Results and conjectures on the number of standard strong marked tableaux

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Abstract. Many results involving Schur functions have analogues involving k-Schur functions. Standard strong marked tableaux play a role for k-Schur functions similar to the role standard Young tableaux play for Schur functions. We discuss results and conjectures toward an analogue of the hook length formula.

Résumé. De nombreux résultats impliquant les fonctions de Schur possèdent des analogues pour les fonctions de k-Schur. Les tableaux standard fortement marqués jouent un rôle pour les fonctions de k-Schur semblable á celui joué par les tableaux de Young pour les fonctions de Schur. Nous proposons ici des résultats et conjectures vers un analogue de la formule des équerres.

Keywords: k-Schur functions, strong marked tableaux, enumeration

1   Introduction

In 1988, Macdonald introduced a new class of polynomials and conjectured that they expand positively in terms of Schur functions. This conjecture, verified in Haiman (2001), has led to an enormous amount of work, including the development of the k-Schur functions. The k-Schur functions were defined in Lapointe et al. (2003). Lapointe, Lascoux, and Morse conjectured that they form a basis for a certain subspace of the space of symmetric functions and that the Macdonald polynomials indexed by partitions whose first part is not larger than k expand positively in terms of the k-Schur functions, leading to a refinement of the Macdonald conjecture. The k-Schur functions have since been found to arise in other contexts; for example, as the Schubert cells of the cohomology of affine Grassmannian permutations Lam (2006), and they are related to the quantum cohomology of the affine permutations Lapointe and Morse (2008).

One of the intriguing features of standard Young tableaux is the Frame-Thrall-Robinson hook-length formula, which enumerates them. It has many different proofs and many generalizations, see e.g. (Stanley, 1999, Chapter 7), Greene et al. (1979), Ciocan-Fontanine et al. (2011) and the references therein.

In this extended abstract, we partially succeed in finding an analogue of the hook-length formula for standard strong marked tableaux (or starred tableaux for short), which are a natural generalization of standard Young tableaux in the context of k-Schur functions. For a fixed n, the shape of a starred tableau (see Subsection 2.5 for a definition) is necessarily an n-core, a partition for which all hook-lengths are different from n. In Lam et al. (2010), a formula is given for the number of starred tableaux for $n = 3$.

Proposition 1.1 (Lam et al. (2010), Proposition 9.17) *For a 3-core λ, the number of starred tableaux of shape λ equals*

$$\frac{m!}{2^{\left\lfloor \frac{m}{2} \right\rfloor}},$$

where m is the number of boxes of λ with hook-length < 3.   □

The number of 2-hooks is $\left\lfloor \frac{m}{2} \right\rfloor$. Therefore we can rewrite the result as

$$\frac{m!}{\displaystyle\prod_{\substack{i,j \in \lambda \\ h_{ij} < 3}} h_{ij}}.$$

Note that this is reminiscent of the classical hook-length formula.

The authors left the enumeration for $n > 3$ as an open problem. The main result (Theorem 3.1) of this extended abstract implies the existence, for each n, of $(n-1)!$ rational numbers which we call *correction factors*. Once the corrections factors have been calculated by enumerating all starred tableaux for certain shapes, the number of starred tableaux of shape λ for any n-core λ can be easily computed. In fact, Theorem 3.1 is a t-analogue of the hook formula. The theorem is "incomplete" in the sense that we were not able to find explicit formulas for the (weighted) correction factors. We have, however, been able to state some of their properties (some conjecturally), the most interesting of these properties being unimodality (Conjecture 3.7).

Another result of interest is a new, alternative description of strong marked covers via simple triangular arrays of integers which we call *residue tables* and *quotient tables* (Theorem 4.2).

The extended abstract is structured as follows. In Section 2, we give the requisite background, notation, definitions, and results. In Section 3, we state the main results and conjectures. In Section 4, we give an alternative description of strong covers directly in terms of bounded partitions (instead of via cores, abacuses or affine permutations). We envision this description as the first steps toward an inductive proof of the main formula We finish with some remarks and open questions in Section 5.

2   Preliminaries

Here we introduce notation and review some constructions. Please see Macdonald for the definitions of integer partitions, ribbons, hook lengths, etc., which we omit in this extended abstract.

2.1 Cores and bounded partitions

Let n be a positive integer. An *n-core* is a partition λ such that $h_{ij}^\lambda \neq n$ for all $(i,j) \in \lambda$. Core partitions were introduced by Nakayama to describe when two ordinary irreducible representations of the symmetric group belong to the same block. There is a close connection between $(k+1)$-cores and *k-bounded partitions*, which are partitions whose first part (and hence every part) is $\leq k$. Indeed, in Lapointe and Morse (2005), a simple bijection between $(k+1)$-cores and k-bounded partitions is presented. Given a $(k+1)$-core λ, whose diagram has λ_i boxes in row i, let π_i be the number of boxes in row i of λ with hook-length $\leq k$. The resulting $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ is a k-bounded partition, we denote it $\mathfrak{b}(\lambda)$. Conversely, given a k-bounded partition π, move from the last row of π upwards, and in row i, shift the π_i boxes of the diagram of π to the right until their hook-lengths are at most k. The resulting $(k+1)$-core is denoted $\mathfrak{c}(\pi)$. In this extended abstract, we will always use n as shorthand for $k+1$.

Example 2.1 *On the left-hand side of Figure 1, the hook-lengths of the boxes of the 5-core $\lambda = 953211$ are shown, with the ones that are < 5 in bold. That means that $\mathfrak{b}(\lambda) = 432211$.*
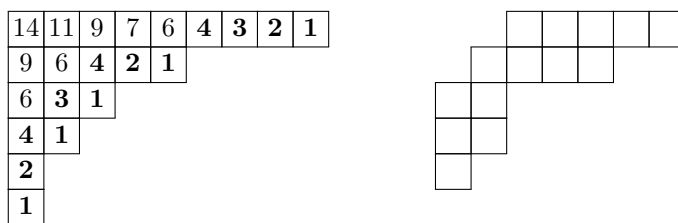

Fig. 1: Bijections $\mathfrak{b}$ and $\mathfrak{c}$.

The right-hand side shows the construction of $\mathfrak{c}(\pi) = 75221$ for the 6-bounded partition $\pi = 54221$. ◇

Of particular importance are k-bounded partitions π that satisfy $m_i(\pi) \leq k - i$ for all $i = 1, \ldots, k$. We call such partitions *k-irreducible partitions*, see Lapointe et al. (2003). The number of k-irreducible partitions is $k!$.

2.2 Young tableaux and the hook-length formula

Young's lattice $\mathcal{Y}$ takes as its vertices all integer partitions, and the relation is containment. If λ and μ are partitions, then μ covers λ if and only if $\lambda \subseteq \mu$ and $|\mu| = |\lambda| + 1$. The rank of a partition is given by its size.

A *semistandard Young tableau* T of shape λ is a Young diagram of shape λ whose boxes have been filled with positive integers satisfying the following: the integers must be nondecreasing as we read a row from left to right, and increasing as we read a column from top to bottom. The *weight* of T is the composition $(\alpha_1, \alpha_2, \ldots)$, where α_i is the number of i's in T. The tableau T is a *standard Young tableau* if the entries are $1, \ldots, |\lambda|$ in some order, i.e. if the weight is $(1, \ldots, 1)$. A standard Young tableau of shape λ represents a saturated chain in the interval $[\emptyset, \lambda]$ of the Young's lattice. Let $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(m)})$, $\lambda^{(0)} = \emptyset$, $\lambda^{(m)} = \lambda$, be such a chain. Then in the tableau corresponding to this chain, i is the entry in the box added in moving from $\lambda^{(i-1)}$ to $\lambda^{(i)}$.

The Frame-Thrall-Robinson hook-length formula shows how to compute f_λ, the number of standard Young tableaux of shape λ. We have:

$$f_\lambda = \frac{|\lambda|!}{\prod_{i,j \in \lambda} h_{ij}^\lambda}. \tag{2.1}$$

This formula has a well-known weighted version, see (Stanley, 1999, Corollary 7.21.5). For a standard Young tableau T, define a *descent* to be an integer i such that $i + 1$ appears in a lower row of T than i, and define the *descent set* $D(T)$ to be the set of all descents of T. Define the *major index* of T as $\mathrm{maj}(T) = \sum_{i \in D(T)} i$, and the polynomial

$$f_\lambda(t) = \sum t^{\mathrm{maj}(T)},$$

where the sum is over all standard Young tableaux of shape λ. Then

$$f_\lambda(t) = \frac{t^{b(\lambda)}(|\boldsymbol{\lambda}|)!}{\prod_{i,j \in \lambda}(\boldsymbol{h_{ij}^\lambda})}. \tag{2.2}$$

Here $b(\lambda) = \sum_i (i-1)\lambda_i = \sum_i \binom{\lambda_i'}{2}$, $(\boldsymbol{i}) = 1 + t + \ldots + t^{i-1}$ and $(\boldsymbol{i})! = (\boldsymbol{1}) \cdot (\boldsymbol{2}) \cdots (\boldsymbol{i})$.

2.3 Strong marked and starred tableaux

The *strong n-core poset* $\mathcal{C}_n$ is the subposet of $\mathcal{Y}$ induced by the set of all n-core partitions. That is, its vertices are n-core partitions and $\lambda \leq \mu$ in $\mathcal{C}_n$ if $\lambda \subseteq \mu$. The cover relations are trickier to describe in $\mathcal{C}_n$ than in $\mathcal{Y}$.

Proposition 2.2 (Lam et al. (2010), Proposition 9.5) *Suppose $\lambda \leq \mu$ in $\mathcal{C}_n$, and let $C_1, \ldots, C_m$ be the connected components of μ/λ. Then μ covers λ (denoted $\lambda \lessdot \mu$) if and only if each C_i is a ribbon, and all the components are translates of each other with heads on consecutive diagonals with the same residue.*

The rank of an n-core is the number of boxes of its diagram with hook-length $< n$. If $\lambda \lessdot \mu$ and μ/λ consists of m ribbons, we say that μ *covers* λ *in the strong order with multiplicity* m. Figure 2 shows the strong marked covers for 4-cores with rank at most 6. Only multiplicities $\neq 1$ are marked.

A *strong marked cover* is a triple (λ, μ, c) such that $\lambda \lessdot \mu$ and that c is the content of the head of one of the ribbons. We call c the *marking* of the strong marked cover. A *strong marked horizontal strip of size r and shape μ/λ* is a sequence $(\nu^{(i)}, \nu^{(i+1)}, c_i)_{i=0}^{r-1}$ of strong marked covers such that $c_i < c_{i+1}$, $\nu^{(0)} = \lambda$, $\nu^{(r)} = \mu$. If λ is an n-core, a *strong marked tableau T of shape λ* is a sequence of strong marked horizontal strips of shapes $\mu^{(i+1)}/\mu^{(i)}$, $i = 0, \ldots, m-1$, such that $\mu^{(0)} = \emptyset$ and $\mu^{(m)} = \lambda$. The *weight* of T is the composition $(r_1, \ldots, r_m)$, where r_i is the size of the strong marked horizontal strip $\mu^{(i)}/\mu^{(i-1)}$. If all strong marked horizontal strips are of size 1, we call T a *standard strong marked tableau* or a *starred tableau* for short. For a k-bounded partition π (recall that $n = k+1$), denote the number of starred tableaux of shape $\mathfrak{c}(\pi)$ by $F_\pi^{(k)}$.

Figure 3 illustrates $F_{211}^{(3)} = 6$.

If λ is a k-bounded partition that is also an n-core (i.e., if $\lambda_1 + \ell(\lambda) \leq k+1$), then strong marked covers on the interval $[\emptyset, \lambda]$ are equivalent to the covers in the Young lattice, strong marked tableaux of shape λ are equivalent to semistandard Young tableaux of shape λ, and starred tableaux of shape λ are equivalent to standard Young tableaux of shape λ.
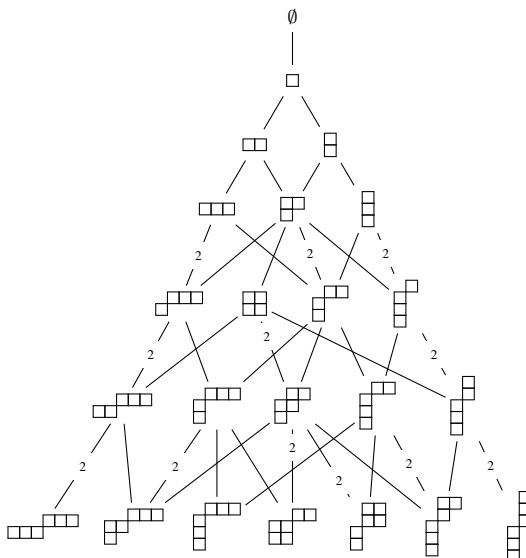
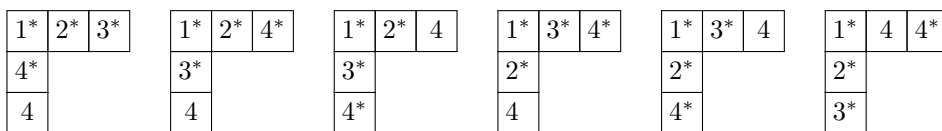**Fig. 2:** The 4-core lattice up to rank 6. Only boxes with hook-lengths ¡ 4 are drawn.

1*	2*	3*
4*		
4		

1*	2*	4*
3*		
4		

1*	2*	4
3*		
4*		

1*	3*	4*
2*		
4		

1*	3*	4
2*		
4*		

1*	4	4*
2*		
3*		

Fig. 3: All starred tableaux of shape 311.

2.4 Schur functions

For the definition of Λ, the ring of symmetric functions, see Macdonald or Stanley (1999). For a partition λ, define the *monomial symmetric function*

$$m_\lambda = m_\lambda(x_1, x_2, \ldots) = \sum_\alpha x^\alpha,$$

where the sum is over all weak compositions α that are a permutation of λ, and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. For partitions λ and μ of the same size, define the *Kostka number* $K_{\lambda\mu}$ as the number of semistandard Young tableaux of shape λ and weight μ. Define the *Schur function*

$$s_\lambda = \sum K_{\lambda\mu} m_\mu$$

with the sum over all partitions μ. The Schur functions form the most important basis of Λ and have numerous beautiful properties. See for example (Stanley, 1999, Chapter 7) and (Macdonald, Chapter 1).

2.5 k-Schur functions

There are at least three conjecturally equivalent definitions of k-Schur functions. Here, we give the definition from Lam et al. (2010) via strong marked tableaux. For k-bounded partitions π and τ, define the k-*Kostka number* $K_{\pi\tau}^{(k)}$ as the number of strong marked tableaux of shape $\mathfrak{c}(\pi)$ and weight τ. Then we define the k-*Schur function*

$$s_\pi^{(k)} = \sum_\tau K_{\pi\tau}^{(k)} m_\tau, \tag{2.3}$$

where the sum is over all k-bounded partitions τ.

If π is also a $(k+1)$-core, then strong marked tableaux of shape π are equivalent to semistandard Young tableaux of shape π, and therefore in this case $s_\pi^{(k)} = s_\pi$.

The original definition of k-Schur functions was via *atoms* Lapointe et al. (2003), which we will not use here (but see **5.2**). Note that in full generality, the k-Schur functions (in any definition) have a parameter t. In this extended abstract, $t = 1$.

3 Main results and conjectures

For a starred tableau T, define the *descent set of T*, $D(T)$, as the set of all i for which the marked box at i is strictly above the marked box at $i + 1$. Define the *major index of T*, $\mathrm{maj}(T)$, by $\sum_{i \in D(T)} i$. For a k-bounded partition π, define the polynomial

$$F_\pi^{(k)}(t) = \sum_T t^{\mathrm{maj}(T)}, \tag{3.1}$$

where the sum is over all starred tableaux of shape $\mathfrak{c}(\pi)$. Recall that $F_\pi^{(k)}$ denotes the number of such starred tableaux, i.e. $F_\pi^{(k)} = F_\pi^{(k)}(1)$.

Our main result is the following theorem.

Theorem 3.1 *Let π be a k-bounded partition, and write*

$$\pi = \langle k^{a_1 + 1 \cdot w_1}, (k-1)^{a_2 + 2 \cdot w_2}, \ldots, 1^{a_k + k \cdot w_k} \rangle,$$

for $0 \le a_i < i$. Then

$$F_\pi^{(k)}(t) = \frac{t^{\sum_{i=1}^k w_i \binom{i}{2}(k-i+1)} (|\boldsymbol{\pi}|)!\, F_\sigma^{(k)}(t)}{(|\sigma|)!\, \prod_{j=1}^k (j)^{\sum_{i=1}^k w_i \min\{i, j, k+1-i, k+1-j\}}},$$

where $\sigma = \langle k^{a_1}, (k-1)^{a_2}, \ldots, 1^{a_k} \rangle$.

By plugging in $t = 1$, we get the following.

Corollary 3.2 *Let π be a k-bounded partition, and write*

$$\pi = \langle k^{a_1 + 1 \cdot w_1}, (k-1)^{a_2 + 2 \cdot w_2}, \ldots, 1^{a_k + k \cdot w_k} \rangle,$$

for $0 \le a_i < i$. Then

$$F_\pi^{(k)} = \frac{|\pi|! \, F_\sigma^{(k)}}{|\sigma|! \, \prod_{j=1}^k j^{\sum_{i=1}^k w_i \min\{i,j,k+1-i,k+1-j\}}},$$

where $\sigma = \langle k^{a_1}, (k-1)^{a_2}, \ldots, 1^{a_k} \rangle$.   □

The theorem (respectively, corollary) implies that in order to compute $F_\pi^{(k)}(t)$ (resp., $F_\pi^{(k)}$) for all k-bounded partitions π, it suffices to compute $F_\sigma^{(k)}(t)$ (resp., $F_\sigma^{(k)}$) only for k-irreducible partitions σ; recall that there are $k!$ such partitions.

The proof is omitted in the extended abstract. Let us just mention that it uses the expansion of k-Schur functions in terms of fundamental quasisymmetric functions, and the stable principal specialization (i.e., evaluation at $1, t, t^2, \ldots$) of fundamental quasisymmetric functions.

Example 3.3 *The following gives the formulas for $k \le 3$.*

1. *For $k = 1$, we have $F_{1^0}^{(1)}(t) = 1$ and therefore*

$$F_{1^{w_1}}^{(1)}(t) = \frac{(w_1)! \cdot 1}{(0)! \cdot (1)^{w_1}} = (w_1)!.$$

   This is consistent with (Lam et al., 2010, §9.4.1), which states that $F_{1^{w_1}}^{(1)} = w_1!$.

2. *For $k = 2$, we have $F_{2^0 1^0}^{(2)}(t) = 1$ and $F_{2^0 1^1}^{(2)}(t) = 1$. Therefore,*

$$F_{2^{w_1} 1^{2w_2}}^{(2)}(t) = \frac{t^{w_2}(2w_1 + 2w_2)! \cdot 1}{(0)! \cdot (2)^{w_1 + w_2}} = \frac{t^{w_2}(2w_1 + 2w_2)!}{(2)^{w_1 + w_2}}.$$

$$F_{2^{w_1} 1^{1+2w_2}}^{(2)}(t) = \frac{t^{w_2}(2w_1 + 2w_2 + 1)! \cdot 1}{(0)! \cdot (2)^{w_1 + w_2}} = \frac{t^{w_2}(2w_1 + 2w_2 + 1)!}{(2)^{w_1 + w_2}}.$$

   This is consistent with (Lam et al., 2010, Proposition 9.17), reprinted here as Proposition 1.1.

3. *For $k = 3$, we have $F_{3^0 2^0 1^0}^{(3)} = F_{3^0 2^0 1^1}^{(3)} = F_{3^0 2^1 1^0}^{(3)} = 1$, $F_{3^0 2^0 1^2}^{(3)} = t$, $F_{3^0 2^1 1^1}^{(3)} = t(1 + t)$ and, $F_{3^0 2^1 1^2}^{(3)} = t(t^2 + 1)(t^2 + t + 1)$. So, among other formulas, we have*

$$F_{3^{w_1} 2^{1+2w_2} 1^{1+3w_3}}^{(3)}(t) = \frac{t^{2w_2 + 3w_3 + 1} \cdot (3w_3 + 4w_2 + 3w_1 + 3)!}{(2)^{w_1 + 2w_2 + w_3} \cdot (3)^{w_1 + w_2 + w_3 + 1}}.$$

Using a computer, it is easy to obtain formulas for larger k.   ◇

We now introduce weighted correction factors. For a k-bounded partition π, let $H_\pi^{(k)}(t) = \prod (h_{ij})$, where the product is over all boxes (i, j) of the $(k+1)$-core $\mathfrak{c}(\pi)$ with hook-lengths at most k, and let $H_\pi^{(k)} = H_\pi^{(k)}(1)$ be the product of all hook-lengths $\le k$ of $\mathfrak{c}(\pi)$. Furthermore, if b_j is the number of boxes in the j-column of $\mathfrak{c}(\pi)$ with hook-length at most k, write $b_\pi^{(k)} = \sum_j \binom{b_j}{2}$.

Example 3.4 *For the* 6*-bounded partition* $\pi = 54211$ *from Example 2.1, we have*
$H_\pi^{(6)}(t) = (\mathbf{1})^4(\mathbf{2})^3(\mathbf{3})^2(\mathbf{4})^2(\mathbf{5})(\mathbf{6})^2$, $H_\pi^{(6)} = 207360$ *and* $b_\pi^{(6)} = 2\binom{3}{2} + 3\binom{2}{2} + 2\binom{1}{2} = 9$. ◇

By introducing *weighted correction factors* $C_\sigma^{(k)}(t)$ for a k-irreducible partition σ, we can, by Theorem 3.1, express $F_\pi^{(k)}(t)$ (for all k-bounded partitions π) in another way which is reminiscent of the classical hook-length formula. More precisely, define a rational function $C_\sigma^{(k)}(t)$ so that

$$F_\sigma^{(k)}(t) = \frac{t^{b_\sigma^{(k)}}(|\boldsymbol{\sigma}|)!C_\sigma^{(k)}(t)}{H_\sigma^{(k)}(t)}. \tag{3.2}$$

Note that this implies, in the notation of Theorem 3.1, that

$$F_\pi^{(k)}(t) = \frac{t^{b_\sigma^{(k)} + \sum_{i=1}^k w_i \binom{i}{2}(k+1-i)}(|\boldsymbol{\pi}|)!\, C_\sigma^{(k)}(t)}{H_\sigma^{(k)}(t) \cdot \prod_{j=1}^k (\boldsymbol{j})^{\sum_{i=1}^k w_i \min\{i,j,k+1-i,k+1-j\}}}.$$

The *correction factor* $C_\sigma^{(k)}$ is defined as $C_\sigma^{(k)}(1)$.

For $k \leq 3$, all weighted correction factors are 1. For $k = 4$, all but four of the 24 weighted correction factors–for 4-bounded partitions 2211, 321, 3211 and 32211–are 1, and the ones different from 1 are

$$\frac{1 + 2t + t^2 + t^3}{(\mathbf{2})(\mathbf{3})}, \ \frac{1 + t + 2t^2 + t^3}{(\mathbf{2})(\mathbf{3})}, \ \frac{1 + 2t + 2t^2 + 2t^3 + t^4}{(\mathbf{3})^2}, \ \frac{1 + t + 3t^2 + t^3 + t^4}{(\mathbf{3})^2},$$

respectively.

We state some results and conjectures about the weighted correction factors. For a k-bounded partition π, denote by $\partial_k(\pi)$ the boxes of $\mathfrak{c}(\pi)$ with hook-length $\leq k$. If $\partial_k(\pi)$ is not connected, we say that π *splits*. Each of the connected components of $\partial_k(\pi)$ is a horizontal translate of $\partial_k(\pi^i)$ for some k-bounded partition π^i. Call $\pi^1, \pi^2, \dots$ the *components* of π.

Proposition 3.5 *The weighted correction factors are multiplicative in the following sense. If a k-irreducible partition σ splits into $\sigma^1, \sigma^2, \dots, \sigma^m$, then $C_\sigma^{(k)}(t) = \prod_{i=1}^m C_{\sigma^i}^{(k)}(t)$.*

Conjecture 3.6 *For a k-irreducible partition σ, the weighted correction factor is 1 if and only if σ splits into $\sigma^1, \sigma^2, \dots, \sigma^l$, where each σ^i is a k-bounded partition that is also a $(k+1)$-core.*

The "if" direction is easy: if a k-bounded partition σ is also a $(k+1)$-core, then strong covers on the interval $[0, \sigma]$ are precisely the regular covers in the Young lattice, the starred tableaux of shape σ are standard Young tableaux of shape σ, and the major index of a starred tableau of shape σ is the classical major index for standard Young tableaux; the fact that the weighted correction factor is 1 then follows from the classical weighted version of the hook-length formula (2.2). If σ splits into cores, we can use (Denton, 2012, Theorem 1.1).

The most interesting conjecture about the weighted correction factors is the following. Recall that a sequence $(\alpha_i)_i$ is *unimodal* if there exists I so that $\alpha_i \leq \alpha_{i+1}$ for $i < I$ and $\alpha_i \geq \alpha_{i+1}$ for $i \geq I$, and a *unimodal polynomial* is a polynomial whose sequence of coefficients is unimodal.

Conjecture 3.7 *For a k-irreducible partition σ, we can write*

$$1 - C_\sigma^{(k)}(t) = \frac{P_1(t)}{P_2(t)},$$

where $P_1(t)$ is a unimodal polynomial with non-negative integer coefficients and $P_2(t)$ is a polynomial of the form $\prod_{i=1}^{k-1} (j)^{w_j}$ for some non-negative integers w_j.
In particular, we have $0 < C_\sigma^{(k)} \leq 1$ for all σ.

4 Strong covers and k-bounded partitions

Our proof of Theorem 3.1, omitted in the extended abstract, closely follows one of the possible proofs of the classical (non-weighted and weighted) hook-length formula, see e.g. (Stanley, 1999, §7.21). Note, however, that the truly elegant proofs (for example, the celebrated probabilistic proof due to Greene, Nijenhuis and Wilf Greene et al. (1979)) are via induction. In this section, we show the first steps toward such a proof.

In the process, we present a new description of strong marked covers in terms of bounded partitions (previous descriptions included cores – at least implicitly, via k-conjugation – affine permutations and abacuses). See the definition of residue and quotient tables below, and Theorem 4.2.

We identify a bounded partition $\pi = \langle k^{p_1}, (k-1)^{p_2}, \ldots, 1^{p_k} \rangle$ with the sequence $p = (p_1, \ldots, p_k)$. Given i, j, m, $0 \leq m < i \leq j \leq k$, define $p^{i,j,m}$ as follows.

$$\text{For } i < j, p_h^{i,j,m} = \begin{cases} p_h + m & \text{if } h = i - 1 \\ p_h - m & \text{if } h = i \\ p_h - m - 1 & \text{if } h = j \\ p_h + m + 1 & \text{if } h = j + 1 \\ P_h & \text{otherwise.} \end{cases} \quad \text{For } i = j, p_h^{i,i,m} = \begin{cases} p_h + m & \text{if } h = i - 1 \\ p_h - 2m - 1 & \text{if } h = i \\ p_h + m + 1 & \text{if } h = i + 1 \\ P_h & \text{otherwise.} \end{cases}$$

If $j = k$, then we are adding $m + 1$ copies of $k - j = 0$, which does not change the partition. If $i = 1$, we have $m = 0$, so adding m copies of $k + 2 - i = k + 1$ also does not change the partition. To put it another way: to get $p^{i,j,m}$ from p, increase the first m copies of $k + 1 - i$ by 1, and decrease the last $m + 1$ copies of $k + 1 - j$ by 1. See Example 4.3.

Define upper-triangular arrays $\mathcal{R} = (r_{ij})_{1 \leq i \leq j \leq k}$, $\mathcal{Q} = (q_{ij})_{1 \leq i \leq j \leq k}$ by

- $r_{jj} = p_j \bmod j$, $r_{ij} = (p_i + r_{i+1,j}) \bmod i$ for $i < j$,

- $q_{jj} = p_j \operatorname{div} j$, $q_{ij} = (p_i + r_{i+1,j}) \operatorname{div} i$ for $i < j$.

We call $\mathcal{R}$ the *residue table* and $\mathcal{Q}$ the *quotient table*.

Example 4.1 *Take $k = 4$ and $p = (1, 3, 2, 5)$. Then the residue and quotient tables are given by*

0	0	0	0
	1	1	1
		2	0
			1

1	2	2	2
	1	2	1
		0	1
			1

It is easy to reconstruct p from the diagonals of $\mathcal{R}$ *and* $\mathcal{Q}$*:* $p_1 = 0 + 1 \cdot 1$*,* $p_2 = 1 + 1 \cdot 2$*,* $p_3 = 2 + 0 \cdot 3$*,*
$p_4 = 1 + 1 \cdot 4$*.*   ◇

It turns out that the residue and quotient tables determine strong marked covers (and probably other important relations as well, see **5.5**).

Theorem 4.2 *Take* $p = (p_1, \ldots, p_k)$ *and* $1 \le i \le j \le k$*. If* $r_{ij} < r_{i+1,j}, \ldots, r_{jj}$*, then p covers* $p^{i,j,r_{ij}}$ *in the strong order with multiplicity* $q_{ij} + \ldots + q_{jj}$*. Furthermore, these are precisely all strong covers. In particular, an element of the* $(k+1)$*-core lattice covers at most* $\binom{k+1}{2}$ *elements.*

Example 4.3 *Take* $k = 4$ *and* $p = (1, 3, 2, 5)$ *as before. Let us underline the entries* r_{ij} *in the residue table* $\mathcal{R}$ *for which* $r_{ij} < r_{i+1,j}, \ldots, r_{jj}$*.*

$$
\begin{array}{|cccc|}
\hline
\underline{0} & \underline{0} & \underline{0} & 0 \\
 & \underline{1} & \underline{1} & 1 \\
 & & \underline{2} & 0 \\
 & & & \underline{1} \\
\hline
\end{array}
$$

By Theorem 4.2, p covers (exactly) the following elements in the strong order:
$p^{1,1,0} = (0, 4, 2, 5)$ *with multiplicity* 1*,* $p^{1,2,0} = (1, 2, 3, 5)$ *with multiplicity* $2+1 = 3$*,* $p^{2,2,1} = (2, 0, 4, 5)$ *with multiplicity* 1*,* $p^{1,3,0} = (1, 3, 1, 6)$ *with multiplicity* $2+2+0 = 4$*,* $p^{2,3,1} = (2, 2, 0, 7)$ *with multiplicity* $2 + 0 = 2$*,* $p^{3,3,2} = (1, 5, -3, 8)$ *with multiplicity* 0*,* $p^{3,4,0} = (1, 3, 2, 4)$ *with multiplicity* $1 + 1 = 2$*, and* $p^{4,4,1} = (1, 3, 3, 2)$ *with multiplicity* 1*.*

Note that while $(1, 5, -3, 8)$ *does not represent a valid partition, the multiplicity of the cover is* 0*, so we can ignore this cover relation.*   ◇

For a k-bounded partition π, we clearly have

$$
F_\pi^{(k)} = \sum_\tau m_{\tau\pi} F_\tau^{(k)},
$$

where the sum is over all k-bounded τ that are covered by π, and $m_{\tau\pi}$ is the multiplicity of the cover. Therefore Theorem 4.2 can be used to prove Corollary 3.2 for small values of k by induction. First, we need the following corollary.

Corollary 4.4 *Let* $p = (p_1, \ldots, p_k)$*,* $p_i < i$*, with corresponding residue and quotient tables* $\mathcal{R}$ *and* $\mathcal{Q}$*. Assume that for* $1 \le i \le j \le k$*, we have* $r_{ij} < r_{i+1,j}, \ldots, r_{jj}$*. For* $s_i \in \mathbb{N}$*, write* $s = (s_1, 2s_2, \ldots, ks_k)$*. Then* $p + s$ *covers* $p^{i,j,r_{ij}} + s$ *with multiplicity* $q_{ij} + \ldots + q_{jj} + s_i + \ldots + s_j$*.*

The corollary implies that in order to prove Corollary 3.2, all we have to do is check $k!$ equalities. The authors did all such calculations with a computer for small k ($k \le 8$).

5 Final remarks

5.1

There are also notions of *weak horizontal strips* and *weak tableaux*. For n-cores λ and μ, $\lambda \subseteq \mu$, we say that μ/λ is a weak horizontal strip if $\mathfrak{b}(\mu)/\mathfrak{b}(\lambda)$ is a horizontal strip and $\mathfrak{b}(\mu')/\mathfrak{b}(\lambda')$ is a vertical strip. If in addition $|\mathfrak{b}(\mu)| = |\mathfrak{b}(\lambda)| + 1$, we say that μ *covers* λ *in the weak order*. A weak tableau of shape λ is a sequence of weak horizontal strips $\mu^{(i+1)}/\mu^{(i)}$, $i = 0, \ldots, m-1$, such that $\mu^{(0)} = \emptyset$ and $\mu^{(m)} = \lambda$.

Define $f_\pi^{(k)}$ to be the number of weak tableaux of shape $\mathfrak{c}(\pi)$. In Lam et al. (2010), it was proved that
$$f^{(2)}_{2^{w_1}1^{2w_2}} = f^{(2)}_{2^{w_1}1^{1+2w_2}} = \frac{(w_1+w_2)!}{w_1!w_2!}.$$

It is not hard to prove by induction that

$$f^{(3)}_{3^{w_1}2^{2w_2}1^{3w_3}} = \frac{2^{2w_2}(w_1+w_2)!(w_2+w_3)!(w_1+2w_2+w_3-1)!(2w_1+2w_2+2w_3)!}{w_1!w_2!w_3!(w_1+w_2+w_3-1)!(2w_1+2w_2)!(2w_2+2w_3)!};$$

similar formulas exist for

$$f^{(3)}_{3^{w_1}2^{2w_2}1^{1+3w_3}}, f^{(3)}_{3^{w_1}2^{2w_2}1^{2+3w_3}}, f^{(3)}_{3^{w_1}2^{1+2w_2}1^{3w_3}}, f^{(3)}_{3^{w_1}2^{1+2w_2}1^{1+3w_3}}, f^{(3)}_{3^{w_1}2^{1+2w_2}1^{2+3w_3}}.$$

We were unable to find formulas for $k \geq 4$, and it seems unlikely that simple formulas exist. For example, the simplest recurrence relation that $g(i,j) = f^{(4)}_{2^{3i}1^{4j}}$ seem to satisfy is

$$a(i,j)g(i,j) + b(i,j)g(i,j+1) - c(i,j)g(i+1,j) = 0,$$

where a and b are fourth degree polynomials in i and j with rational coefficients and c, also fourth degree, is a polynomial with integer coefficients.

5.2

Our work has led us to consider (weighted) correction factors. They seem to be mysterious objects that deserve further study. The unimodality conjecture (Conjecture 3.7) is certainly intriguing and could hint that the factors have some geometric meaning.

Let us give another perspective on these factors. Since k-Schur functions are symmetric, they can be expanded in terms of Schur functions; in fact, the original definition of k-Schur functions via atoms gives precisely such an expansion. For example, $s^{(4)}_{2211} = s_{2211} + s_{321}$. Take the stable principal specialization and multiply by $(6)!(1-t)^6$. By calculations done in our proof of Theorem 3.1 and (Stanley, 1999, Proposition 7.19.11), we have
$$F^{(4)}_{2211}(t) = f_{2211}(t) + f_{321}(t).$$
Then, by (3.2) and (Stanley, 1999, Corollary 7.21.5),

$$C^{(k)}_{2211}(t) = (2)(3)(4)\left(\frac{t^3}{(2)^2(4)(5)} + \frac{1}{(3)^2(5)}\right) = \frac{1+2t+t^2+t^3}{(2)(3)}.$$

5.3

There is also a formula for the principal specialization of s_λ of order i (i.e. evaluation at $1, t, \ldots, t^{i-1}$, see e.g. (Stanley, 1999, Theorem 7.21.2)), in which both hook-lengths and contents of boxes appear. By imitating **5.2**, we can get rational functions (which depend on i) which converge to the weighted correction factors as $i \to \infty$. These rational functions also seem interesting and worthy of further study.

5.4

As we already mentioned, it would be preferable to prove Corollary 3.2 by induction, using the cover relations in Section 4 for a general k and in a way that would make apparent the meaning of hook-lengths and correction factors (the ideal being a variant of the probabilistic proof from Greene et al. (1979)). It seems likely that one would need to know a formula for the correction factors before such a proof would be feasible.

5.5

We showed (in Theorem 4.2) how to interpret the residue and quotient table to find strong covers. We feel that residue (and quotient) tables could prove important in other aspects of the k-Schur function theory. These tables can also be used to describe weak covers, weak horizontal and vertical strips and at least one of the possible cases of LLMS insertion for standard strong marked tableaux (see Lam et al. (2010)).

Acknowledgments

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