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The partition algebra and the Kronecker product (Extended abstract)

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Abstract. We propose a new approach to study the Kronecker coefficients by using the Schur–Weyl duality between the symmetric group and the partition algebra.

Résumé. Nous proposons une nouvelle approche pour l'étude des coéfficients de Kronecker via la dualité entre le groupe symétrique et l'algèbre des partitions.

Keywords: Kronecker coefficients, tensor product, partition algebra, representations of the symmetric group

1 Introduction

A fundamental problem in the representation theory of the symmetric group is to describe the coefficients in the decomposition of the tensor product of two Specht modules. These coefficients are known in the literature as the *Kronecker coefficients*. Finding a formula or combinatorial interpretation for these coefficients has been described by Richard Stanley as 'one of the main problems in the combinatorial representation theory of the symmetric group'. This question has received the attention of Littlewood [Lit58], James [JK81, Chapter 2.9], Lascoux [Las80], Thibon [Thi91], Garsia and Remmel [GR85], Kleshchev and Bessenrodt [BK99] amongst others and yet a combinatorial solution has remained beyond reach for over a hundred years.

Murnaghan discovered an amazing limiting phenomenon satisfied by the Kronecker coefficients; as we increase the length of the first row of the indexing partitions the sequence of Kronecker coefficients obtained stabilises. The limits of these sequences are known as the *reduced Kronecker coefficients*.

The novel idea of this paper is to study the Kronecker and reduced coefficients through the Schur–Weyl duality between the symmetric group, $\mathfrak{S}_n$, and the partition algebra, $P_r(n)$. The key observation being that the tensor product of Specht modules corresponds to the restriction of simple modules in $P_r(n)$ to a Young subalgebra. The combinatorics underlying the representation theory of both objects is based on

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partitions. The duality results in a Schur functor, $\mathrm{F} : \mathfrak{S}_n\text{-mod} \to P_r(n)\text{-mod}$, which acts by *first row removal* on the partitions labelling the simple modules. We exploit this functor along with the following three key facts concerning the representation theory of the partition algebra: (a) it is semisimple for large n (b) it has a stratification by symmetric groups (c) its non-semisimple representation theory is well developed.

Using our method we explain the limiting phenomenon of tensor products and bounds on stability, we also re-interpret the Kronecker and reduced Kronecker coefficients and the passage between the two in terms of the representation theory of the partition algebra. One should note that our proofs are surprisingly elementary.

The paper is organised as follows. In Sections 2 and 3 we recall the combinatorics underlying the representation theories of the symmetric group and partition algebra. In Section 4 we show how to pass the Kronecker problem through Schur–Weyl duality and phrase it as a question concerning the partition algebra. We then summarise results concerning the Kronecker and reduced Kronecker coefficients that have a natural interpretation (and very elementary proofs) in this setting. Section 5 contains an extended example.

2 Symmetric group combinatorics

The combinatorics underlying the representation theory of the symmetric group, $\mathfrak{S}_n$, is based on partitions. A *partition* λ of n, denoted $\lambda \vdash n$, is defined to be a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of non-negative integers such that the sum $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ equals n. The *length* of a partition is the number of nonzero parts, we denote this by $\ell(\lambda)$. We let Λ_n denote the set of all partitions of n.

With a partition, λ, is associated its *Young diagram*, which is the set of nodes

$$[\lambda] = \left\{ (i,j) \in \mathbb{Z}^2_{>0} \mid j \leq \lambda_i \right\}.$$

Given a node specified by $i, j \geq 1$, we say the node has *content* $j - i$. We let $\mathrm{ct}(\lambda_i)$ denote the content of the last node in the ith row of $[\lambda]$, that is $\mathrm{ct}(\lambda_i) = \lambda_i - i$.

Over the complex numbers, the irreducible *Specht* modules, $\mathbf{S}(\lambda)$, of $\mathfrak{S}_n$ are indexed by the partitions, λ, of n. An explicit construction of these modules is given in [JK81].

2.1 The classical Littlewood–Richardson rule

The Littlewood–Richardson rule is a combinatorial description of the coefficients in the restriction of a Specht module to a Young subgroup of the symmetric group. Through Schur–Weyl duality, the rule also computes the coefficients in the decomposition of a tensor product of two simple modules of $\mathrm{GL}_n(\mathbb{C})$.

The following is a simple restatement of this rule as it appears in [JK81, Section 2.8.13].

Theorem 2.1.1 (The Littlewood–Richardson Coefficients) *For $\lambda \vdash r_1$, $\mu \vdash r_2$ and $\nu \vdash r_1 + r_2$,*

$$\mathbf{S}(\nu)\big\downarrow^{\mathfrak{S}_{r_1+r_2}}_{\mathfrak{S}_{r_1}\times\mathfrak{S}_{r_2}} \cong \bigoplus_{\lambda\vdash r_1, \mu\vdash r_2} c^{\nu}_{\lambda,\mu} \mathbf{S}(\lambda) \boxtimes \mathbf{S}(\mu)$$

where the $c^{\nu}_{\lambda,\mu}$ are the Littlewood–Richardson coefficients.

The Littlewood–Richardson rule calculates the coefficients, $c^{\nu}_{\lambda,\mu}$, by counting tableaux, see [Mac95, Chapter I.9]. By transitivity of induction we have that the Littlewood–Richardson rule determines the structure of the restriction of a Specht module to any Young subgroup. Of particular importance in this paper is the three-part case

$$\mathbf{S}(\nu)\downarrow^{\mathfrak{S}_{r_1+r_2+r_3}}_{\mathfrak{S}_{r_1}\times\mathfrak{S}_{r_2}\times\mathfrak{S}_{r_3}} \cong \bigoplus_{\substack{\lambda\vdash r_1,\\ \mu\vdash r_2,\nu\vdash r_3}} \left(\sum_{\xi\vdash r_1+r_2} c^{\xi}_{\lambda,\mu} c^{\nu}_{\xi,\eta}\right) \mathbf{S}(\lambda)\boxtimes\mathbf{S}(\mu)\boxtimes\mathbf{S}(\eta).$$

We therefore set $c^{\nu}_{\lambda,\mu,\eta} = \sum_{\xi} c^{\xi}_{\lambda,\mu} c^{\nu}_{\xi,\eta}$.

2.2 Tensor products of Specht modules of the symmetric group

In this section we define the Kronecker coefficients and the reduced Kronecker coefficients as well as set some notation. Let λ and μ be two partitions of n, then

$$\mathbf{S}(\lambda)\otimes\mathbf{S}(\mu) = \bigoplus_{\nu\vdash n} g^{\nu}_{\lambda,\mu}\mathbf{S}(\nu),$$

the coefficients $g^{\nu}_{\lambda,\mu}$ are known as the *Kronecker coefficients*. These coefficients satisfy an amazing stability property illustrated in the following example.

Example 2.2.1 *We have the following tensor products of Specht modules:*

$$\begin{aligned} \mathbf{S}(1^2)\otimes\mathbf{S}(1^2) &= \mathbf{S}(2) \\ \mathbf{S}(2,1)\otimes\mathbf{S}(2,1) &= \mathbf{S}(3)\oplus\mathbf{S}(2,1)\oplus\mathbf{S}(1^3) \\ \mathbf{S}(3,1)\otimes\mathbf{S}(3,1) &= \mathbf{S}(4)\oplus\mathbf{S}(3,1)\oplus\mathbf{S}(2,1^2)\oplus\mathbf{S}(2^2) \end{aligned}$$

at which point the product stabilises, i.e. for all $n \geq 4$, we have

$$\mathbf{S}(n-1,1)\otimes\mathbf{S}(n-1,1) = \mathbf{S}(n)\oplus\mathbf{S}(n-1,1)\oplus\mathbf{S}(n-2,1^2)\oplus\mathbf{S}(n-2,2).$$

Let $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_\ell)$ be a partition and n be an integer, define $\lambda_{[n]} = (n-|\lambda|,\lambda_1,\lambda_2,\ldots,\lambda_\ell)$. Note that all partitions of n can be written in this form.

For $\lambda_{[n]},\mu_{[n]},\nu_{[n]} \in \Lambda_n$ we let

$$g^{\nu_{[n]}}_{\lambda_{[n]},\mu_{[n]}} = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathfrak{S}_n}(\mathbf{S}(\lambda_{[n]})\otimes\mathbf{S}(\mu_{[n]}),\mathbf{S}(\nu_{[n]}))),$$

denote the multiplicity of $\mathbf{S}(\nu_{[n]})$ in the tensor product $\mathbf{S}(\lambda_{[n]})\otimes\mathbf{S}(\mu_{[n]})$. Murnaghan showed (see [Mur38, Mur55]) that if we allow the first parts of the partitions to increase in length then we obtain a limiting behaviour as follows. For $\lambda_{[N]},\mu_{[N]},\nu_{[N]} \in \Lambda_N$ and N sufficiently large we have that

$$g^{\nu_{[N+k]}}_{\lambda_{[N+k]},\mu_{[N+k]}} = \overline{g}^{\nu}_{\lambda,\mu}$$

for all $k \geq 1$; the integers $\overline{g}^{\nu}_{\lambda,\mu}$ are called the *reduced Kronecker coefficients*. Bounds for this stability have been given in [Bri93, Val99, Kly04, BOR11].

Remark 2.2.2 *The reduced Kronecker coefficients are also the structural constants for a linear basis for the polynomials in countably many variables known as the* character polynomials, *see [Mac95].*

3 The partition algebra

The partition algebra was originally defined by Martin in [Mar91]. All the results in this section are due to Martin and his collaborators, see [Mar96] and references therein.

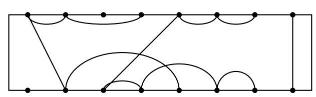
3.1 Definitions

For $r \in \mathbb{Z}_{>0}$, $\delta \in \mathbb{C}$, we let $P_r(\delta)$ denote the complex vector space with basis given by all set-partitions of $\{1, 2, \ldots, r, \bar{1}, \bar{2}, \ldots, \bar{r}\}$. A part of a set-partition is called a *block*. For example,

$$d = \{\{1, 2, 4, \bar{2}, \bar{5}\}, \{3\}, \{5, 6, 7, \bar{3}, \bar{4}, \bar{6}, \bar{7}\}, \{8, \bar{8}\}, \{\bar{1}\}\},$$

is a set-partition (for $r = 8$) with 5 blocks.

A set-partition can be represented *uniquely* by an (r, r)-*partition diagram* consisting of a frame with r distinguished points on the northern and southern boundaries, which we call vertices. We number the northern vertices from left to right by $1, 2, \ldots, r$ and the southern vertices similarly by $\bar{1}, \bar{2}, \ldots, \bar{r}$. Any block in a set-partition is of the form $A \cup B$ where $A = \{i_1 < i_2 < \ldots < i_p\}$ and $B = \{\bar{j_1} < \bar{j_2} < \ldots < \bar{j_q}\}$ (and A or B could be empty). We draw this block by putting an arc joining each pair (i_l, i_{l+1}) and $(\bar{j_l}, \bar{j}_{l+1})$ and if A and B are non-empty we draw a strand from i_1 to $\bar{j_1}$, that is we draw a single propagating line on the leftmost vertices of the block. Blocks containing a northern and a southern vertex will be called *propagating blocks*; all other blocks will be called *non-propagating blocks*. For d as in the example above, the partition diagram of d is given by:

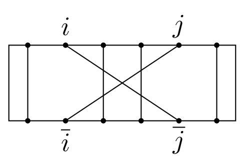


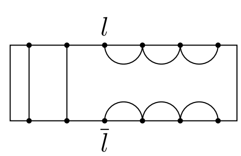
We can generalise this definition to (r, m)-partition diagrams as diagrams representing set-partitions of $\{1, \ldots, r, \bar{1}, \ldots, \bar{m}\}$ in the obvious way.

We define the product $x \cdot y$ of two diagrams x and y using the concatenation of x above y, where we identify the southern vertices of x with the northern vertices of y. If there are t connected components consisting only of middle vertices, then the product is set equal to δ^t times the diagram with the middle components removed. Extending this by linearity defines a multiplication on $P_r(\delta)$.

Assumption: *We assume throughout the paper that* $\delta \neq 0$.

The following elements of the partition algebra will be of importance.

$s_{i,j} =$ 

$e_l = \dfrac{1}{\delta}$ 

In particular, note that e_r corresponds to the set-partition $\{1, \bar{1}\}\{2, \bar{2}\} \cdots \{r-1, \overline{r-1}\}\{r\}\{\bar{r}\}$.

3.2 Filtration by propagating blocks and standard modules

Fix $\delta \in \mathbb{C}^{\times}$ and write $P_r = P_r(\delta)$. Note that the multiplication in P_r cannot increase the number of propagating blocks. More precisely, if x, respectively y, is a partition diagram with p_x, respectively

p_y, propagating blocks then $x \cdot y$ is equal to $\delta^t z$ for some $t \geq 0$ and some partition diagram z with $p_z \leq \min\{p_x, p_y\}$. This gives a filtration of the algebra P_r by the number of propagating blocks. This filtration can be realised using the idempotents e_l. We have

$$\mathbb{C} \cong P_r e_1 P_r \subset \ldots \subset P_r e_{r-1} P_r \subset P_r e_r P_r \subset P_r.$$

It is easy to see that

$$e_r P_r e_r \cong P_{r-1} \tag{3.2.1}$$

and that this generalises to $P_{r-l} \cong e_{r-l+1} P_r e_{r-l+1}$ for $1 \leq l \leq r$. We also have

$$P_r / (P_r e_r P_r) \cong \mathbb{C}\mathfrak{S}_r. \tag{3.2.2}$$

Using equation (3.2.2), we get that any $\mathbb{C}\mathfrak{S}_r$-module can be *inflated* to a P_r-module. We also get from equations (3.2.1) and (3.2.2), by induction, that the simple P_r-modules are indexed by the set $\Lambda_{\leq r} = \bigcup_{0 \leq i \leq r} \Lambda_i$.

For any $\nu \in \Lambda_{\leq r}$ with $\nu \vdash r - l$, we define a P_r-module, $\Delta_r(\nu)$, by

$$\Delta_r(\nu) = P_r e_{r-l+1} \otimes_{P_{r-l}} \mathbf{S}(\nu),$$

where the action of P_r is given by left multiplication. (Note that we have identified P_{r-l} with $e_{r-l+1} P_r e_{r-l+1}$ using the isomorphism given in equation (3.2.1).)

For $\delta \notin \{0, 1, \ldots, 2r-2\}$ the algebra $P_r(\delta)$ is semisimple and the set $\{\Delta_r(\nu) : \nu \in \Lambda_{\leq r}\}$ forms a complete set of non-isomorphic simple modules.

In general, the algebra $P_r(\delta)$ is quasi-hereditary with respect to the partial order on $\Lambda_{\leq r}$ given by $\lambda < \mu$ if $|\lambda| > |\mu|$ (see [Mar96]). The modules $\Delta_r(\nu)$ are the standard modules, each of which has a simple head $L_r(\nu)$, and the set $\{L_r(\nu) : \nu \in \Lambda_{\leq r}\}$ forms a complete set of non-isomorphic simple modules.

3.3 Non-semisimple representation theory of the partition algebra

We assume that $\delta = n \in \mathbb{Z}_{>0}$ (as otherwise the algebra is semisimple).

Definition 3.3.1 *Let λ and μ be partitions. We say that (μ, λ) is an n-pair, and write $\mu \hookrightarrow_n \lambda$, if $\mu \subset \lambda$ and the Young diagram of λ differs from the Young diagram of μ by a horizontal row of boxes of which the last (rightmost) one has content $n - |\mu|$.*

Example 3.3.2 *For example, $((2,1),(4,1))$ is a 6-pair. We have that $6 - |\mu| = 3$ and the Young diagrams (with contents) are as follows:*

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline -1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \subset \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

note that they differ by $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}$.

Recall that the set of simple (or standard) modules for $P_r(n)$ are labelled by the set $\Lambda_{\leq r}$. This set splits into $P_r(n)$-blocks. The set of labels in each block forms a maximal chain of n-pairs

$$\lambda^{(0)} \hookrightarrow_n \lambda^{(1)} \hookrightarrow_n \lambda^{(2)} \hookrightarrow_n \ldots \hookrightarrow_n \lambda^{(t)}.$$

Moreover, for $1 \leq i \leq t$ we have that $\lambda^{(i)}/\lambda^{(i-1)}$ consists of a strip of boxes in the ith row. Now we have an exact sequence of $P_r(n)$-modules

$$0 \to \Delta_r(\lambda^{(t)}) \to \ldots \to \Delta_r(\lambda^{(2)}) \to \Delta_r(\lambda^{(1)}) \to \Delta_r(\lambda^{(0)}) \to L_r(\lambda^{(0)}) \to 0$$

with the image of each homomorphism being simple. Each standard module $\Delta_r(\lambda^{(i)})$ (for $0 \leq i \leq t-1$) has Loewy structure

$$\begin{array}{c} L_r(\lambda^{(i)}) \\ L_r(\lambda^{(i+1)}) \end{array}$$

and so in the Grothendieck group we have

$$[L_r(\lambda^{(i)})] = \sum_{j=i}^{t} (-1)^{j-i} [\Delta_r(\lambda^{(j)})]. \tag{3.3.1}$$

Note that each block is totally ordered by the size of the partitions.

Proposition 3.3.3 *Let $\nu \in \Lambda_{\leq r}$ and assume that $\nu_{[n]}$ is a partition. Then we have that (i) ν is the minimal element in its $P_r(n)$-block, and (ii) ν is the unique element in its block if and only if $n+1-\nu_1 > r$.*

Proof: (i) Observe that for $\nu_{[n]}$ to be a partition we must have $n - |\nu| \geq \nu_1$. This implies that $\mathrm{ct}(\nu_1) = \nu_1 - 1 \leq n - |\nu| - 1$. So we have $\nu \hookrightarrow_n \mu$ for some partition μ with μ/ν being a single strip in the first row. Thus we have $\nu = \nu^{(0)}$ and $\mu = \nu^{(1)}$.
(ii) Now as $\nu^{(1)}/\nu$ is a single strip in the first row with last box having content $n - |\nu|$, we have that $|\nu^{(1)}/\nu| = n - |\nu| + 1 - \nu_1$ and thus $|\nu^{(1)}| = n + 1 - \nu_1$. Thus if $n + 1 - \nu_1 > r$ then $\nu^{(1)} \notin \Lambda_{\leq r}$ and we have that ν is the only partition in its $P_r(n)$-block. □

4 Schur–Weyl duality

Classical Schur–Weyl duality is the relationship between the general linear and symmetric groups over tensor space. To be more specific, let V_n be an n-dimensional complex vector space and let $V_n^{\otimes r}$ denote its rth tensor power.

We have that the symmetric group $\mathfrak{S}_r$ acts on the right by permuting the factors. The general linear group, GL_n, acts on the left by matrix multiplication on each factor. These two actions commute and moreover GL_n and $\mathfrak{S}_r$ are full mutual centralisers in $\mathrm{End}(V_n^{\otimes r})$.

The partition algebra, $P_r(n)$, plays the role of the symmetric group, $\mathfrak{S}_r$, when we restrict the action of GL_n to the subgroup of permutation matrices, $\mathfrak{S}_n$.

4.1 Schur-Weyl duality between $\mathfrak{S}_n$ and $P_r(n)$

Let V_n denote an n-dimensional complex space. Then $\mathfrak{S}_n$ acts on V_n via the permutation matrices.

$$\sigma \cdot v_i = v_{\sigma(i)} \qquad \text{for } \sigma \in \mathfrak{S}_n. \tag{4.1.1}$$

Notice that we are simply restricting the GL_n action in the classical Schur-Weyl duality to the permutation matrices. Thus, $\mathfrak{S}_n$ acts diagonally on the basis of simple tensors of $V_n^{\otimes r}$ as follows

$$\sigma \cdot (v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}) = v_{\sigma(i_1)} \otimes v_{\sigma(i_2)} \otimes \cdots \otimes v_{\sigma(i_r)}.$$

For each (r,r)-partition diagram d and each integer sequence $i_1 \ldots, i_r, i_{\bar{1}}, \ldots, i_{\bar{r}}$ with $1 \leq i_j, i_{\bar{j}} \leq n$, define

$$\phi_{r,n}(d)^{i_1,\ldots,i_r}_{i_{\bar{1}},\ldots,i_{\bar{r}}} = \begin{cases} 1 & \text{if } i_t = i_s \text{ whenever vertices } t \text{ and } s \text{ are connected in } d \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.2}$$

A partition diagram $d \in P_r(n)$ acts on the basis of simple tensors of $V_n^{\otimes r}$ as follows

$$\Phi_{r,n}(d)(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}) = \sum_{i_{\bar{1}},\ldots i_{\bar{r}}} \phi_{r,n}(d)^{i_1,\ldots,i_r}_{i_{\bar{1}},\ldots,i_{\bar{r}}} v_{i_{\bar{1}}} \otimes v_{i_{\bar{2}}} \otimes \cdots \otimes v_{i_{\bar{r}}}.$$

Theorem 4.1.1 (Jones [Jon94]) *$\mathfrak{S}_n$ and $P_r(n)$ generate the full centralisers of each other in $\mathrm{End}(V_n^{\otimes r})$.*

(a) $P_r(n)$ generates $\mathrm{End}_{\mathfrak{S}_n}(V_n^{\otimes r})$, and when $n \geq 2r$, $P_r(n) \cong \mathrm{End}_{\mathfrak{S}_n}(V_n^{\otimes r})$.

(b) $\mathfrak{S}_n$ generates $\mathrm{End}_{\mathfrak{S}_n}(V_n^{\otimes r})$.

We will denote $E_r(n) = \mathrm{End}_{\mathfrak{S}_n}(V_n^{\otimes r})$.

Theorem 4.1.2 ([Mar96] see also [HR05]) *We have a decomposition of $V_n^{\otimes r}$ as a $(\mathfrak{S}_n, P_r(n))$-bimodule*

$$V_n^{\otimes r} = \bigoplus \mathbf{S}(\lambda_{[n]}) \otimes L_r(\lambda)$$

where the sum is over all partitions $\lambda_{[n]}$ of n such that $|\lambda| \leq r$.

Using [GW98, Theorem 9.2.2] we have, for $\lambda_{[n]}, \mu_{[n]}, \nu_{[n]} \vdash n$ with $\lambda \vdash r$ and $\mu \vdash s$,

$$\mathrm{Hom}_{\mathfrak{S}_n}(\mathbf{S}(\nu_{[n]}), \mathbf{S}(\lambda_{[n]}) \otimes \mathbf{S}(\mu_{[n]})) \tag{4.1.3}$$

$$\cong \begin{cases} \mathrm{Hom}_{E_r(n)\otimes E_s(n)}(L_r(\lambda) \boxtimes L_s(\mu), L_{r+s}(\nu){\downarrow}_{E_r(n)\otimes E_s(n)}) & \text{if } \nu \in \Lambda_{\leq r+s} \\ 0 & \text{otherwise.} \end{cases}$$

4.2 Kronecker product via the partition algebra

Going back to the formula in (4.1.3) we need to consider $L_{r+s}(\nu){\downarrow}_{E_r(n)\otimes E_s(n)}$. Now $L_{r+s}(\nu)$ is a simple $P_{r+s}(n)$-module annihilated by $\ker \Phi_{r+s,n}$ and hence also by $\ker \Phi_{r,n} \otimes \ker \Phi_{s,n}$. Thus $L_{r+s}(\nu){\downarrow}_{P_r(s)\otimes P_s(n)}$ is semisimple and has the same simple factors as $L_{r+s}(\nu){\downarrow}_{E_r(s)\otimes E_s(n)}$.

Now combining (4.1.3) with (3.3.1) we have the following result.

Theorem 4.2.1 *Let $\lambda_{[n]}, \mu_{[n]}, \nu_{[n]} \vdash n$ with $\lambda \vdash r$ and $\mu \vdash s$. Then we have*

$$g^{\nu_{[n]}}_{\lambda_{[n]},\mu_{[n]}} = \begin{cases} \sum_{i=0}^{t} (-1)^i [\Delta_{r+s}(\nu^{(i)}){\downarrow}_{P_r(n)\otimes P_s(n)} \colon L_r(\lambda) \boxtimes L_s(\mu)] & \textit{if } \nu \in \Lambda_{\leq(r+s)} \\ 0 & \textit{otherwise} \end{cases}$$

where $\nu = \nu^{(0)} \hookrightarrow_n \nu^{(1)} \hookrightarrow_n \ldots \hookrightarrow \nu^{(t)}$ is the $P_{r+s}(n)$-block of ν.

For sufficiently large values of n the partition algebra is semisimple. Therefore Theorem 4.2.1 reproves the limiting behaviour of tensor products observed by Murnaghan. It also offers the following concrete representation theoretic interpretation of the $\overline{g}^{\nu}_{\lambda,\mu}$.

Corollary 4.2.2 *Let $\lambda \vdash r$ and $\mu \vdash s$ and suppose $|\nu| \leq r+s$. Then we have*

$$\overline{g}^{\nu}_{\lambda,\mu} = [\Delta_{r+s}(\nu)\downarrow_{P_r(n)\otimes P_s(n)} \colon L_r(\lambda) \boxtimes L_s(\mu)].$$

Using the semisimplicity criterion for the partition algebra given in Section 3.2 and Proposition 3.3.3 we immediately obtain

Corollary 4.2.3 *We have that $\overline{g}^{\nu}_{\lambda,\mu} = g^{\nu_{[n]}}_{\lambda_{[n]},\mu_{[n]}}$ if either $|\lambda| + |\mu| \leq \frac{n+1}{2}$, or $|\lambda| + |\mu| < n+1-\nu_1$.*

The second part of Corollary 4.2.3 is a new proof of Brion's bound [Bri93] for the stability of the Kronecker coefficients using the partition algebra.

By constructing an explicit filtration of the restriction of a standard module to a Young subalgebra of the partition algebra and identifying the corresponding subquotients we obtain

Theorem 4.2.4 *Write $m = r+s$ and let $\nu \vdash m-l$, $\lambda \vdash r-l_r$ and $\mu \vdash s-l_s$ for some non-negative integers l, l_r, l_s. Then $\Delta_m(\nu)\downarrow_{P_r\otimes P_s}$ has a filtration by standard modules with multiplicities given by*

$$[\Delta_m(\nu)\downarrow_{P_r\otimes P_s} :\ \Delta_r(\lambda) \boxtimes \Delta_s(\mu)] = \sum_{\substack{l_1,l_2 \\ l_1+2l_2 = l-l_r-l_s}} \ \sum_{\substack{\alpha \vdash r-l_r-l_1-l_2 \\ \beta \vdash s-l_s-l_1-l_2 \\ \pi,\rho,\sigma \vdash l_1 \\ \gamma \vdash l_2}} c^{\nu}_{\alpha,\beta,\pi} c^{\lambda}_{\alpha,\gamma,\rho} c^{\mu}_{\beta,\gamma,\sigma} g^{\pi}_{\rho,\sigma}.$$

By Corollary 4.2.2, we recover the formula for the reduced coefficients given in terms of Littlewood–Richardson coefficients and Kronecker coefficients in [BOR11, Lemma 2.1]

Corollary 4.2.5 *Let λ, μ, ν be any partitions with $|\lambda| = r$, $|\mu| = s$ and $|\nu| = r+s-l$. Then we have*

$$\overline{g}^{\nu}_{\lambda,\mu} = \sum_{\substack{l_1,l_2 \\ l = l_1+2l_2}} \ \sum_{\substack{\alpha \vdash r-l_1-l_2 \\ \beta \vdash s-l_1-l_2}} \ \sum_{\substack{\pi,\rho,\sigma \vdash l_1 \\ \gamma \vdash l_2}} c^{\nu}_{\alpha,\beta,\pi} c^{\lambda}_{\alpha,\gamma,\rho} c^{\mu}_{\beta,\gamma,\sigma} g^{\pi}_{\rho,\sigma}$$

Remark 4.2.6 *We also recover the Murnaghan–Littlewood Theorem, namely, for partitions λ, μ, ν with $|\lambda| + |\mu| = |\nu|$ we have that $\overline{g}^{\nu}_{\lambda,\mu} = c^{\nu}_{\lambda,\mu}$.*

4.3 Passing between the Kronecker and reduced Kronecker coefficients

In [BOR11] a formula is given for passing between the Kronecker and reduced Kronecker coefficients. We shall now interpret this formula in the Grothendieck group of the partition algebra by showing that it coincides with the formula in Theorem 4.2.1.

Let $\nu_{[n]}$ be a partition of n. We make the convention that $\nu_0 = n - |\nu|$ is the 0th row of $\nu_{[n]}$. For $i \in \mathbb{Z}_{\geq 0}$ define $\nu^{\dagger i}_{[n]}$ to be the partition obtained from $\nu_{[n]}$ by adding 1 to its first $i-1$ rows and erasing its ith row. In particular we have $\nu^{\dagger 0}_{[n]} = \nu$.

Theorem 4.3.1 (Theorem 1.1 of [BOR11]) *Let $\lambda_{[n]}, \mu_{[n]}, \nu_{[n]} \vdash n$. Then*

$$g_{\lambda_{[n]},\mu_{[n]}}^{\nu_{[n]}} = \sum_{i=0}^{l} (-1)^i \bar{g}_{\lambda,\mu}^{\nu_{[n]}^{\dagger i}}$$

where $l = \ell(\lambda_{[n]})\ell(\mu_{[n]}) - 1$.

Relating this to the partition algebra, we have the following.

Proposition 4.3.2 *Let $\nu_{[n]} \vdash n$ and let $\nu = \nu^{(0)} \hookrightarrow_n \nu^{(1)} \hookrightarrow_n \dots$ be a chain of n-pairs. Then for all $i \geq 0$ we have*

$$\nu_{[n]}^{\dagger i} = \nu^{(i)}$$

Proof: The $i = 0$ case is clear from the definitions. We proceed by induction. Assume that

$$\nu_{[n]}^{\dagger i} = \nu^{(i)}.$$

Then $(\nu^{(i)})_1 = n - |\nu| + 1$, $(\nu^{(i)})_j = \nu_{j-1} + 1$ for $j \leq i$, and $(\nu^{(i)})_j = \nu_j$ for $j > i$. Therefore

$$|\nu^{(i)}| = n - |\nu| + 1 + \sum_{j \neq i} \nu_j + i - 1 = n - \nu_i + i$$

We have that $\nu^{(i+1)}/\nu^{(i)}$ is a skew partition consisting of a strip in the $(i+1)$th row. By definition of an n-pair the content, $\mathrm{ct}(\nu_{i+1}^{(i+1)})$, of the last node is $n - |\nu^{(i)}|$. Therefore

$$\mathrm{ct}(\nu_{i+1}^{(i+1)}) := \nu_{i+1}^{(i+1)} - (i+1) = n - (n - \nu_i + i) = \nu_i - i$$

and $\nu_{i+1}^{(i+1)} = \nu_i + 1$, therefore $\nu_{[n]}^{\dagger(i+1)} = \nu^{(i+1)}$.

□

In Theorem 4.2.1, t is chosen so that $|\nu^{(t)}| \leq |\lambda| + |\mu|$ and $|\nu^{(t+1)}| > |\lambda| + |\mu|$. So Theorem 4.2.1 and 4.3.1 seem to give a different number of terms in the sum. For example consider

$$g_{(1^2),(1^2)}^{(2)} = 1 \quad g_{(1^2),(1^2)}^{(1^2)} = 0$$

these are given as a sum of one, respectively two terms in Theorem 4.2.1, both cases have four terms in Theorem 4.3.1. Now consider

$$\lambda_{[n]} = \mu_{[n]} = \nu_{[n]} = (10, 10, 10)$$

then $\ell(\lambda_{[n]})\ell(\mu_{[n]}) = 9$. We have $\nu_{[n]}^{\dagger 8} = (11^3, 1^5)$ with $|\nu_{[n]}^{\dagger 8}| = 38$. But $r + s = 40$, so we have two more terms in Theorem 4.2.1, corresponding to $\nu^{(9)} = (11^3, 1^6)$ and $\nu^{(10)} = (11^3, 1^7)$. However, we can show that in fact the two theorems give the same sum.

First assume that $\ell(\lambda_{[n]})\ell(\mu_{[n]}) - 1 > t$, then for all $i > t$ we have

$$\bar{g}_{\lambda,\mu}^{\nu_{[n]}^{\dagger i}} = 0$$

as $|\nu^{\dagger i}_{[n]}| > |\lambda| + |\mu|$. And so the two sums coincide.

Now assume that $\ell(\lambda_{[n]})\ell(\mu_{[n]}) - 1 < t$. Then for all $i > \ell(\lambda_{[n]})\ell(\mu_{[n]}) - 1$, we have

$$i \geq \ell(\lambda_{[n]})\ell(\mu_{[n]}) \geq |\lambda_{[n]} \cap (\mu_{[n]})'|.$$

where $(\mu_{[n]})'$ denotes the conjugate partition of $\mu_{[n]}$. (To see this observe that the Young diagram of $\lambda_{[n]} \cap (\mu_{[n]})'$ fits in a rectangle of size $\ell(\lambda_{[n]}) \times \ell(\mu_{[n]})$). Now we have

$$\ell(\nu^{(i)}) \geq i \geq |\lambda_{[n]} \cap (\mu_{[n]})'|.$$

But this implies that $\bar{g}^{\nu^{(i)}}_{\lambda,\mu} = 0$ by [Dvi93].

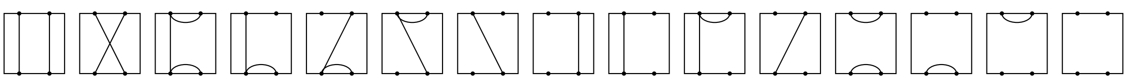
5 Example

In this section, we shall compute the tensor square of the Specht module, $\mathbf{S}(n-1,1)$ for $n \geq 2$, labelled by the first non-trivial hook, via the partition algebra. We have that

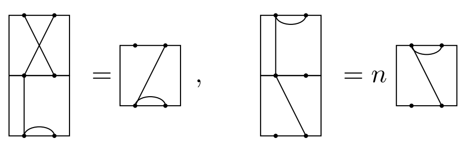
$$\mathrm{Hom}_{\mathfrak{S}_n}(\mathbf{S}(\nu_{[n]}), \mathbf{S}(n-1,1) \otimes \mathbf{S}(n-1,1)) \cong \mathrm{Hom}_{P_1(n)\otimes P_1(n)}(L_1(1) \otimes L_1(1), L_2(\nu)\downarrow^{P_2(n)}_{P_1(n)\otimes P_1(n)})$$

if $\nu \in \Lambda_{\leq 2}$ and zero otherwise. Therefore, it is enough to consider the restriction of simple modules from $P_2(n)$ to the Young subalgebra $P_1(n) \otimes P_1(n)$.

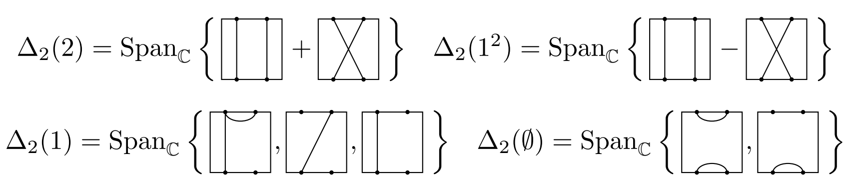
The partition algebra $P_2(n)$ is a 15-dimensional algebra with basis:



and multiplication defined by concatenation. For example:



There are four standard modules corresponding to the partitions of degree less than or equal to 2; these are obtained by inflating the Specht modules from the symmetric groups of degree $0, 1, 2$. These modules have bases:



The action of $P_2(n)$ is given by concatenation. If the resulting diagram has fewer propagating lines than the original, we set the product equal to zero. The algebra $P_1(n) \otimes P_1(n)$ is the 4-dimensional subalgebra spanned by the diagrams with no lines crossing an imagined vertical wall down the centre of the diagram.

The restriction of the standard modules to this subalgebra is as follows:

$$\Delta_2(2)\!\downarrow_{P_1\otimes P_1}\cong \Delta_1(1)\boxtimes\Delta_1(1),\quad \Delta_2(1^2)\!\downarrow_{P_1\otimes P_1}\cong \Delta_1(1)\boxtimes\Delta_1(1),$$

$$\Delta_2(1)\!\downarrow_{P_1\otimes P_1}\cong \Delta_1(1)\boxtimes\Delta_1(1)\oplus\Delta_1(\emptyset)\boxtimes\Delta_1(1)\oplus\Delta_1(1)\boxtimes\Delta_1(\emptyset),$$

$$\Delta_2(\emptyset)\!\downarrow_{P_1\otimes P_1}\cong \Delta_1(1)\boxtimes\Delta_1(1)\oplus\Delta_1(\emptyset)\boxtimes\Delta_1(\emptyset).$$

In particular, note that $\bar{g}^{\nu}_{(1),(1)} = [\Delta_2(\nu)\!\downarrow_{P_1\otimes P_1}\colon \Delta_1(1)\boxtimes\Delta_1(1)] = 1$ for $\nu=\emptyset,1,1^2,2$.

The partition algebra $P_2(n)$ is semisimple for $n>2$. For $\nu=\emptyset,(1),(1^2)$ or (2) we have that $\nu_{[n]} = (n),(n-1,1),(n-2,1^2)$, or $(n-2,2)$ and $\nu_{[n]}$ is a partition for $n\geq 0,2,3,4$ respectively. Therefore the Kronecker coefficients

$$g^{\nu_{[n]}}_{(n-1,1),(n-1,1)}$$

stabilise for $n\geq 4$ and are non-zero for $n\geq 4$ if and only $\nu_{[n]}$ is one of the partitions above.

Now consider the case $n=2$. Neither $\nu=(1^2)$, nor (2) correspond to partitions of 2, we therefore consider $\nu=\emptyset$ and (1). We have that $(1)\subset(2)$ is the unique 2-pair of partitions of degree less than or equal to 2 (see Section 3.3). Therefore the only standard $P_2(2)$-module which is not simple is $\Delta_2(1)$ and we have an exact sequence

$$0\to L_2(2)\to\Delta_2(1)\to L_2(1)\to 0.$$

Thus in the Grothendieck group we have that $[L_2(1)] = [\Delta_2(1)] - [\Delta_2(2)]$. Therefore we have that $[L_2(1)\!\downarrow_{P_1(2)\otimes P_1(2)}\colon L_1(1)\boxtimes L_1(1)] = 0$. We conclude that $g^{(1^2)}_{(1^2),(1^2)} = 0$ and $g^{(2)}_{(1^2),(1^2)} = 1$ as expected.

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