\{d\} \cup \{\alpha_i^\vee \mid i \in I_{\mathrm{af}}\}$ the dual basis of $X_{\mathrm{af}}^\vee$. The natural projection $\mathrm{cl} : X_{\mathrm{af}} \to X$ has kernel $\mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\delta$ and sends $\Lambda_i \mapsto \omega_i$ for $i \in I$.

The affine Weyl group W_{af} acts on X_{af} and $X_{\mathrm{af}}^\vee$ by

$$r_i\lambda = \lambda - \langle \alpha_i^\vee\,,\,\lambda\rangle\alpha_i \qquad \text{and} \qquad r_i\mu = \mu - \langle \mu\,,\,\alpha_i\rangle\alpha_i^\vee\,,$$

for $i \in I_{\mathrm{af}}$, $\lambda \in X_{\mathrm{af}}$, and $\mu \in X_{\mathrm{af}}^\vee$. For $\beta \in \Phi^{\mathrm{af}}$, let $w \in W_{\mathrm{af}}$ and $i \in I_{\mathrm{af}}$ be such that $\beta = w\alpha_i$. Define the associated reflection $r_\beta \in W_{\mathrm{af}}$ and associated coroot $\beta^\vee \in X_{\mathrm{af}}^\vee$ by $r_\beta = wr_iw^{-1}$ and $\beta^\vee = w\alpha_i^\vee$.

The null root is the unique element $\delta \in \bigoplus_{i\in I_{\mathrm{af}}} \mathbb{Z}_{>0}\alpha_i$ which generates the rank 1 sublattice $\{\lambda \in X_{\mathrm{af}} \mid \langle \alpha_i^\vee\,,\,\lambda\rangle = 0 \text{ for all } i \in I_{\mathrm{af}}\}$. We have $\delta = \alpha_0 + \theta$, where θ is the highest root for $\mathfrak{g}$. The canonical central element is the unique element $c \in \bigoplus_{i\in I_{\mathrm{af}}} \mathbb{Z}_{>0}\alpha_i^\vee$ which generates the rank 1 sublattice $\{\mu \in X_{\mathrm{af}}^\vee \mid \langle \mu\,,\,\alpha_i\rangle = 0 \text{ for all } i \in I_{\mathrm{af}}\}$. The level of a weight $\lambda \in X_{\mathrm{af}}$ is defined by $\mathrm{level}(\lambda) = \langle c\,,\,\lambda\rangle$. Let $X_{\mathrm{af}}^0 \subset X_{\mathrm{af}}$ be the sublattice of level-zero elements.

We denote by $\ell(w)$ for $w \in W_{\mathrm{af}}$ (resp. W) the length of w and by $\lessdot$ the Bruhat cover. The element $t_\mu \in W_{\mathrm{af}}$ is the translation by the element μ in the coroot lattice $Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$.

2.2 Affinization of a weight stabilizer

Let $\lambda \in X$ be a dominant weight, which is fixed throughout the remainder of Sections 2 and 3. Let W_J be the stabilizer of λ in W. It is a parabolic subgroup, being generated by r_i for $i \in J$, where $J = \{i \in I \mid \langle \alpha_i^\vee , \lambda \rangle = 0\}$. Let $Q_J^\vee = \bigoplus_{i \in J} \mathbb{Z}\alpha_i^\vee$ be the associated coroot lattice, W^J the set of minimum-length coset representatives in W/W_J, $\Phi_J \supset \Phi_J^+$ the set of roots and positive roots respectively, and $\rho_J = (1/2)\sum_{\alpha \in \Phi_J^+} \alpha$ (if $J = \emptyset$, then ρ_J is denoted by ρ). Define

$$(W_J)_{\mathrm{af}} = W_J \ltimes Q_J^\vee = \{wt_\mu \in W_{\mathrm{af}} \mid w \in W_J \, , \mu \in Q_J^\vee\} \, , \tag{1}$$

$$\Phi_J^{\mathrm{af}+} = \{\beta \in \Phi^{\mathrm{af}+} \mid \mathrm{cl}(\beta) \in \Phi_J\} = \Phi_J^+ \cup (\mathbb{Z}_{>0}\delta + \Phi_J) \, , \tag{2}$$

$$(W^J)_{\mathrm{af}} = \{x \in W_{\mathrm{af}} \mid x \cdot \beta > 0 \text{ for all } \beta \in \Phi_J^{\mathrm{af}+}\}. \tag{3}$$

By [LS, Lemma 10.5] [P], any $w \in W_{\mathrm{af}}$ factors uniquely as $w = w_1 w_2$, where $w_1 \in (W^J)_{\mathrm{af}}$ and $w_2 \in (W_J)_{\mathrm{af}}$. Therefore, we can define $\pi_J : W_{\mathrm{af}} \to (W^J)_{\mathrm{af}}$ by $w \mapsto w_1$. We say that $\mu \in Q^\vee$ is *J-adjusted* if $\pi_J(t_\mu) = z_\mu t_\mu$ with $z_\mu \in W$. Say that $\mu \in Q^\vee$ is *J-superantidominant* if μ is antidominant (i.e., $\langle \mu , \alpha \rangle \le 0$ for all $\alpha \in \Phi^+$), and $\langle \mu , \alpha \rangle \ll 0$ for $\alpha \in \Phi^+ \setminus \Phi_J^+$.

2.3 The parabolic quantum Bruhat graph

The *parabolic quantum Bruhat graph* $\mathrm{QB}(W^J)$ is a directed graph with vertex set W^J, whose directed edges have the form $w \xrightarrow{\alpha} \lfloor wr_\alpha \rfloor$ for $w \in W^J$ and $\alpha \in \Phi^+ \setminus \Phi_J^+$. Here we denote by $\lfloor v \rfloor$ the minimum-length representative in the coset vW_J for $v \in W$. There are two kinds of edges:

1. (Bruhat edge) $w \lessdot wr_\alpha$. (One may deduce that $wr_\alpha \in W^J$.)
2. (Quantum edge) $\ell(\lfloor wr_\alpha \rfloor) = \ell(w) + 1 - \langle \alpha^\vee , 2\rho - 2\rho_J \rangle$.

If $J = \emptyset$, then we recover the quantum Bruhat graph $\mathrm{QB}(W)$ defined in [BFP].

3 Two lifts of the parabolic quantum Bruhat graph

3.1 Lifting $\mathrm{QB}(W^J)$ to W_{af}

We construct a parabolic analogue of the lift of $\mathrm{QB}(W)$ to W_{af} given in [LS].

Let $\Omega_J^\infty \subset W_{\mathrm{af}}$ be the subset of elements of the form $w\pi_J(t_\mu)$ with $w \in W^J$ and $\mu \in Q^\vee$ J-superantidominant and J-adjusted. We have $\Omega_J^\infty \subset (W^J)_{\mathrm{af}} \cap W_{\mathrm{af}}^-$, where W_{af}^- is the set of minimum-length coset representatives in W_{af}/W. Impose the Bruhat covers in Ω_J^∞ whenever the connecting root has classical part in $\Phi \setminus \Phi_J$. Then Ω_J^∞ is a subposet of the Bruhat poset W_{af}.

Proposition 3.1 *Every edge in $\mathrm{QB}(W^J)$ lifts to a downward Bruhat cover in Ω_J^∞, and every cover in Ω_J^∞ projects to an edge in $\mathrm{QB}(W^J)$. More precisely:*

1. *For any edge $\lfloor wr_\alpha \rfloor \xleftarrow{\alpha} w$ in $\mathrm{QB}(W^J)$, and $\mu \in Q^\vee$ that is J-superantidominant and J-adjusted with $\pi_J(t_\mu) = zt_\mu$, there is a covering relation $y \lessdot x$ in Ω_J^∞ where*

$$x = wzt_\mu \, , \quad y = xr_{\widetilde{\alpha}} = wr_\alpha t_{\chi\alpha^\vee} zt_\mu \, , \quad \widetilde{\alpha} = z^{-1}\alpha + (\chi + \langle \mu , z^{-1}\alpha \rangle)\delta \in \Phi^{\mathrm{af}-} \, ,$$

and χ is 0 or 1 according as the arrow in $\mathrm{QB}(W^J)$ is of Bruhat or quantum type respectively.

2. *Suppose $y \lessdot x$ is an arbitrary covering relation in Ω_J^∞. Then we can write $x = wzt_\mu$ with $w \in W^J$, $z = z_\mu \in W_J$, and $\mu \in Q^\vee$ J-superantidominant and J-adjusted, as well as $y = xr_\gamma$ with $\gamma = z^{-1}\alpha + n\delta \in \Phi^{\mathrm{af}}$, $\alpha \in \Phi^+ \setminus \Phi_J^+$, and $n \in \mathbb{Z}$. With the notation $\chi := n - \langle \mu\, ,\, z^{-1}\alpha \rangle$, we have*

$$\chi \in \{0,1\}\,, \quad \gamma = z^{-1}\alpha + (\chi + \langle \mu\, ,\, z^{-1}\alpha \rangle)\delta \in \Phi^{\mathrm{af}-}\,;$$

furthermore, there is an edge $wr_\alpha z \xleftarrow{z^{-1}\alpha} wz$ in $\mathrm{QB}(W)$ and an edge $\lfloor wr_\alpha \rfloor \xleftarrow{\alpha} w$ in $\mathrm{QB}(W^J)$, where both edges are of Bruhat type if $\chi = 0$ and of quantum type if $\chi = 1$.

3.2 The Diamond Lemmas

In the following, a dotted (resp. plain) edge represents a quantum (resp. Bruhat) edge in $\mathrm{QB}(W^J)$, whereas a dashed edge can be of both types. Given $w \in W^J$ and $\gamma \in \Phi^+$, define $z \in W_J$ by $r_\theta w = \lfloor r_\theta w \rfloor z$. We now state the Diamond Lemmas for $\mathrm{QB}(W^J)$. They are proved based on the lift of $\mathrm{QB}(W^J)$ to W_{af} in Proposition 3.1 and the fact that such a lemma holds for any Coxeter group [BB].

Lemma 3.2 *Let $\alpha \in \Phi$ be a simple root, $\gamma \in \Phi^+ \setminus \Phi_J^+$, and $w \in W^J$. Then we have the following cases, in each of which the bottom two edges imply the top two edges, and vice versa.*

1. *In the left diagram we assume $\gamma \neq w^{-1}(\alpha)$. In both cases we have $r_\alpha \lfloor wr_\gamma \rfloor = \lfloor r_\alpha wr_\gamma \rfloor$.*

$$\begin{tikzcd}[column sep=small] & r_\alpha\lfloor wr_\gamma\rfloor & \\ r_\alpha w \arrow[ur, "\gamma"] & & \lfloor wr_\gamma\rfloor \arrow[ul, "\lfloor wr_\gamma\rfloor^{-1}(\alpha)"'] \\ & w \arrow[ul, "w^{-1}(\alpha)"] \arrow[ur, "\gamma"'] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & r_\alpha\lfloor wr_\gamma\rfloor & \\ r_\alpha w \arrow[ur, dotted, "\gamma"] & & \lfloor wr_\gamma\rfloor \arrow[ul, "\lfloor wr_\gamma\rfloor^{-1}(\alpha)"'] \\ & w \arrow[ul, "w^{-1}(\alpha)"] \arrow[ur, dotted, "\gamma"'] & \end{tikzcd} \tag{4}$$

2. *Here z is defined as above. We assume $\gamma \neq -w^{-1}(\theta)$ whenever both of the hypothesized edges are quantum ones. In the left diagram, the dashed edge is a quantum (resp. a Bruhat) edge depending on $\langle \gamma^\vee, w^{-1}(\theta) \rangle$ being nonzero (resp. zero). In the right diagram, the dashed edge is a Bruhat (resp. a quantum) edge depending on $\langle \gamma^\vee, w^{-1}(\theta) \rangle$ being nonzero (resp. zero).*

$$\begin{tikzcd}[column sep=small] & \lfloor r_\theta wr_\gamma\rfloor & \\ \lfloor r_\theta w\rfloor \arrow[ur, dashed, "z(\gamma)"] & & \lfloor wr_\gamma\rfloor \arrow[ul, dotted, "-\lfloor wr_\gamma\rfloor^{-1}(\theta)"'] \\ & w \arrow[ul, dotted, "-w^{-1}(\theta)"] \arrow[ur, "\gamma"'] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \lfloor r_\theta wr_\gamma\rfloor & \\ \lfloor r_\theta w\rfloor \arrow[ur, dashed, "z(\gamma)"] & & \lfloor wr_\gamma\rfloor \arrow[ul, dotted, "-\lfloor wr_\gamma\rfloor^{-1}(\theta)"'] \\ & w \arrow[ul, dotted, "-w^{-1}(\theta)"] \arrow[ur, dotted, "\gamma"'] & \end{tikzcd} \tag{5}$$

3.3 Lifting $\mathrm{QB}(W^J)$ to the level-zero weight poset

In [Li], Littelmann introduced a poset related to LS paths for arbitrary (not necessarily dominant) integral weights. Littelmann did not give a precise local description of it. We consider this poset for level-zero weights and characterize its cover relations in terms of the PQBG.

Let $\lambda \in X$ be a fixed dominant weight (cf. Section 2.2 and the notation thereof, e.g., W_J is the stabilizer of λ). We view X as a sublattice of X^0_{af}. Let $X^0_{\mathrm{af}}(\lambda)$ be the orbit $W_{\mathrm{af}}\lambda$.

Definition 3.3 (Level-zero weight poset [Li]) *A poset structure is defined on $X^0_{\mathrm{af}}(\lambda)$ as the transitive closure of the relation*

$$\mu < r_\beta(\mu) \quad \Leftrightarrow \quad \langle \beta^\vee, \mu\rangle > 0\,,$$

where $\beta \in \Phi^{\mathrm{af}+}$. This poset is called the level-zero weight poset for λ.

The cover $\mu \lessdot \nu = r_\beta(\mu)$ of $X^0_{\mathrm{af}}(\lambda)$ is labeled by the root $\beta \in \Phi^{\mathrm{af}+}$. The projection map cl restricts to the map $\mathrm{cl} : X^0_{\mathrm{af}}(\lambda) \to W\lambda$. We identify $W\lambda \simeq W/W_J \simeq W^J$, and consider $\mathrm{QB}(W^J)$. Our main result is the construction of a lift of $\mathrm{QB}(W^J)$ to $X^0_{\mathrm{af}}(\lambda)$. The proof is based on Lemma 3.2.

Theorem 3.4 *Let $\mu \in X^0_{\mathrm{af}}(\lambda)$ and $w := \mathrm{cl}(\mu) \in W^J$. If $\mu \lessdot \nu$ is a cover in $X^0_{\mathrm{af}}(\lambda)$, then its label β is in Φ^+ or $\delta - \Phi^+$. Moreover, $w \to \mathrm{cl}(\nu)$ is an up (respectively down) edge in $\mathrm{QB}(W^J)$ labeled by $w^{-1}(\beta) \in \Phi^+ \setminus \Phi^+_J$ (respectively $w^{-1}(\beta - \delta)$), depending on $\beta \in \Phi^+$ (respectively $\beta \in \delta - \Phi^+$). Conversely, if $w \xrightarrow{\gamma} wr_\gamma = w'$ (respectively $w \overset{\gamma}{\dashrightarrow} \lfloor wr_\gamma \rfloor = w'$) in $\mathrm{QB}(W^J)$ for $\gamma \in \Phi^+ \setminus \Phi^+_J$, then there exists a cover $\mu \lessdot \nu$ in $X^0_{\mathrm{af}}(\lambda)$ labeled by $w(\gamma)$ (respectively $\delta + w(\gamma)$) with $\mathrm{cl}(\nu) = w'$.*

4 Models for KR crystals and the energy function

4.1 Quantum LS paths

Throughout this section, we fix a dominant integral weight $\lambda \in X$, and set $J := \{ i \in I \mid \langle \alpha_i^\vee\,,\, \lambda\rangle = 0\}$.

Definition 4.1 *Let $x, y \in W^J$, and let $\sigma \in \mathbb{Q}$ be such that $0 < \sigma < 1$. A directed σ-path from y to x is, by definition, a directed path*

$$x = w_0 \overset{\gamma_1}{\leftarrow} w_1 \overset{\gamma_2}{\leftarrow} w_2 \overset{\gamma_3}{\leftarrow} \cdots \overset{\gamma_n}{\leftarrow} w_n = y$$

from y to x in $\mathrm{QB}(W^J)$ such that $\sigma\langle \gamma_k^\vee\,,\, \lambda\rangle \in \mathbb{Z}$ for all $1 \le k \le n$.

A quantum LS path of shape λ is a pair $\eta = (\underline{x}\,;\, \underline{\sigma})$ of a sequence $\underline{x} \,:\, x_1, x_2, \ldots, x_s$ of elements in W^J with $x_u \ne x_{u+1}$ for $1 \le u \le s-1$ and a sequence $\underline{\sigma} \,:\, 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ of rational numbers such that there exists a directed σ_u-path from x_{u+1} to x_u for each $1 \le u \le s-1$. Denote by $\mathrm{QLS}(\lambda)$ the set of quantum LS paths of shape λ. We identify an element $\eta = (x_1, x_2, \ldots, x_s\,;\, \sigma_0, \sigma_1, \ldots, \sigma_s) \in \mathrm{QLS}(\lambda)$ with the following piecewise linear, continuous map $\eta : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} X$:

$$\eta(t) = \sum_{u'=1}^{u-1} (\sigma_{u'} - \sigma_{u'-1}) x_{u'} \cdot \lambda + (t - \sigma_{u-1}) x_u \cdot \lambda \quad \text{for } \sigma_{u-1} \le t \le \sigma_u,\ 1 \le u \le s,$$

and set $\mathrm{wt}(\eta) =: \eta(1)$. Following [Li], we define the root operators e_i and f_i for $i \in I_{\mathrm{af}} = I \sqcup \{0\}$ as follows. For $\eta \in \mathrm{QLS}(\lambda)$ and $i \in I_{\mathrm{af}}$, we set

$$H(t) = H_i^\eta(t) := \langle \alpha_i^\vee\,,\, \eta(t)\rangle \quad \text{for } t \in [0,1],$$

$$m = m_i^\eta := \min\{ H_i^\eta(t) \mid t \in [0,1]\};$$

in fact, $m \in \mathbb{Z}_{\le 0}$. If $m = 0$, then $e_i\eta := \mathbf{0}$. If $m \le -1$, then we define $e_i\eta$ by:

$$(e_i\eta)(t) = \begin{cases} \eta(t) & \text{if } 0 \le t \le t_0, \\ r_{i,\,m+1}(\eta(t)) = \eta(t_0) + s_i(\eta(t) - \eta(t_0)) & \text{if } t_0 \le t \le t_1, \\ r_{i,\,m+1} r_{i,\,m}(\eta(t)) = \eta(t) + \widetilde{\alpha}_i & \text{if } t_1 \le t \le 1, \end{cases}$$

where

$$t_1 := \min\bigl\{ t \in [0,1] \mid H(t) = m \bigr\},$$
$$t_0 := \max\bigl\{ t \in [0,t_1] \mid H(t) = m+1 \bigr\},$$

$r_{i,\,n}$ is the reflection with respect to the hyperplane $H_{i,\,n} := \bigl\{ \mu \in \mathbb{R} \otimes_{\mathbb{Z}} X \mid \langle \alpha_i^\vee\,,\,\mu \rangle = n \bigr\}$ for each $n \in \mathbb{Z}$, and

$$\widetilde{\alpha}_i := \begin{cases} \alpha_i & \text{if } i \ne 0, \\ -\theta & \text{if } i = 0, \end{cases} \qquad s_i := \begin{cases} r_i & \text{if } i \ne 0, \\ r_\theta & \text{if } i = 0. \end{cases}$$

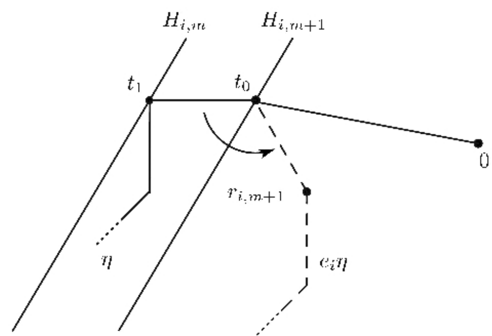

Fig. 1: Root operator e_i.

The definition of $f_i\eta$ is similar. The following theorem is one of our main results.

Theorem 4.2 *(1) The set* $\mathrm{QLS}(\lambda)$ *together with crystal operators* e_i, f_i *for* $i \in I_{\mathrm{af}}$ *and weight function* wt, *becomes a regular crystal with weight lattice* X.

(2) For each $i \in I$, *the crystal* $\mathrm{QLS}(\omega_i)$ *is isomorphic to the crystal basis of* $W(\omega_i)$, *the fundamental representation of level zero, introduced by Kashiwara* [Ka].

(3) Let $\mathbf{i} = (i_1, i_2, \ldots, i_p)$ *be an arbitrary sequence of elements of* I, *and set* $\lambda_{\mathbf{i}} := \omega_{i_1} + \omega_{i_2} + \cdots + \omega_{i_p}$. *There exists a crystal isomorphism* $\Psi_{\mathbf{i}} : \mathrm{QLS}(\lambda_{\mathbf{i}}) \xrightarrow{\sim} \mathrm{QLS}(\omega_{i_1}) \otimes \mathrm{QLS}(\omega_{i_2}) \otimes \cdots \otimes \mathrm{QLS}(\omega_{i_p})$.

Remark 4.3 It is known that the fundamental representation $W(\omega_i)$ (of level zero) is isomorphic to the KR module $W_1^{(i)}$ in the sense of [HKOTT, §2.3] (but for the explicit form of the Drinfeld polynomials of $W(\omega_i)$, see [N, Remark 3.3]), and that it has a global crystal basis (see [Ka, Theorem 5.17]). Furthermore, the crystal basis of $W(\omega_i) \cong W_1^{(i)}$ is unique, up to a nonzero constant multiple (see also [NS3, Lemma 1.5.3]); we call it a (one-column) KR crystal. By the theorem above, the crystal $\mathrm{QLS}(\lambda)$ is a model for the corresponding tensor product of KR crystals.

4.2 Sketch of the proof of Theorem 4.2

First, let us recall the definition of LS paths of shape λ from [Li].

Definition 4.4 *For $\mu, \nu \in X^0_{\mathrm{af}}(\lambda)$ with $\mu > \nu$ (see Definition 3.3) and a rational number $0 < \sigma < 1$, a σ-chain for (μ, ν) is, by definition, a sequence $\mu = \xi_0 \gtrdot \xi_1 \gtrdot \cdots \gtrdot \xi_n = \nu$ of covers in $X^0_{\mathrm{af}}(\lambda)$ such that $\sigma\langle \gamma_k^\vee, \xi_{k-1}\rangle \in \mathbb{Z}$ for all $k = 1, 2, \dots, n$, where γ_k is the label for $\xi_{k-1} \gtrdot \xi_k$.*

Definition 4.5 *An LS path of shape λ is, by definition, a pair $(\underline{\nu}\,;\,\underline{\sigma})$ of a sequence $\underline{\nu} : \nu_1 > \nu_2 > \cdots > \nu_s$ of elements in $X^0_{\mathrm{af}}(\lambda)$ and a sequence $\underline{\sigma} : 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ of rational numbers such that there exists a σ_u-chain for (ν_u, ν_{u+1}) for each $u = 1, 2, \dots, s-1$.*

We denote by $\mathbb{B}(\lambda)$ the set of all LS paths of shape λ. We identify an element

$$\pi = (\nu_1, \nu_2, \dots, \nu_s\,;\,\sigma_0, \sigma_1, \dots, \sigma_s) \in \mathbb{B}(\lambda)$$

with the following piecewise linear, continuous map $\eta : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} X_{\mathrm{af}}$:

$$\pi(t) = \sum_{u'=1}^{u-1} (\sigma_{u'} - \sigma_{u'-1})\nu_{u'} + (t - \sigma_{u-1})\nu_u \quad \text{for } \sigma_{u-1} \le t \le \sigma_u,\ 1 \le u \le s.$$

Let $\mathbb{B}(\lambda)_{\mathrm{cl}} := \{\mathrm{cl}(\pi) \mid \pi \in \mathbb{B}(\lambda)\}$, where $\mathrm{cl}(\pi)$ is the piecewise linear, continuous map $[0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} X$ defined by $(\mathrm{cl}(\pi))(t) := \mathrm{cl}(\pi(t))$ for $t \in [0,1]$. For $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ and $i \in I_{\mathrm{af}}$, we define $e_i\eta$ and $f_i\eta$ in exactly the same way as above. Then it is known from [NS1, NS2] that the same statement as in Theorem 4.2 holds for $\mathbb{B}(\lambda)_{\mathrm{cl}}$. Thus, Theorem 4.2 follows immediately from the following proposition.

Proposition 4.6 *The affine crystals $\mathrm{QLS}(\lambda)$ and $\mathbb{B}(\lambda)_{\mathrm{cl}}$ are isomorphic.*

This proposition is a consequence of Theorem 3.4. Let us show that if $\eta \in \mathrm{QLS}(\lambda)$, then $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$. Here, for simplicity, we assume that $\eta = (x, y\,;\,0, \sigma, 1) \in \mathrm{QLS}(\lambda)$, and $x = w_0 \xleftarrow{\gamma_1} w_1 \xleftarrow{\gamma_2} w_2 = y$ is a directed σ-path from y to x. Take an arbitrary $\mu \in X^0_{\mathrm{af}}(\lambda)$ such that $\mathrm{cl}(\mu) = y$. By applying Theorem 3.4 to $w_1 \xleftarrow{\gamma_2} w_2 = y$, we obtain a cover $\nu_1 \gtrdot \mu$ for some $\nu_1 \in X^0_{\mathrm{af}}(\lambda)$ with $\mathrm{cl}(\nu_1) = w_1$. Then, by applying Theorem 3.4 to $x = w_0 \xleftarrow{\gamma_1} w_1$, we obtain a cover $\nu \gtrdot \nu_1$ for some $\nu \in X^0_{\mathrm{af}}(\lambda)$ with $\mathrm{cl}(\nu) = x$. Thus we get a sequence $\nu \gtrdot \nu_1 \gtrdot \mu$ of covers in $X^0_{\mathrm{af}}(\lambda)$. It can be easily seen that this is a σ-chain for (ν, μ), which implies that $\pi = (\nu, \mu\,;\,0, \sigma, 1) \in \mathbb{B}(\lambda)$. Therefore, $\eta = \mathrm{cl}(\pi) \in \mathbb{B}(\lambda)_{\mathrm{cl}}$. The reverse inclusion can be shown similarly.

4.3 Description of the energy function in terms of quantum LS paths

Recall the notation in Theorem 4.2 (2); for simplicity, we set $\lambda := \lambda_{\mathbf{i}}$. In [NS5], Naito and Sagaki introduced a degree function $\mathrm{Deg}_\lambda : \mathbb{B}(\lambda)_{\mathrm{cl}} = \mathrm{QLS}(\lambda) \to \mathbb{Z}_{\le 0}$, and proved that Deg_λ is identical to the *energy function* [HKOTT, HKOTY] on the tensor product $\mathbb{B}(\omega_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\omega_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\omega_{i_p})_{\mathrm{cl}} = \mathrm{QLS}(\omega_{i_1}) \otimes \mathrm{QLS}(\omega_{i_2}) \otimes \cdots \otimes \mathrm{QLS}(\omega_{i_p})$ (which is isomorphic to the corresponding tensor product of

KR crystals; see Remark 4.3) via the isomorphism $\Psi_{\mathbf{i}}$. The function $\mathrm{Deg}_\lambda : \mathbb{B}(\lambda)_{\mathrm{cl}} \to \mathbb{Z}_{\leq 0}$ is described in terms of $\mathrm{QB}(W^J)$ as follows. For $x,\, y \in W^J$ let

$$\mathbf{d} : x = w_0 \xleftarrow{\beta_1} w_1 \xleftarrow{\beta_2} \cdots \xleftarrow{\beta_n} w_n = y$$

be a shortest directed path from y to x, and define

$$\mathrm{wt}(\mathbf{d}) := \sum_{\substack{1 \leq k \leq n \text{ such that} \\ w_{k-1} \xleftarrow{\beta_k} w_k \text{ is a down arrow}}} \beta_k^\vee .$$

The value $\langle \mathrm{wt}(\mathbf{d})\,,\, \lambda \rangle$ does not depend on the choice of a shortest directed path $\mathbf{d}$ from y to x, and

Theorem 4.7 *Let* $\eta = (x_1,\, x_2,\, \ldots,\, x_s\,;\, \sigma_0,\, \sigma_1,\, \ldots,\, \sigma_s) \in \mathrm{QLS}(\lambda) = \mathbb{B}(\lambda)_{\mathrm{cl}}$. *Then,*

$$\mathrm{Deg}(\eta) = -\sum_{u=1}^{s-1} (1 - \sigma_u) \big\langle \lambda,\, \mathrm{wt}(\mathbf{d}_u) \big\rangle, \tag{6}$$

where $\mathbf{d}_u$ *is a shortest directed path from* x_{u+1} *to* x_u.

4.4 The quantum alcove model

The quantum alcove model is a generalization of the alcove model of the first author and Postnikov [LP1, LP2], which, in turn, is a discrete counterpart of the Littelmann path model [Li]. For the affine Weyl group terminology below, we refer to [H]. Fix a dominant weight λ. We say that two alcoves are adjacent if they are distinct and have a common wall. Given a pair of adjacent alcoves A and B, we write $A \xrightarrow{\beta} B$ for $\beta \in \Phi$ if the common wall is orthogonal to β and β points in the direction from A to B.

Definition 4.8 [LP1] *The sequence of roots* $(\beta_1, \beta_2, \ldots, \beta_m)$ *is called a* λ-chain *if*

$$A_0 = A_\circ \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_m} A_m = A_\circ - \lambda$$

is a shortest sequence of alcoves from the fundamental alcove $A_\circ$ *to its translation by* $-\lambda$.

The *lex* λ*-chain* Γ_{lex} is a particular λ-chain defined in [LP2, Section 4]. Given an arbitrary λ-chain $\Gamma = (\beta_1, \beta_2, \ldots, \beta_m)$, let $r_i = r_{\beta_i}$ and define the *level sequence* $(l_1, \ldots, l_m)$ of Γ by $l_i = |\{ j \geq i \mid \beta_j = \beta_i \}|$.

Definition 4.9 [LeL1] *A (possibly empty) finite subset* $J = \{ j_1 < j_2 < \cdots < j_s \}$ *of* $\{1, \ldots, m\}$ *is a* Γ-admissible subset *if we have the following path in* $\mathrm{QB}(W)$*:*

$$1 \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} r_{j_1} r_{j_2} \xrightarrow{\beta_{j_3}} \cdots \xrightarrow{\beta_{j_s}} r_{j_1} r_{j_2} \cdots r_{j_s} = \kappa(J)\,. \tag{7}$$

Let $\mathcal{A}^\Gamma(\lambda)$ *be the collection of* Γ*-admissible subsets. Define the level of* $J \in \mathcal{A}^\Gamma(\lambda)$ *by* $\mathrm{level}(J) = \sum_{j \in J^-} l_j$*, where* $J^- \subseteq J$ *corresponds to the down steps in Bruhat order in* (7)*.*

Theorem 4.10 *There is an isomorphism of graded classical crystals* $\Xi : \mathcal{A}^{\Gamma_{\mathrm{lex}}}(\lambda) \to \mathrm{QLS}(\lambda)$.

The map Ξ is the following forgetful map. Given the path (7), based on the structure of Γ_{lex} we select a subpath, and project its elements under $W \to W/W_\lambda$ where W_λ is the stabilizer of λ in W, thereby obtaining a quantum LS path. The inverse map is more subtle, and is based on the so-called *tilted Bruhat theorem* of [LNSSS1]; this is a QBG analogue of the minimum-length Deodhar lift [De] $W/W_\lambda \to W$.

An affine crystal structure was defined on $\mathcal{A}^{\Gamma_{\text{lex}}}(\lambda)$ in [LeL1]. We show that Ξ is an affine crystal isomorphism, up to removing some f_0-arrows from $\text{QLS}(\lambda)$. The remaining arrows are called *Demazure arrows* in [ST], as they correspond to the arrows of a certain affine Demazure crystal, cf. [FSS]. Moreover the above bijection sends the level statistic to the Deg statistic of (6). Thus, we have proved that $\mathcal{A}^{\Gamma_{\text{lex}}}(\lambda)$ is also a model for KR crystals.

In [LeL2], we show that all the affine crystals $\mathcal{A}^{\Gamma}(\lambda)$, for various Γ, are isomorphic. Making particular choices in classical types, we can translate the level statistic into a so-called *charge statistic* on sequences of the corresponding *Kashiwara–Nakashima columns* [KN]. In type A, we recover the classical Lascoux–Schützenberger charge [LSc]. Type C was worked out in [Le2, LeS], while type B is considered in [BL].

5 Macdonald polynomials

The symmetric Macdonald polynomials $P_\lambda(x;q,t)$ [M] are a remarkable family of orthogonal polynomials associated to any affine root system (where λ is a dominant weight for the canonical finite root system), which depend on parameters q, t. They generalize the corresponding irreducible characters, which are recovered upon setting $q = t = 0$. For untwisted affine root systems the Ram–Yip formula [RY] expresses $P_\lambda(x;q,t)$ in terms of all subsequences of any λ-chain Γ (cf. Definition 4.8). In [Le2], it was shown that the Ram–Yip formula takes the following simple form for $t = 0$:

$$P_\lambda(x;q,0) = \sum_{J \in \mathcal{A}^{\Gamma}(\lambda)} q^{\text{level}(J)}\, x^{\text{wt}(J)}\,, \tag{8}$$

where $\text{wt}(J)$ is a weight associated with J. By using the results in Section 4.4, we can write the right-hand side of (8) as a sum over the corresponding tensor product of KR crystals. It follows that $P_\lambda(x;q,0) = X_\lambda(q)$. Furthermore, we can express the nonsymmetric Macdonald polynomial $E_{w\lambda}(x;q,0)$, for $w \in W$, in a similar way to (8), by summing over those $J \in \mathcal{A}^{\Gamma}(\lambda)$ with $\kappa(J) \leq w$ (in Bruhat order), where $\kappa(J)$ was defined in (7). This formula can be derived both by induction, based on Demazure operators, and from the Ram–Yip formula (in this case, for nonsymmetric Macdonald polynomials); however, the latter derivation is more involved than the one of (8), as it uses the transformation on admissible subsets defined in [Le1, Section 5.1].

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