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Diagrams of affine permutations, balanced labellings, and affine Stanley symmetric functions (Extended Abstract)

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Abstract. We study the *diagrams* of affine permutations and their *balanced labellings*. As in the finite case, which was investigated by Fomin, Greene, Reiner, and Shimozono, the balanced labellings give a natural encoding of reduced decompositions of affine permutations. In fact, we show that the sum of weight monomials of the *column strict* balanced labellings is the affine Stanley symmetric function defined by Lam and we give a simple algorithm to recover reduced words from balanced labellings. Applying this theory, we give a necessary and sufficient condition for a diagram to be an affine permutation diagram. Finally, we conjecture that if two affine permutations are *diagram equivalent* then their affine Stanley symmetric functions coincide.

Résumé. Nous étudions les schémas de permutations affines et de leurs étiquetages équilibrés. Comme ce fut le cas fini, qui a été étudiée par Fomin, Greene, Reiner, et Shimozono, les étiquetages équilibrés donner un codage naturel des décompositions réduites de permutations affines. En fait, nous montrons que l'addition des monômes poids de la colonne strictes étiquetages équilibrés est le symétrique affine de Stanley fonction définie par Lam, et nous donnons un algorithme simple pour récupérer des mots réduits étiquetages équilibrés. Sur l'application de cette théorie, nous donnons une condition nécessaire et suffisante pour qu'un diagramme soit un schéma affine permutation. Enfin, nous supposons que si deux permutations affines sont les schémas équivalents puis leurs fonctions symétriques affines Stanley coïncident.

Keywords: affine permutations, permutation diagrams, balanced labellings, reduced words, Stanley symmetric functions

1 Introduction

The *diagram*, or the *Rothe diagram* of a permutation is a widely used technique to visualize the inversions of the permutation on the plane. It is well known that there is a one-to-one correspondence between the permutations and the set of their inversions.

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Balanced labellings are labellings of the diagram $D(w)$ of a permutation $w \in \Sigma_n$ such that each cell of the diagram is *balanced*. They are defined in [FGRS97] to encode reduced decompositions of the permutation w. There is a notion of *injective labellings* which generalize both standard Young tableaux and Edelman-Greene's balanced tableaux [EG87], and *column strict labellings* which generalize semi-standard Young tableaux. Column strict labellings yield symmetric functions in the same way semi-standard Young tableaux yield Schur functions. In fact, these symmetric functions $F_w(x)$ are the *Stanley symmetric functions*, which were introduced to calculate the number of reduced decompositions of $w \in \Sigma_n$ [Sta84]. The Stanley symmetric function coincides with the Schur function when w is a *Grassmannian* permutation. Furthermore, if one imposes flag conditions on column strict labellings, they yield Schubert polynomial of Lascoux and Schützenberger [LS85]. One can directly observe the limiting behaviour of Schubert polynomials (e.g. stability, convergence to $F_w(x)$, etc.) in this context. In [FGRS97] it was also shown that the balanced flagged labellings form a basis of the Schubert modules whose character is the Schubert polynomial.

The main purpose of this paper is to extend the idea of diagrams and balanced labellings to affine permutations. We first define the diagrams of affine permutations and balanced labellings on them. Following the footsteps of [FGRS97], we show that the column strict labellings on affine permutation diagrams yield the affine Stanley symmetric function defined by Lam in [Lam06]. When a permutation is 321-*avoiding affine Grassmannian*, the balanced labellings coincide with semi-standard cylindric tableaux, and they yield the cylindric Schur function of Postnikov [Pos05].

Also as a byproduct of balanced labellings, we give the complete characterization of diagrams of affine permutations using the notions of *content*. We will introduce the notion of a *wiring diagram* of an affine permutation diagram in the process, which generalizes Postnikov's wiring diagram of Grassmannian permutations [Pos06].

2 Balanced labellings and reduced words

In this section we study balanced labellings and their relations with reduced words and affine Stanley symmetric functions. Our terms, lemmas, and theorems will be in parallel with [FGRS97], extending them from finite to affine permutations. Although most of the definitions in [FGRS97] will remain the same with slight modifications, we state them here for the sake of completeness.

Let $\widetilde{\Sigma}_n$ denote the affine symmetric group generated by $s_0, s_1, \ldots, s_{n-1}$ satisfying the relations

$$\begin{aligned} s_i^2 &= 1 && \text{for all } i \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for all } i \\ s_i s_j &= s_j s_i && \text{for } |i-j| \geq 2 \end{aligned}$$

where the indices are taken modulo n. An element w of $\widetilde{\Sigma}_n$ is called an *affine permutation (of period n)*. A *reduced decomposition* of w is a decomposition $w = s_{i_1} \cdots s_{i_\ell}$ where ℓ is the minimal number for which such a decomposition exists. In this case, ℓ is called the *length* of w and denoted by $\ell(w)$. The word $i_1 i_2 \cdots i_\ell$ is called a *reduced word* of w.

Another way to realize $\widetilde{\Sigma}_n$ is as the set of bijections $w : \mathbb{Z} \to \mathbb{Z}$ such that $w(i+n) = w(i) + n$ and $\sum_{i=1}^{n} w(i) = n(n+1)/2$. In this alternative realization, one can write each element of $\widetilde{\Sigma}_n$ in the *window notation*: $w = [w(1), \cdots, w(n)]$, since these n numbers are sufficient for identifying w. The (finite) symmetric group Σ_n generated by $s_1, \cdots, s_{n-1}$ can be naturally embedded in $\widetilde{\Sigma}_n$ and the window notation for elements of Σ_n is the usual one-line notation for finite permutations. We will call w *finite* if it is contained in this subgroup.

The *diagram*, or *affine permutation diagram*, of $w \in \widetilde{\Sigma}_n$ is the set

$$D(w) = \{(i, w(j)) \mid i < j, w(i) > w(j)\} \subseteq \mathbb{Z} \times \mathbb{Z}.$$

This is a natural generalization of the *Rothe diagram* for finite permutations. When w is finite, $D(w)$ consists of infinite number of identical copies of the Rothe diagram of w diagonally.

Throughout this paper, we will use a matrix-like coordinate system on $\mathbb{Z} \times \mathbb{Z}$: The vertical axis corresponds to the first coordinate increasing as one moves toward south, and the horizontal axis corresponds to the second coordinate increasing as one moves toward east. We will visualize $D(w)$ as the collection of unit square lattice boxes on $\mathbb{Z} \times \mathbb{Z}$ whose positions are given by $D(w)$.

From the construction it is clear that $(i, j) \in D(w) \Leftrightarrow (i+n, j+n) \in D(w)$. We will call a collection D of unit square lattice boxes on $\mathbb{Z}\times\mathbb{Z}$ an *affine diagram* (*of period* n) if there are finite number of cells on each row and column, and $(i, j) \in D \Leftrightarrow (i+n, j+n) \in D$. Obviously $D(w)$ is an affine diagram of period n. For an affine diagram D, we will call the collection of boxes $\{(i + rn, j + rn) \mid r \in \mathbb{Z}\}$ a *cell* of D, and denote it by $\overline{(i, j)}$. From the periodicity, we can take the representatives of each cell $\overline{(i, j)}$ in the first n rows $\{1, 2, \ldots, n\} \times \mathbb{Z}$, called the *fundamental window*. Each horizontal strip $\{1 + rn, \cdots, n + rn\} \times \mathbb{Z}$ for some $r \in \mathbb{Z}$ will be called a *window*. The intersection of D and the fundamental window will be denoted by $[D]$. The boxes in $[D]$ are the natural representatives of the cells of D. An affine diagram D is said to be of size ℓ if the number of boxes in $[D]$ is ℓ. Note that the size of $D(w)$ for $w \in \widetilde{\Sigma}_n$ is the length of w.

2.1 Balanced labellings: basic definitions and results

In this section, we define the notion of balanced labellings of affine diagrams.

To each cell (i, j) of an affine diagram D, we associate the *hook* $H_{i,j} := H_{i,j}(D)$ consisting of the cells (i', j') of D such that either $i' = i$ and $j' \geq j$ or $i' \geq i$ and $j' = j$. The cell (i, j) is called the *corner* of $H_{i,j}$.

Definition 2.1 (Balanced hooks) *A labelling of the cells of $H_{i,j}$ with positive integers is called* balanced *if it satisfies the following condition: if one rearranges the labels in the hook so that they weakly increase from right to left and from top to bottom, then the corner label remains unchanged.*

A labelling of an affine diagram is a map $T : D \to \mathbb{Z}_{>0}$ from the boxes of D to the positive integers such that $T(i, j) = T(i + n, j + n)$ for all $(i, j) \in D$. In other words, it sends each cell $\overline{(i, j)}$ to some positive integer. Therefore if D has size ℓ, there can be at most ℓ different numbers for the labels of the boxes in D.

Definition 2.2 (Balanced labellings) *Let D be an affine diagram of size ℓ.*

1. *A labelling of D is* balanced *if each hook $H_{i,j}$ is balanced for all $(i, j) \in D$.*

2. *A balanced labelling is* injective *if each of the labels* $1, \cdots, \ell$ *appears exactly once in* $[D]$.

3. *A balanced labelling is* column strict *if no column contains two equal labels.*

Given $w \in \widetilde{\Sigma}_n$ and its reduced decomposition $w = s_{a_1} \cdots s_{a_\ell}$, we read from left to right and interpret s_k as adjacent transpositions switching the numbers at $(k + rn)$-th and $(k + 1 + rn)$-th positions, for all $r \in \mathbb{Z}$. In other words, w can be obtained from applying the sequence of transpositions $s_{a_1}, s_{a_2}, \ldots, s_{a_\ell}$ to the identity permutation. It is clear that each s_i corresponds to a unique inversion of w. Here, an *inversion* of w is a family of pairs $\{(w(i+rn), w(j+rn)) \mid r \in \mathbb{Z}\}$ where $i < j$ and $w(i) > w(j)$. Note that $w(i + rn) > w(j + rn) \Leftrightarrow w(i) > w(j)$. Often we will ignore r and use a representative of pairs when we talk about the inversions. On the other hand, each cell of $D(w)$ also corresponds to a unique inversion of w. In fact, $(i, j) \in D(w)$ if and only if $(w(i), j)$ is an inversion of w.

Definition 2.3 (Canonical labelling) *Let* $w \in \widetilde{\Sigma}_n$ *be of length* ℓ*, and* $a = a_1 a_2 \cdots a_\ell$ *be a reduced word of* w*. Let* $T_a : D \to \{1, \cdots, \ell\}$ *be the injective labelling defined by setting* $T_a(i, w(j)) = k$ *if* s_{a_k} *transposes* $w(i)$ *and* $w(j)$ *in the partial product* $s_{a_1} \cdots s_{a_k}$ *where* $w(i) > w(j)$*. Then* T_a *is called the* canonical labelling *of* $D(w)$ *induced by* a*.*

Theorem 2.4 *Let* $\mathcal{R}(w)$ *denote the set of reduced words of* $w \in \widetilde{\Sigma}_n$*, and* $\mathcal{B}(D)$ *denote the set of injective balanced labellings of the affine diagram* D*. The correspondence* $a \mapsto T_a$ *is a bijection between* $\mathcal{R}(w)$ *and* $\mathcal{B}(D(w))$*.*

Theorem 2.4 follows as a corollary from more general results in next sections, namely Lemma 2.11 and Theorem 2.12. Algorithm to decode the reduced word from a balanced labelling will be given in Section 2.3. The following lemma will be useful in the proof.

Lemma 2.5 (Localization) *Let* $w \in \widetilde{\Sigma}_n$ *and let* T *be an injective labelling of* $D(w)$*. Then* T *is balanced if and only if for all integers* $i < j < k$ *the restriction of* T *to the sub-diagram of* $D(w)$ *determined by the intersections of rows* i, j, k *and columns* $w(i), w(j), w(k)$ *is balanced.*

2.2 Column strict balanced labellings and affine Stanley symmetric functions

In this section we consider column strict balanced labellings of affine permutation diagrams. We show that they give us the affine Stanley symmetric function in the same way the semi-standard Young tableaux give us the Schur function.

Affine Stanley symmetric functions are symmetric functions parametrized by affine permutations. They are defined in [Lam06] as an affine counterpart of the Stanley symmetric function [Sta84]. Like Stanley symmetric functions, they play an important role in combinatorics of reduced words. The affine Stanley symmetric functions also have a natural geometric interpretation [Lam08], namely they are pullbacks of the cohomology Schubert classes of the affine flag variety $LSU(n)/T$ to the affine Grassmannian $\Omega SU(n)$ under the natural map $\Omega SU(n) \to LSU(n)/T$. There are various ways to define the affine Stanley symmetric function, including the geometric one above. For our purpose, we use one of the two combinatorial definitions in [Lam10].

A word $a_1 a_2 \cdots a_\ell$ with letters in $\mathbb{Z}/n\mathbb{Z}$ is called *cyclically decreasing* if (1) each letter appears at most once, and (2) whenever i and $i+1$ both appears in the word, $i+1$ precedes i. An affine permutation $w \in \widetilde{\Sigma}_n$

is called cyclically decreasing if it has a cyclically decreasing reduced word. We call $w = v_1 v_2 \cdots v_r$ *cyclically decreasing factorization* of w if each $v_i \in \widetilde{\Sigma}_n$ is cyclically decreasing, and $\ell(w) = \sum_{i=1}^{r} \ell(v_i)$.

Definition 2.6 ([Lam10]) *Let $w \in \widetilde{\Sigma}_n$ be an affine permutation. The affine Stanley symmetric function $\widetilde{F}_w(x)$ corresponding to w is defined by*

$$\widetilde{F}_w(x) := \widetilde{F}_w(x_1, x_2, \cdots) = \sum_{w = v_1 v_2 \cdots v_r} x_1^{\ell(v_1)} x_2^{\ell(v_2)} \cdots x_r^{\ell(v_r)},$$

where the sum is over all cyclically decreasing factorization of w.

Given an affine diagram D, let $\mathcal{CB}(D)$ denote the set of column strict balanced labellings of D. Now we can state our first main theorem.

Theorem 2.7 *Let $w \in \widetilde{\Sigma}_n$ be an affine permutation. Then*

$$\widetilde{F}_w(x) = \sum_{T \in \mathcal{CB}(D(w))} x^T$$

where x^T denotes the monomial $\prod_{(i,j) \in [D(w)]} x_{T(i,j)}$

Remark 2.8 *In the case of finite permutations, Fomin, Greene, Reiner, and Shimozono also showed that the generating function for the balanced labellings under certain flag condition is the Schubert polynomial $\mathfrak{S}_w$. In fact, if $\mathcal{CFB}(D(w))$ is the set of all column strict balanced labellings T such that $T(i,j) \leq i$ for all $(i,j) \in D(w)$, then $\mathfrak{S}_w = \sum_{T \in \mathcal{CFB}(D(w))} x^T$. One may regard this formula as a direct translation of the result of Billey, Jockusch, and Stanley [BJS93] to the language of balanced labellings. It would be really interesting if we could extend this result to affine permutations.*

Comparing the coefficients of $x_1 x_2 \cdots x_{\ell(w)}$ in both sides of Theorem 2.7, we see that the set of reduced words of w and the set of injective balanced labellings of $D(w)$ have the same cardinality. See Theorem 2.4 and Section 2.3 for an explicit bijection between them.

Definition 2.9 (Border cell) *Let $w \in \widetilde{\Sigma}_n$ and $\overline{(i,j)}$ be a cell of $D(w)$. If $w(i+1) = j$ then the cell $\overline{(i,j)}$ is called a* border cell *of $D(w)$.*

The border cells correspond to the (right) descents of w, i.e. the simple reflections that can appear at the end of some reduced decomposition of w. When we multiply a descent of w to w from right side, we get an affine permutations whose length is $\ell(w) - 1$. It is immediate that this operation changes the diagram in the following manner:

Lemma 2.10 *Let s_i be a descent of w, and $\alpha = \overline{(i,j)}$ be the corresponding border cell of $D(w)$. Let $D(w) \setminus \alpha$ denote the diagram obtained from $D(w)$ by deleting every cell $(i+rn, j+rn)$ and exchanging rows $(i+rn)$ and $(i+1+rn)$, for all $r \in \mathbb{Z}$. Then the diagram $D(ws_i)$ is $D(w) \setminus \alpha$.*

Lemma 2.11 *Let T be a columns strict balanced labelling of $D(w)$ with largest label M. Then every row containing an M must contain an M in a border cell. In particular, if i is the index of such row, then i must be a descent of w.*

Theorem 2.12 *Let T be any labelling of $D(w)$, and assume some border cell α contains the largest label M in T. Then T is balanced if and only if $T \setminus \alpha$ is balanced.*

Note that Theorem 2.4 in the previous section follows directly from these results. We also obtain a recurrence relation on the number of injective balanced labellings.

Corollary 2.13 *Let $b_{D(w)}$ denote the number of injective balanced labellings of $D(w)$. Then,*

$$b_{D(w)} = \sum_{\alpha} b_{D(w)\backslash\alpha},$$

where the sum is over all border cells α of $D(w)$.

2.3 Encoding and decoding of reduced decompositions

In this section we present a direct combinatorial formula for decoding reduced words from injective balanced labellings of affine permutation diagrams. Again, the theorem in [FGRS97] extends to the affine case naturally.

Theorem 2.14 *Let $w \in \widetilde{\Sigma}_n$ and T be an injective balanced labelling of $D(w)$. Let α be the box in $[D]$ labelled by k. Let*

$$\begin{aligned} I(k) &:= \text{ the row index of } \alpha, \\ R^+(k) &:= \text{ the number of entries } k' > k \text{ in the same row of } \alpha, \\ U^+(k) &:= \text{ the number of entries } k' > k \text{ above } \alpha \text{ in the same column, and} \\ a_k &:= (I(k) + R^+(k) - U^+(k)) \text{ modulo } n. \end{aligned}$$

Then $a = a_1 a_2 \cdots a_{\ell(w)}$ is a reduced word of w, and T is the canonical labelling T_a induced by a.

3 Characterization of affine permutation diagrams

One unexpected application of balanced labellings is a nice characterization of affine permutation diagrams. We will introduce the notion of the *content map* of an affine diagram, which generalizes the classical notion of content of a Young diagram. We will conclude that the existence of such map, along with the *North-West property*, completely characterizes the affine permutation diagrams.

3.1 The content map

Given an affine diagram D of size n, the *oriental labelling* of D will denote the injective labelling of the diagram with numbers from 1 to n such that the numbers increases as we read the boxes in $[D]$ from top to bottom, and from right to left. See Figure 1. (This reading order reminds us of the traditional way to write and read a book in some East Asian countries such as Korea, China, or Japan, and hence the term "oriental".)

Lemma 3.1 *The oriental labelling of an affine (or finite) diagram is a balanced labelling.*

Now, suppose we start from an affine permutations and we construct the oriental labelling of the diagram of the permutation. For example, let $w = [2,6,1,4,3,7,8,5] \in \Sigma_8 \subset \widetilde{\Sigma}_8$. Figure 1 shows the oriental labelling of the diagram of w, where the box labelled by 7 is at the (1,1)-coordinate.

Following the spirit of Theorem 2.14, for each box with label k in the diagram, let us write down the integer a_k where $a_k = I(k) + R^+(k) - U^+(k)$. Recall that $I(k)$ is the row index, $R^+(k)$ the number of entries greater than k in the same row, $U^+(k)$ the number of entries greater than k and located above k in the same column. The formula is actually much simpler in the case of the oriental labelling, since $U^+(k)$ vanishes and $R^+(k)$ is simply the number of boxes to the left of the box labelled by k. Figure 2 illustrates the diagram filled with a_k instead of k. From Theorem 2.14, we already know that we can recover the affine permutation we started with by a_k's. For example, $w = [2,6,1,4,3,7,8,5] = s_5s_6s_7s_4s_3s_4s_1s_2$, where the right hand side comes from reading the Figure 2 "orientally" modulo 8.

Motivated by this example, we define a special way of assigning integers to each box of a diagram, which will take a crucial role in the rest of this section.

Definition 3.2 *Let D be an affine diagram with period n. A map $\mathcal{C} : D \to \mathbb{Z}$ is called a* content map *if it satisfies the following four conditions.*

(C1) If boxes b_1 and b_2 are in the same row (respectively, column), b_2 being to the east (resp., south) to b_1, and there are no boxes between b_1 and b_2, then $\mathcal{C}(b_2) - \mathcal{C}(b_1) = 1$.

(C2) If b_2 is strictly to the southeast of b_1, then $\mathcal{C}(b_2) - \mathcal{C}(b_1) \geq 2$.

(C3) If $b_1 = (i,j)$ and $b_2 = (i+n, j+n)$ coordinate-wise, then $\mathcal{C}(b_2) - \mathcal{C}(b_1) = n$.

(C4) For each row (resp., column), the content of the leftmost (resp., topmost) box is equal to the row (resp., column) index.

Proposition 3.3 *Let D be the diagram of an affine permutation $w \in \widetilde{\Sigma}_n$. Then, D has a unique content map.*

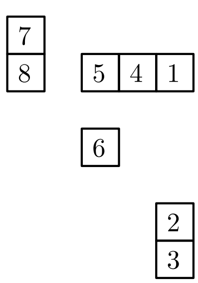

Fig. 1: oriental labelling of a finite diagram

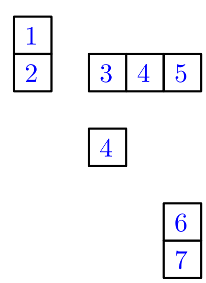

Fig. 2: a_k's of the oriental labelling

3.2 The wiring diagram and the bijection

We start this section by recalling a well-known property of (affine) permutation diagrams.

Definition 3.4 *An affine diagram is called* North-West *(or* NW*) if, whenever there is a box at* (i, j) *and at* (k, ℓ) *with the condition* $i < k$ *and* $j > \ell$*, there is a box at* (i, ℓ)*.*

It is easy to see that every affine permutation diagram is NW. In fact, if $(i, w^{-1}(j))$ and $(k, w^{-1}(\ell))$ is an inversion and $i < k$, $j > \ell$, then $(i, w^{-1}(\ell))$ is also an inversion since $i < k < w^{-1}(\ell)$ and $w(i) > j > \ell$. The main theorem of this section is that the content map and the NW property completely characterize the affine permutation diagrams.

Theorem 3.5 *An affine diagram is an affine permutation diagram if and only if it is NW and admits a content map.*

In fact, given a NW affine diagram D of period n with a content map, we will introduce a combinatorial algorithm to recover the affine permutation $w \in \widetilde{\Sigma}_n$ corresponding to D. This will turn out to be a generalization of the *wiring diagram* appeared in the section 19 of [Pos06], which gave a bijection between Grassmannian permutations and the partitions.

Let D be a NW affine diagram of period n with a content map. A northern edge of a box b in D will be called a *N-boundary* of D if

(1) b is the northeast-most box among all the boxes with the same content and

(2) there is no box above b on the same column.

Similarly, an eastern edge of a box b in D will be called a *E-boundary* of D if

(1) b is the northeast-most box among all the boxes with the same content and

(2) there is no box to the right of b on the same row.

A northern or eastern edge of a box in D will be called a *NE-boundary* if it is either a N-boundary or an E-boundary. We can define an *S-boundary*, *W-boundary*, and *SW-boundary* in the same manner by replacing "north" by "south", "east" by "west", "above" by "below", "right" by "left", etc.

Now, from the midpoint of each NE-boundary, we draw an infinite ray to NE-direction (red rays in Figure 3) and index the ray "i" if it is a N-boundary of a box of content i, and "$i + 1$" if it is an E-boundary of a box of content i. We call such rays *NE-rays*. Similarly, a *SW-ray* is an infinite ray from the midpoint of each SW-boundary to SW-direction (blue rays in Figure 3), indexed "w_i" if it is a W-boundary of a box of content i, and "w_{i+1}" if it is a S-boundary of a box of content i.

Lemma 3.6 *No two NE-rays (respectively, SW-rays) have the same index, and the indices increase as we read the rays from NW to SE direction.*

Lemma 3.7 *There is no NE-ray of index* k *if and only if there is no SW-ray of index* w_k*.*

Now, given a NW affine diagram D with a content map, we construct the *wiring diagram* of D through the following procedure.

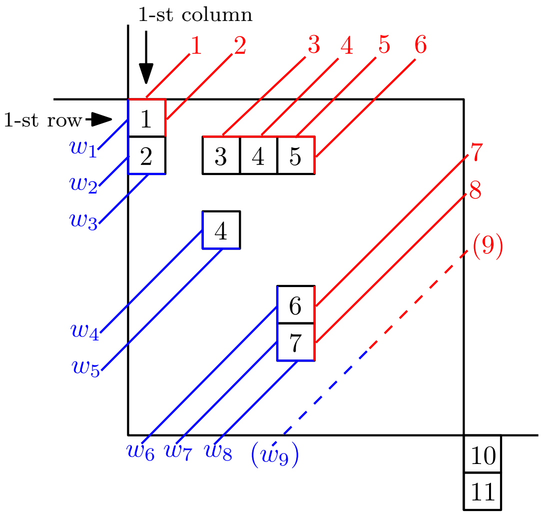

Fig. 3: content, (NE/SW-) boundaries, and rays

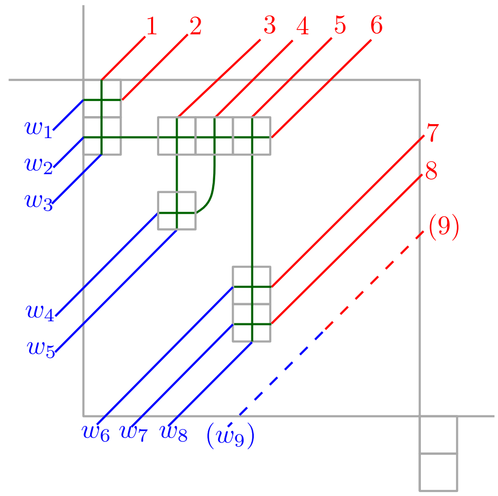

Fig. 4: wiring diagram

(a) (Rays) Draw NE- and SW-rays.

(b) (The "Crosses") Draw a "+" sign inside each box, i.e., connect the midpoint of the western edge to the midpoint of the eastern edge, and the midpoint of the northern edge to the midpoint of the southern edge of each box.

(c) (Horizontal Movement) If the box a and the box b are in the same row (a is to the left of b) and there are no boxes between them, then connect the midpoint of the eastern edge of a to the midpoint of the western edge of b.

(d) (Vertical Movement) If the box a and the box b are in the same column (a is above b) and there are no boxes between them, then connect the midpoint of the southern edge of a to the midpoint of the northern edge of b.

(e) (The "Tunnels") Suppose that the box a of content k is not the northeast-most box among all the boxes with content k and that there is no box on the same row to the right of a. Let b be the closest box to a, which is to the northeast of b and has content k. For every such pair a and b, connect the midpoint of the eastern edge of a to the midpoint of the southern edge of b.

Lemma 3.8 *Each midpoint of an edge of a box in D is connected to exactly two line segments of* (a)*,* (b)*,* (c)*,* (d)*, and* (e)*.*

Figure 4 illustrates the wiring diagram of the affine diagram of period 9 in Figure 2. Note that the curved line connecting two boxes of content 4 is a "tunnel". Once we draw this wiring diagram of a NW affine diagram with a content, it is very easy to recover the affine permutation corresponding to the diagram. From a NE-ray indexed by i, proceed to the southwest direction following the lines in the wiring diagram until we meet a SW-ray of index w_j. This translates to $w_j = i$ in the corresponding affine permutation. If there is no NE-ray of index i (equivalently, no SW-ray of index w_i), then let $w_i = i$. For instance, Figure 4 corresponds to the affine permutation $w = [w_1, w_2, \dots, w_9] = [2, 6, 1, 4, 3, 7, 8, 5, 9] \in \Sigma_9 \subset \widetilde{\Sigma}_9$

Proposition 3.9 *The wiring diagram gives a bijection between the NW affine diagrams of period n with a content map, and the affine permutations in $\widetilde{\Sigma}_n$.*

Our main result of this section, Theorem 3.5, is a direct consequence of Proposition 3.9.

4 Diagram equivalence conjecture

In this section we consider some sufficient condition for affine permutations to have the same affine Stanley symmetric function.

For finite permutations, Stanley symmetric functions are the Frobenius character of the diagram Specht module of the permutation diagram [RS95]. By definition, the Specht module is invariant under permuting rows and columns of the diagram, hence so is the Stanley symmetric function. We call this invariance property *diagram equivalence*. We extend the notion to the affine permutations.

4.1 Diagram graphs and equivalence

To each affine diagram D, let us associate a bipartite graph G_D.

Definition 4.1 *The* diagram graph G_D *of D is defined as follows:*

1. *$G_D = (V, E_D)$ where $V = V_L \sqcup V_R$ is the vertex set, and each edge in E_D connects a vertex in V_L with a vertex in V_R. Here V_L and V_R are called the left and right vertices, respectively.*
2. *Both V_L and V_R are indexed by $\mathbb{Z}$, and $(i, j) \in E_D$ denotes the edge connecting $i \in V_L$ with $j \in V_R$.*
3. *$(i, j) \in E_D$ if and only if $(i, j) \in D$.*

It is immediate from the definition of affine diagram that G_D has the following properties: (1) $(i, j) \in E \Leftrightarrow (i+n, j+n) \in E$, (2) every vertex has a finite degree. We will call a graph that has these properties *n-periodic*.

Definition 4.2 *Two n-periodic graphs G and H are* isomorphic *if there is a pair of bijections (ϕ, ψ) between vertices of G and H such that*

1. *ϕ is a bijection between the left vertices, and ψ is between the right vertices,*
2. *$(i, j) \in E_G \Leftrightarrow (\phi(i), \psi(j)) \in E_H$.*

Definition 4.3 *Two affine permutations $u, v \in \widetilde{\Sigma}_n$ are said to be* diagram equivalent *if $G_{D(u)}$ and $G_{D(v)}$ are isomorphic.*

Conjecture 4.4 *$\widetilde{F}_u(x) = \widetilde{F}_v(x)$ if u and v are diagram equivalent.*

Remark 4.5 *Note that the converse is not true. The Stanley symmetric functions may coincide even if the permutations are not diagram equivalent. For finite skew diagrams, more precise conditions are studied in [RSvW07].*

There is a natural dihedral symmetry of the Dynkin diagram of affine type A. This symmetry gives us an obvious relation between the affine Stanley symmetric functions of certain permutations. The $*$-operator in [Lam06] is an example of this symmetry. Obviously w and $(w^*)^{-1}$ have the same affine Stanley symmetric function, and it is easy to check that they actually are diagram equivalent.

When $G_{D(w)}$ does not have any cycle, the number of reduced decompositions of w turns to be the same as the normalized volume of certain polytope called *periodic matching polytope*. Since the periodic matching polytope is invariant under the diagram equivalence, this result supports our conjecture. Precise definitions and the results on periodic matching polytopes will appear in a separate paper [Yoo13].

5 Further questions

1. Can we define the affine Schubert polynomial using balanced labellings? Geometrically, they have to form a basis of the cohomology ring of the affine flag variety, and combinatorially, they would have to satisfy an analogue of the transition equation of Schubert polynomials.

2. It is possible to define set-valued balanced labellings of affine diagrams and describe affine stable Grothen-dieck polynomials using them. Also we were able to use flagged column strict set-valued balanced labellings to describe Grothendieck polynomials. As in the previous question, we wonder if this result can be extended to affine Grothendieck polynomials.

3. One motivation of studying the diagram of affine permutations is to answer the question posed by Lam in [Lam06]. How can we characterize the "affine vexillary permutations"?

4. Can we prove Conjecture 4.4 by finding a bijection between column strict balanced labellings of two diagram equivalent permutations? For dihedral symmetry, it is easy to find such a bijection but it does not extend to the general situation.

5. Our Theorem 3.5 is not local since we have to find a global datum like content. Is there local criterion of affine permutation diagrams? More precisely, can we characterize the affine permutation diagrams by avoiding some diagram patterns?

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