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Descent sets for oscillating tableaux

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Abstract. The descent set of an oscillating (or up-down) tableau is introduced. This descent set plays the same role in the representation theory of the symplectic groups as the descent set of a standard tableau plays in the representation theory of the general linear groups. In particular, we show that the descent set is preserved by Sundaram's correspondence.

This gives a direct combinatorial interpretation of the branching rules for the defining representations of the symplectic groups; equivalently, for the Frobenius character of the action of a symmetric group on an isotypic subspace in a tensor power of the defining representation of a symplectic group.

Résumé. Dans cet article, nous définissons la notion d'ensemble de descentes pour un tableau oscillant. Ces descentes sont analogues aux descentes d'un tableau standard dans la théorie des représentations des groupes généraux linéaires. Nous montrons que la correspondance de Sundaram préserve cet ensemble et nous donnons une interprétation combinatoire directe des règles de branchement pour la représentation des groupes symplectiques. Enfin, nous décrivons combinatoirement les caractères de Frobenius associés à l'action du groupe symétrique sur les composantes isotypiques du produit tensoriel des représentations d'un groupe symplectique.

Keywords: oscillating tableaux, quasisymmetric expansion, Frobenius character

1 Introduction

There is a well developed combinatorial theory associated with the polynomial representations of the algebraic groups $\mathrm{GL}(n)$. For a textbook treatment we refer to Stanley's book [6]. At its core is the Robinson-Schensted correspondence

$$\mathrm{RS} : \{1, \dots, n\}^r \to \bigcup_{\substack{\mu \in P(r) \\ \ell(\mu) \leq n}} SSYT(\mu, n) \times SYT(\mu),$$

where the union is over the set of partitions μ of r into at most n parts, $SSYT(\mu, n)$ is the set of semistandard Young tableaux of shape μ and entries no larger than n and $SYT(\mu)$ is the set of standard Young tableaux of shape μ.

This correspondence can be regarded as a combinatorial counterpart of the direct sum decomposition of the r-th tensor power of the defining representation V of $\mathrm{GL}(n)$ as a $\mathrm{GL}(n) \times \mathfrak{S}_r$ module, where $\mathrm{GL}(n)$

acts diagonally and the symmetric group acts by permuting tensor coordinates:

$$\otimes^r V \cong \bigoplus_{\substack{\mu \in P(r) \\ \ell(\mu) \leq n}} V(\mu) \otimes S(\mu), \tag{1}$$

where $V(\mu)$ and $S(\mu)$ are irreducible representations of $\mathrm{GL}(n)$ and $\mathfrak{S}_r$ and respectively. Namely, the character of $V(\mu)$ is given by the Schur function associated to μ,

$$s_\mu(x_1, \ldots, x_n) = \sum_{T \in SSYT(\mu,n)} \mathbf{x}^T. \tag{2}$$

and there is a basis of $S(\mu)$ indexed by the elements of $SYT(\mu)$.

A remarkable property of the Robinson-Schensted correspondence, due to Schützenberger [5, Remarque 2], is that the descent set of a word equals the descent set of the standard Young tableau to which it is mapped. In this article we prove an analogous result for the symplectic groups $\mathrm{Sp}(2n)$, Theorem 5.1. As a corollary we obtain Theorem 3.8, which is a symplectic version of the expansion of Schur functions in terms of the fundamental quasisymmetric functions L_α:

$$s_\mu = \sum_{T \in SYT(\mu)} L_{\mathrm{co}\,\mathrm{Des}(T)}, \tag{3}$$

where we associate with a subset $D = \{d_1 < d_2 < \cdots < d_{k-1}\}$ of $\{1, \ldots, r-1\}$ the composition $\mathrm{co}\, D = \{d_1, d_2 - d_1, \ldots, r - d_{k-1}\}$ of r, and

$$L_{\mathrm{co}\, D} = \sum_{\substack{i_1 \leq \cdots \leq i_r \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

2 The general setting

Let V be a finite dimensional rational representation of a complex reductive algebraic group G. Let Λ be the set of isomorphism classes of irreducible rational representations of G and let $V(\mu)$ be the representation corresponding to $\mu \in \Lambda$. For example, when G is the general linear group $\mathrm{GL}(n)$ we can identify Λ with the set of non-decreasing sequences of integers of length n. Also, when G is the symplectic group $\mathrm{Sp}(2n)$ we can identify Λ with the set of partitions with at most n parts. In both cases, the trivial representation corresponds to the empty partition and the defining representation corresponds to the partition 1.

Let $\mathfrak{S}_r$ be the symmetric group permuting r elements. We identify the isomorphism classes of its irreducible representations with $P(r)$ and denote the representation corresponding to $\lambda \in P(r)$ by $S(\lambda)$.

For each $r \geq 0$, the tensor power $\otimes^r V$ is a rational representation of $G \times \mathfrak{S}_r$, where G acts diagonally and $\mathfrak{S}_r$ acts by permuting tensor coordinates. This representation is completely reducible. Decomposing it as a representation of G we obtain the following analogue of Equation (1):

$$\otimes^r V \cong \bigoplus_{\mu \in \Lambda} V(\mu) \otimes U(r, \mu). \tag{4}$$

The isotypic space $U(r,\mu)$ inherits the action of $\mathfrak{S}_r$ and therefore decomposes as

$$U(r,\mu) \cong \bigoplus_{\lambda \in P(r)} A(\lambda,\mu) \otimes S(\lambda). \tag{5}$$

Thus, the Frobenius character of $U(r,\mu)$ is

$$\operatorname{ch} U(r,\mu) = \sum_{\lambda \in P(r)} a(\lambda,\mu) s_\lambda, \tag{6}$$

where $a(\lambda,\mu) = \dim A(\lambda,\mu)$ and s_λ is the Schur function associated to λ.

When V is the defining representation of $\mathrm{GL}(n)$ the coefficient $a(\lambda,\mu)$ equals 1 for $\lambda = \mu$ and vanishes otherwise. Thus, in this case the Frobenius character is simply s_μ. For the defining representation of the symplectic group $\mathrm{Sp}(2n)$ the coefficients $a(\lambda,\mu)$ were determined by Sundaram [7]. In general it is a difficult problem to determine these characters explicitly.

We would like to advertise a new approach to describe the Frobenius character, using descent sets. It appears that the proper setting for a general definition of descent set is the combinatorial theory of crystal graphs. This theory is an off-shoot of the representation theory of Drinfeld-Jimbo quantised enveloping algebras. However, for our purposes a few notions from this theory suffice. For a textbook treatment we refer to the book by Hong and Kang [2].

For each rational representation V of a connected reductive algebraic group there is a crystal. A crystal is a combinatorial framework for the representation. The vector space V is replaced by a set of cardinality $\dim(V)$. The raising and lowering operators, which are certain linear operators on V, are replaced by partial functions on the set. It is common practice to represent these partial functions by directed graphs. Each vertex of the crystal has a weight and the sum of these weights is the character of the representation, see Equations (2) and (9). Isomorphic modules correspond to crystal graphs that are isomorphic as coloured digraphs and the module is irreducible if and only if the graph is connected.

For example, the crystal graph corresponding to the defining representation of $\mathrm{GL}(n)$ is

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n. \tag{7}$$

while the crystal graph corresponding to the defining representation of $\mathrm{Sp}(2n)$ is

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n \xrightarrow{n} -n \xrightarrow{n-1} -(n-1) \xrightarrow{n-2} \cdots \xrightarrow{2} -2 \xrightarrow{1} -1. \tag{8}$$

A vertex in a crystal graph with no in-coming arcs is a highest weight vertex. Each connected component contains a unique highest weight vertex. The weight of this vertex is the weight of the representation it corresponds to. Finally, there is a (relatively) simple way to construct the crystal graph of a tensor product of two modules given their individual crystal graphs. Thus, the highest weight vertices of the crystal corresponding to $\otimes^r V$ can be regarded as words of length r with letters being vertices of the crystal corresponding to V.

Many aspects of the classical theory can be generalised at least to the crystals corresponding to the classical groups. In particular, explicit analogues of the Robinson-Schensted correspondence were found for the defining representations of the symplectic groups $\mathrm{Sp}(2n)$ as well as for the odd and even orthogonal groups, see [4].

Analogous to the expansion in Equation (3) of s_μ we can now state the fundamental property we require for a general descent set:

Definition 2.1. A function Des which assigns a subset of $\{1, 2, \ldots, r-1\}$ to each highest weight vertex of the crystal graph corresponding to $\otimes^r V$ is a *descent function* if it satisfies

$$\operatorname{ch} U(r, \mu) = \sum_w L_{\operatorname{co}\operatorname{Des}(w)},$$

where the sum is over all highest weight vertices of weight μ.

In particular, when V is the defining representation of the $\operatorname{GL}(n)$, the highest weight vertices of the corresponding crystal graph can be taken to be the reversals of the Yamanouchi words (or lattice permutations) in $\{1, \ldots, n\}^r$. It is straightforward to check that the restriction of the Robinson-Schensted correspondence to these words is a bijection to standard Young tableaux such that the weight of the word is mapped to the shape of the tableau. Therefore,

$$\operatorname{ch} U(r, \mu) = s_\mu = \sum_{T \in SYT(\mu)} L_{\operatorname{co}\operatorname{Des}(T)} = \sum_w L_{\operatorname{co}\operatorname{Des}(w)},$$

the last summation being over all highest weight vertices of weight μ. Thus the usual descent set of a word, $\operatorname{Des}(w_1 w_2 \ldots w_r) = \{k \,|w_k > w_{k+1}\}$, is a descent set in the sense of Definition 2.1. We remark that in terms of the crystal graph (7) above, a highest weight vertex $w_1 w_2 \ldots w_r$ has a descent at position k if and only if there is a (nontrivial) directed path from w_{k+1} to w_k in the crystal graph.

In the remaining sections we will show that we can define a descent function for the symplectic group in almost the same way. In the next section we define a notion of descent set for oscillating tableaux which will permit us to give the desired combinatorial interpretation of $\operatorname{ch} U(r, \mu)$ in Theorem 3.8. Section 4 describes the tools necessary to prove this theorem, namely symplectic Littlewood-Richardson tableaux and a correspondence due to Sundaram. In Section 5 we give the proof by analysing this correspondence in detail.

3 Oscillating tableaux and descents

In the case of the defining representation of the symplectic group $\operatorname{Sp}(2n)$ the vertices of the crystal graph corresponding to $\otimes^r V$ are words $w_1 w_2 \ldots w_r$ in $\{\pm 1, \ldots, \pm n\}^r$. The weight of a vertex is the tuple $(\mu_1, \ldots, \mu_n)$, where μ_i is the number of letters i minus the number of letters $-i$ in w. The vertex is a highest weight vertex if for any $k \le r$, the weight of $w_1, \ldots, w_k$ is a partition, i.e., $\mu_1 \ge \mu_2 \cdots \ge \mu_n$.

Definition 3.1. A highest weight vertex $w_1 w_2 \ldots w_r$ in the crystal graph corresponding to $\otimes^r V$ has a *descent* at position k if there is a (nontrivial) directed path from w_k to w_{k+1} in the crystal graph (8).

The *descent set* of w is

$$\operatorname{Des}(w) = \{k \mid k \text{ is a descent of } w\}.$$

We can now state our main result:

Theorem 3.2. *Let V be the defining representation of the symplectic group $\operatorname{Sp}(2n)$. Then the Frobenius character of $\otimes^r V$ is*

$$\operatorname{ch} U(r, \mu) = \sum_w L_{\operatorname{co}\operatorname{Des}(w)},$$

where the sum is over all highest weight vertices of the corresponding crystal graph.

To prove this theorem we will first rephrase it in terms of n-symplectic oscillating tableaux, also known as up-down-tableaux, which are in bijection with the highest weight vertices of $\otimes^r V$:

Definition 3.3. An *oscillating tableau* of length r and (final) *shape* μ is a sequence of partitions

$$(\emptyset = \mu^0, \mu^1, \ldots, \mu^r = \mu)$$

such that the Ferrers diagrams of two consecutive partitions differ by exactly one cell.

The k-th step (going from μ^{k-1} to μ^k) is an *expansion* if a cell is added and a *contraction* if a cell is deleted. We will refer to the cell that is added or deleted in the k-th step as b_k.

The oscillating tableau $O = (\mu^0, \mu^1, \ldots, \mu^r)$ is n*-symplectic* if every partition μ^i has at most n non-zero parts.

The oscillating tableau corresponding to a highest weight vertex $w_1 w_2 \ldots w_r$ is given by the sequence of weights of its initial factors $w_1, w_1 w_2, w_1 w_2 w_3, \ldots, w_1 w_2 \ldots w_r$.

Example 3.4. The 1-symplectic oscillating tableaux of length three are

$$(\emptyset, 1, 2, 3), \quad (\emptyset, 1, 2, 1), \quad \text{and} \quad (\emptyset, 1, \emptyset, 1).$$

The corresponding words are

$$1\,1\,1, \quad 1\,1\,\text{-}1, \quad \text{and} \quad 1\,\text{-}1\,1.$$

As a running example, we will use the oscillating tableau

$$O = (\emptyset, 1, 11, 21, 2, 1, 2, 21, 211, 21)$$

which has length 9 and shape 21. It is 3-symplectic (since no partition has four parts) but it is not 2-symplectic (since there is a partition with three parts). The corresponding word is 1 2 1 -2 -1 1 2 3 -3 with descent set $\{1, 3, 4, 6, 7, 8\}$.

Definition 3.5. An oscillating tableau O has a *descent* at position k in any of the following three cases:

1. Step k is an expansion and step $k+1$ is a contraction,
2. steps k and $k+1$ are both expansions and b_k is in a row strictly above b_{k+1}.
3. steps k and $k+1$ are both contractions and b_k is in a row strictly below b_{k+1},

where we view all Ferrers diagrams in English notation. The *descent set* of O is

$$\mathrm{Des}(O) = \{k \mid k \text{ is a descent of } O\}.$$

Example 3.6. The descent set of the oscillating tableau O from Example 3.4 is

$$\mathrm{Des}(O) = \{1, 3, 4, 6, 7, 8\}.$$

The definition for the descent set of an oscillating tableau is such that the bijection between highest weight vertices and oscillating tableaux has the following property:

Proposition 3.7. *The descent set of an n-symplectic oscillating tableau coincides with the descent set of the corresponding highest weight vertex of the crystal graph of $\otimes^r V$.*

Thus it suffices to prove the following variant of Theorem 3.2:

Theorem 3.8. *Let V be the defining representation of the symplectic group $\mathrm{Sp}(2n)$. Then the Frobenius character of $\otimes^r V$ is*

$$\operatorname{ch} U(r,\mu) = \sum_{O} L_{\operatorname{co}\operatorname{Des}(O)},$$

where the sum is over all n-symplectic oscillating tableaux of length r and shape μ.

Let us motivate Definition 3.5 in a second way. Note first that a standard Young tableau T can be regarded as an oscillating tableau O where every step is an expansion and the cell containing k in T is added during the k-th step in O. In this case

$$\operatorname{Des}(O) = \operatorname{Des}(T),$$

so the definition of descents for oscillating tableaux is an extension of the one for standard tableaux.

In Sundaram's correspondence an arbitrary oscillating tableau O is first transformed into a fixed-point-free involution ι and a partial Young tableau T, that is, a filling of a Ferrers shape with all entries distinct and increasing in rows and columns. There is a natural extension of descents for these objects:

Definition 3.9. The *descent set* of an involution ι (or in fact any permutation) defined on a set A is

$$\operatorname{Des}(\iota) = \{k : k, k+1 \in A,\ \iota(k) > \iota(k+1)\}.$$

For a partial Young tableau T whose set of entries is A, the *descent set* is

$$\operatorname{Des}(T) = \{k : k, k+1 \in A,\ k+1 \text{ is in a row below } k\}.$$

The definition for descents in oscillating tableaux is constructed so that the descent set of the oscillating tableau contains the union of the descent set of the associated partial tableau and permutation.

4 The correspondences of Berele and Sundaram

One of our main tools for proving Theorem 3.8 will be a bijection Sun due to Sundaram [7, 8]. In combination with a correspondence due to Berele [1], which is a combinatorial counterpart of the isomorphism in Equation (4) in the case where V is the defining representation of $\mathrm{Sp}(2n)$, Sundaram's bijection can be regarded as a combinatorial counterpart of the isomorphism in Equation (5). In this section we define the objects involved, the bijection itself will be described in detail in the next section.

Definition 4.1. Let u be a word with letters in $\mathbb{N}$. Then u is a *Yamanouchi word* (or *lattice permutation*) if in any initial factor $u_1 u_2 \ldots u_k$ and for each i, there are at least as many occurrences of i in u as there are occurrences of $i+1$.

The *weight* β of a lattice permutation u is the partition $\beta = (\beta_1 \geq \beta_2 \geq \cdots)$, where β_i is the number of occurrences of the letter i in u.

For a skew semistandard Young tableau S the *reverse reading word* $w^{rev}(S)$ is the reversal of the word obtained by concatenating the rows from bottom to top.

A skew semistandard Young tableau S of shape λ/μ is an *n-symplectic Littlewood-Richardson tableau of weight β* if

- its reverse reading word is a lattice permutation of weight β, where β is a partition with all columns having even height, and
- entries in row $n+i+1$ of λ are strictly larger than $2i+1$ for $0 \le i \le \frac{1}{2}\ell(\beta)$.

The number of n-symplectic Littlewood-Richardson tableaux of shape λ/μ and weight β is denoted by $c^{\lambda}_{\mu,\beta}(n)$.

For $\ell(\lambda) \le n+1$ and β a partition with all columns having even height, the number $c^{\lambda}_{\mu,\beta}(n)$ is the usual Littlewood-Richardson coefficient. This is trivial for $\ell(\lambda) \le n$, and follows from the correspondence Sun described below for $\ell(\lambda) = n+1$.

Note that we are only interested in the case that the length of μ is at most n. In this case the restriction on the size of the entries is equivalent to the condition given by Sundaram that $2i+1$ appears no lower than row $n+i$ for $0 \le i \le \frac{1}{2}\ell(\beta)$.

As an example, when $\mu = (1)$ there is a single 1-symplectic Littlewood-Richardson tableau of weight $\beta = (1,1)$:

$$\begin{array}{|c|c|}\hline & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$$

We alert the reader that there is a typo both in [7, Definition 9.5] and [8, Definition 3.9], where the range of indices is stated as $1 \le i \le \frac{1}{2}\ell(\beta)$. With this definition,

$$\begin{array}{|c|}\hline \\ \hline 1 \\ \hline 2 \\ \hline\end{array}$$

would also be 1-symplectic, since $\beta = (1,1)$ and 3 does not appear at all. However, there are only two 1-symplectic oscillating tableaux of length 3 and shape (1), and there are two standard Young tableaux of shape $(2,1)$. Thus, if the tableau above would also be 1-symplectic, Theorem 4.2 below would fail.

We now have all the definitions in place to explain the domain and range of the correspondence Sun.

Theorem 4.2 ([7, Theorem 9.4])**.** *There is an explicit bijection* Sun *between n-symplectic oscillating tableaux of length r and shape μ and pairs (Q,S), where*

- *Q is a standard tableau of shape λ, with $|\lambda| = r$, and*
- *S is an n-symplectic Littlewood-Richardson tableau of shape λ/μ and weight β, where $|\beta| = r - |\mu|$ and has even columns.*

For completeness, let us point out the relation between Sundaram's bijection and Berele's correspondence. This correspondence involves the following objects due to King [3], indexing the irreducible representations of the symplectic group $\mathrm{Sp}(2n)$:

Definition 4.3. An *n-symplectic semistandard tableau of shape μ* is a filling of μ with letters from $1 < -1 < 2 < -2 < \cdots < n < -n$ such that

- entries in rows are weakly increasing,
- entries in columns are strictly increasing, and

- the entries in row i are greater than or equal to i, in the above ordering.

Denoting the set of n-symplectic oscillating tableaux of length r and final shape μ by $Osc(r,n,\mu)$ and the set of n-symplectic semistandard tableaux of shape μ by $K(\mu,n)$, Berele's correspondence is a bijection

$$\mathrm{Ber} : \{\pm 1, \ldots, \pm n\}^r \to \bigcup_{\substack{\mu\in P(r)\\ \ell(\mu)\le n}} K(\mu,n) \times Osc(r,n,\mu).$$

In analogy to Equation (2), the character of the representation $V(\mu)$ of the symplectic group $\mathrm{Sp}(2n)$ is

$$sp_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = \sum_{T\in K(\mu,n)} \mathbf{x}^T. \tag{9}$$

Now consider an n-symplectic oscillating tableau as a word in the ordered alphabet $1 < -1 < 2 < -2 < \cdots < n < -n$, as described just after Definition 3.3. We can then apply the Robinson-Schensted correspondence to obtain a semistandard Young tableau P_{RS} in this alphabet and a (usual) standard Young tableau Q_{RS}. Alternatively, we can compose Berele's correspondence with Sundaram's bijection to obtain a triple $(P_{\mathrm{Ber}}, Q_{\mathrm{Sun}}, S_{\mathrm{Sun}})$. It then turns out that $Q_{\mathrm{RS}} = Q_{\mathrm{Sun}}$. This implies that for each standard Young tableau Q_{Sun} we have a correspondence $P_{\mathrm{RS}} \mapsto (P_{\mathrm{Ber}}, S_{\mathrm{Sun}})$. Moreover, this correspondence is independent of the choice of Q_{Sun}. One can then show the following theorem:

Theorem 4.4 ([7, Theorem 12.1])**.** *For all λ, μ the coefficient $a(\lambda,\mu)$ is given by*

$$a(\lambda,\mu) = \sum_\beta c^\lambda_{\mu,\beta}(n)$$

where the sum is over the partitions β of $|\lambda| - |\mu|$ having only columns of even length.

5 Correspondences and the proof of the main result

In light of Sundaram's results, we claim that to prove Theorem 3.8 it suffices to demonstrate the following:

Theorem 5.1. *Let $\mathrm{Sun}(O) = (Q,S)$. Then*

$$\mathrm{Des}(O) = \mathrm{Des}(Q).$$

Proof of Theorem 3.8: We have

$$\begin{aligned}
\sum_O L_{\mathrm{co\,Des}(O)} &= \sum_{(Q,S)} L_{\mathrm{co\,Des}(Q)} && \text{(by Theorems 4.2 and 5.1)}\\
&= \sum_{|Q|=r} \sum_\beta c^{\mathrm{sh}\,Q}_{\mu,\beta}(n) L_{\mathrm{co\,Des}(Q)} && \text{(by Definiton 4.1)}\\
&= \sum_{|Q|=r} a(\mathrm{sh}(Q),\mu) L_{\mathrm{co\,Des}(Q)} && \text{(by Theorem 4.4)}\\
&= \sum_{\lambda\in P(r)} a(\lambda,\mu) \sum_{\mathrm{sh}(Q)=\lambda} L_{\mathrm{co\,Des}(Q)} &&\\
&= \mathrm{ch}\, U(r,\mu) && \text{(by Equations (3) and (6))}
\end{aligned}$$

which is the desired conclusion. $\square$

In order to prove Theorem 5.1, we will need to analyse the bijection Sun in detail. Sundaram described Sun as the composition of several bijections. Specifically, Sun is the composition

$$O \overset{\mathrm{Sun}_1}{\mapsto} (\iota, T) \mapsto (\mathrm{RS}(\iota), T) \overset{\mathrm{Sun}_2}{\mapsto} (Q, S)$$

where Sun_1 and Sun_2 are described below and RS denotes the Robinson-Schensted correspondence. We will prove Theorem 5.1 by tracking their effect on the descent set.

5.1 Sundaram's first bijection

We now describe Sundaram's first bijection which we will denote Sun_1. It maps an oscillating tableau O to a pair (ι, T) where ι is a fixed-point-free involution, T is a partial Young tableau (that is, a filling of a Ferrers shape with all entries distinct and increasing in rows and columns), and the entries of ι and T are complementary sets.

Let $O = (\emptyset = \mu^0, \mu^1, \dots, \mu^r)$ be an oscillating tableau. We then construct a sequence of pairs (ι_k, T_k) for $0 \le k \le r$, such that $\mathrm{sh}(T_k) = \mu^k$ and the set of entries of the pair (ι_k, T_k) is $\{1, \dots, k\}$, all entries being distinct.

Both ι_0 and T_0 are empty. For $k > 0$ the pair (ι_k, T_k) is constructed from the pair (ι_{k-1}, T_{k-1}) and the k-th step in the oscillating tableau:

- If the k-th step is an expansion then $\iota_k = \iota_{k-1}$ and T_k is obtained from T_{k-1} by putting k in cell b_k.
- Otherwise, if the k-th step is a contraction then take T_{k-1} and bump out (using Robinson-Schensted column deletion) the entry in cell b_k to get a letter x and the standard tableau T_k. In other words, let x and T_k be such that column inserting x into T_k yields T_{k-1}. The involution ι_k is then given by adjoining the transposition (k, x) to ι_{k-1}.

The result of the bijection is the final pair (ι_r, T_r).

Lemma 5.2 (Sundaram [7, Lemma 8.7])**.** *The map Sun_1 is a bijection between oscillating tableaux of length r and shape μ and pairs (ι, T) where*

- *ι is a fixed-point-free involution of a set $A \subseteq \{1, \dots, r\}$, and*
- *T is a partial tableau of shape μ such that its set of entries is $\{1, \dots, r\} \setminus A$.*

Example 5.3. Starting with the oscillating tableau O from Example 3.4 we get the following sequence of pairs (T_k, i_k), where in the diagram below we only list each pair of the involution once when it is produced by the algorithm.

k:	0	1	2	3	4	5	6	7	8	9
T_k:	$\emptyset$	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 3&6\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 3&6\\\hline 7\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 3&6\\\hline 7\\\cline{1-1} 8\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 3&6\\\hline 7\\\cline{1-1}\end{array}$
ι_k:	$\emptyset$				$(2,4)$	$(1,5)$				$(8,9)$

So the final output is the involution

$$\iota = (1,5)(2,4)(8,9) = \begin{array}{cccccc} 1 & 2 & 4 & 5 & 8 & 9 \\ 5 & 4 & 2 & 1 & 9 & 8 \end{array}$$

and the partial tableau

$$T = \begin{array}{|c|c|} \hline 3 & 6 \\ \hline 7 \\ \cline{1-1} \end{array}$$

with $\iota \uplus T = \{1, \ldots, 9\}$.

We wish to define the descent set of a pair (ι, T) in such a way that this map preserves descents. Let us define for any pair A and B

$$\mathrm{Des}(A/B) = \{k : k \in A \text{ and } k+1 \in B\}$$

and define

$$\mathrm{Des}(\iota, T) = \mathrm{Des}(\iota) \uplus \mathrm{Des}(T) \uplus \mathrm{Des}(T/\iota),$$

where we use Definition 3.9 for the descent sets of ι and T. Thus, in our running example, $\mathrm{Des}(T/\iota) = \{3, 7\}$ and $\mathrm{Des}(\iota, T) = \{1, 3, 4, 6, 7, 8\}$, which coincides with $\mathrm{Des}(O)$.

We can now take our first step in proving Theorem 5.1.

Proposition 5.4. *Let O be an oscillating tableau and suppose that* $\mathrm{Sun}_1(O) = (\iota, T)$. *Then*

$$\mathrm{Des}(O) = \mathrm{Des}(\iota, T).$$

Proof: We proceed by analysing the effect of two successive steps in the oscillating tableau.

If step k is an expansion and step $k+1$ is a contraction then $k+1 \in \iota$ and $\iota(k+1) < k+1$. Now k either ends up in T or in ι. In the former case, $k \in \mathrm{Des}(T/\iota)$. In the latter case $\iota(k) > k \geq \iota(k+1)$ and $k \in \mathrm{Des}(\iota)$. In both cases this gives a descent of (ι, T).

If step k is a contraction and step $k+1$ is an expansion then $k \in \iota$ and $\iota(k) < k$. Now either $k+1$ ends up in T or in ι. In the former case $k \in \mathrm{Des}(\iota/T)$ rather than $\mathrm{Des}(T/\iota)$. In the latter case $\iota(k) < k < k+1 < \iota(k+1)$. Neither of these cases gives a descent of (ι, T).

If steps k and $k+1$ are both contractions then $k, k+1 \in \iota$. If b_k is strictly below b_{k+1} then, by well-known properties of RS, the element removed when bumping out b_k will be in a lower row than the one obtained when bumping out b_{k+1}. Thus $\iota(k) > \iota(k+1)$ and $k \in \mathrm{Des}(\iota, T)$ as desired. By a similar argument, if b_k is weakly above b_{k+1} then $\iota(k) < \iota(k+1)$ and $k \notin \mathrm{Des}(\iota, T)$.

Now suppose steps k and $k+1$ are both expansions. If b_k is in a row strictly above b_{k+1}, then any column deletion will keep k in a row strictly above $k+1$. It follows that at the end we have one of three possibilities. The first is that $k, k+1 \in T$ and, as was just observed, we must have $k \in \mathrm{Des}(T)$. If either element is removed, then $k+1$ must be removed first because the row condition forces $k+1$ to always be in a column weakly left of k. So at the end we either have $k \in T$ and $k+1 \in \iota$, or we have $\iota(k) > \iota(k+1)$ and in both cases $k \in \mathrm{Des}(\iota, T)$.

If steps k and $k+1$ are both expansions and b_k is in a row weakly below b_{k+1}, then any column deletion will keep k in a row weakly below $k+1$. Again there are three possibilities. The first is that $k, k+1 \in T$ and, as was just observed, we must have $k \notin \mathrm{Des}(T)$. If either element is removed, then k must be removed first because the row condition forces k to always be in a column strictly left of $k+1$. So at the end either we have $k+1 \in T$ and $k \in \iota$ or we have $\iota(k+1) > \iota(k)$, and neither are descents of (ι, T).

This completes the check of all the cases and the proof. $\square$

5.2 Robinson-Schensted

Next, we apply the Robinson-Schensted correspondence to the fixed-point-free involution ι to obtain a partial Young tableau I with the same set of entries. Let us first recall the following fact:

Lemma 5.5 ([6, Exercise 7.28a]). *Let $\pi \in \mathfrak{S}_r$ and let $\mathrm{RS}(\pi) = (P, Q)$. Then π is a fixed-point-free involution if and only if $P = Q$ and all columns of Q have even length.*

Combining Proposition 5.4 with this lemma we obtain:

Proposition 5.6. *Let O be an oscillating tableau, let $\mathrm{Sun}_1(O) = (\iota, T)$ and let $\mathrm{RS}(\iota) = (I, I)$. Then*

$$\mathrm{Des}(O) = \mathrm{Des}(I) \dot\cup \mathrm{Des}(T) \dot\cup \mathrm{Des}(T/I).$$

Example 5.7. The fixed point free involution from Example 5.3 is mapped to the tableau

$$I = \begin{array}{|c|c|} \hline 1 & 8 \\ \hline 2 & 9 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}$$

5.3 Sundaram's second bijection

Finally, we need a bijection Sun_2 that transforms the pair of partial Young tableaux (I, T) to pairs (Q, S) as in Theorem 4.2. Let Q be the standard Young tableau of shape λ obtained by *column-inserting* the reverse reading word of the tableau I into the tableau T. Also construct a skew semistandard Young tableau S as follows: whenever a letter of $w^{rev}(I)$ is inserted, record its row index in I in the cell which is added. Then $\mathrm{Sun}_2(I, T) = (Q, S)$.

Sundaram actually defines this map for pairs (I, T) of semistandard Young tableaux. But we will not need this level of generality.

Lemma 5.8 (Sundaram [7, Theorem 8.11, Theorem 9.4]). *The map Sun_2 is a bijection from pairs of partial Young tableaux (I, T) of shapes β and μ, respectively, such that $I \dot\cup T = \{1, \dots, r\}$ to pairs (Q, S) such that*

- *Q is a standard Young tableau of shape $\lambda \in P(r)$ and*
- *S is an n-symplectic Littlewood-Richardson tableau of shape λ/μ and weight β, for some n.*

Example 5.9. The tableau I from Example 5.7 has $w^{\mathrm{rev}}(I) = 819245$. Inserting this word into the tableau T from Example 5.3 and recording the row indices yields the pair of tableaux

$$(Q, S) = \left(\begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 4 & 8 \\ \cline{1-2} 5 & 9 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 2 \\ \cline{1-2} 1 & 3 \\ \cline{1-2} 2 & 4 \\ \cline{1-2} \end{array} \right).$$

Note that the $\mathrm{Des}(Q) = \{1, 3, 4, 6, 7, 8\} = \mathrm{Des}(O)$ as desired.

To prove Theorem 5.1 we need two properties of column insertion:

Lemma 5.10. *Let π be a permutation and suppose that $\mathrm{RS}^{col}(\pi) = T$. Then*

- $\mathrm{RS}^{col}\left(w^{rev}(T)\right) = T$
- $\mathrm{Des}(T) = \{k : k$ *is left of* $k+1$ *in the* 1*-line notation of* $\pi\}$.

Proof of Theorem 5.1: By the first assertion of Lemma 5.10 we have that Q can also be obtained by column-inserting the concatenation of $w^{\mathrm{rev}}(T)$ and $w^{\mathrm{rev}}(I)$. By the second assertion of Lemma 5.10, the descent set of Q equals $\mathrm{Des}(I) \uplus \mathrm{Des}(T) \uplus \mathrm{Des}(T/I)$, which in turn equals the descent set of O by Proposition 5.6. $\square$

References

[1] Allan Berele, *A Schensted-type correspondence for the symplectic group*, J. Combin. Theory Ser. A **43** (1986), no. 2, 320–328. MR 867655 (88b:20027)

[2] Jin Hong and Seok-Jin Kang, *Introduction to quantum groups and crystal bases*, Graduate Studies in Mathematics, vol. 42, American Mathematical Society, Providence, RI, 2002. MR 1881971 (2002m:17012)

[3] Ron C. King, *Weight multiplicities for the classical groups*, Group theoretical methods in physics (Fourth Internat. Colloq., Nijmegen, 1975), Springer, Berlin, 1976, pp. 490–499. Lecture Notes in Phys., Vol. 50. MR 0480895 (58 #1044)

[4] Cédric Lecouvey, *Combinatorics of crystal graphs for the root systems of types* A_n, B_n, C_n, D_n *and* G_2, Combinatorial aspect of integrable systems, MSJ Mem., vol. 17, Math. Soc. Japan, Tokyo, 2007, pp. 11–41. MR 2269126 (2008b:05183)

[5] Marcel Paul Schützenberger, *Quelques remarques sur une construction de Schensted*, Mathematica Scandinavica **12** (1963), 117–128. MR MR0190017 (32 #7433)

[6] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR MR1676282 (2000k:05026)

[7] Sheila Sundaram, *On the combinatorics of representations of the symplectic group*, ProQuest LLC, Ann Arbor, MI, 1986, Thesis (Ph.D.)–Massachusetts Institute of Technology. MR 2941115

[8] ———, *Tableaux in the representation theory of the classical Lie groups*, Invariant theory and tableaux (Minneapolis, MN, 1988), IMA Vol. Math. Appl., vol. 19, Springer, New York, 1990, pp. 191–225. MR 1035496 (91e:22022)