On the poset of weighted partitions

Rafael S. González D'León † and Michelle L. Wachs ‡

Department of Mathematics, University of Miami, Coral Gables, FL 33124

Abstract. In this extended abstract we consider the poset of weighted partitions Π_n^w, introduced by Dotsenko and Khoroshkin in their study of a certain pair of dual operads. The maximal intervals of Π_n^w provide a generalization of the lattice Π_n of partitions, which we show possesses many of the well-known properties of Π_n. In particular, we prove these intervals are EL-shellable, we compute the Möbius invariant in terms of rooted trees, we find combinatorial bases for homology and cohomology, and we give an explicit sign twisted $\mathfrak{S}_n$-module isomorphism from cohomology to the multilinear component of the free Lie algebra with two compatible brackets. We also show that the characteristic polynomial of Π_n^w has a nice factorization analogous to that of Π_n.

Résumé. Dans ce résumé étendu, nous considèrons l'ensemble ordonné des partitions pondérées Π_n^w, introduit par Dotsenko et Khoroshkin dans leur étude d'une certaine paire d'opérades duales. Les intervalles maximaux de Π_n^w généralisent le treillis Π_n des partitions et, comme nous le montrons, possèdent beaucoup de propriétés classiques de Π_n. En particulier, nous prouvons que ces intervalles sont "EL-shellable", nous exprimons leur invariant de Möbius en fonction d'arbres enracinés, nous trouvons des bases combinatoires pour l'homologie et la cohomologie et nous donnes un morphisme explicite de $\mathfrak{S}_n$-modules gauches entre la cohomologie et la composante multilinéaire de l'algèbre de Lie libre avec deux crochets de Lie compatibles. Nous montrons aussi que le polynôme caractéristique de Π_n^w admet une factorisation sympathique, analogue à celle de Π_n.

Keywords: poset topology, partitions, free Lie algebra, rooted trees

1 Introduction

We recall some combinatorial, topological and representation theoretic properties of the lattice Π_n of partitions of the set $[n] := \{1, 2, \dots, n\}$ ordered by refinement. The Möbius invariant of Π_n is given by $\mu(\Pi_n) = (-1)^{n-1}(n-1)!$ and the characteristic polynomial by $\chi_{\Pi_n}(x) = (x-1)(x-2)\dots(x-n+1)$ (see [18, Example 3.10.4]). The order complex $\Delta(P)$ of a poset P is the simplicial complex whose faces are the chains of P; and the proper part $\bar{P}$ of a bounded poset P is the poset obtained by removing the minimum element $\hat{0}$ and the maximum element $\hat{1}$. It was proved by Björner [2], using an edge labeling of Stanley [16], that Π_n is EL-shellable; consequently the order complex $\Delta(\overline{\Pi}_n)$ has the homotopy type of a wedge of $(n-1)!$ spheres of dimension $n-3$. The falling chains of the EL-labeling provide a basis for cohomology of $\Delta(\overline{\Pi}_n)$ called the Lyndon basis. See [21] for a discussion of this basis and other nice bases for the homology and cohomology of the partition lattice.

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The symmetric group $\mathfrak{S}_n$ acts naturally on Π_n and this induces isomorphic representations of $\mathfrak{S}_n$ on the unique nonvanishing reduced simplicial homology $\tilde{H}_{n-3}(\overline{\Pi}_n)$ of the order complex $\Delta(\overline{\Pi}_n)$ and on the unique nonvanishing simplicial cohomology $\tilde{H}^{n-3}(\overline{\Pi}_n)$. (Throughout this paper homology and cohomology are taken over $\mathbb{C}$.) Joyal [12] observed that a formula of Stanley [17] for the character of this representation is a sign twisted version of an earlier formula of Brandt [5] for the character of the representation of $\mathfrak{S}_n$ on the multilinear component $Lie(n)$ of the free Lie algebra on n generators. Hence the following $\mathfrak{S}_n$-module isomorphism holds,

$$\tilde{H}_{n-3}(\overline{\Pi}_n) \simeq_{\mathfrak{S}_n} \mathcal{L}ie(n) \otimes \mathrm{sgn}_n, \tag{1.1}$$

where sgn_n is the sign representation of $\mathfrak{S}_n$. Joyal [12] gave a species theoretic proof of the isomorphism. The first purely combinatorial proof was obtained by Barcelo [1] who gave a bijection between known bases for the two $\mathfrak{S}_n$-modules (Björner's NBC basis for $\tilde{H}_{n-3}(\bar{\Pi}_n)$ and the Lyndon basis for $\mathcal{L}ie(n)$). Later Wachs [21] gave a more general combinatorial proof by providing a natural bijection between generating sets of $\tilde{H}^{n-3}(\overline{\Pi}_n)$ and $\mathcal{L}ie(n)$, which revealed the strong connection between the two $\mathfrak{S}_n$-modules.

In this paper we explore analogous properties for a weighted version of Π_n, introduced by Dotsenko and Khoroshkin [6] in their study of Koszulness of certain quadratic binary operads. A weighted partition of $[n]$ is a set $\{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\}$ where $\{B_1, B_2, ..., B_t\}$ is a partition of $[n]$ and $v_i \in \{0, 1, 2, ..., |B_i| - 1\}$ for all i. The *poset of weighted partitions* Π_n^w is the set of weighted partitions of $[n]$ with covering relation given by $\{A_1^{w_1}, A_2^{w_2}, ..., A_s^{w_s}\} \lessdot \{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\}$ if the following conditions hold

- $\{A_1, A_2, \dots, A_s\} \lessdot \{B_1, B_2, \dots, B_t\}$ in Π_n
- if $B_k = A_i \cup A_j$, where $i \neq j$, then $v_k - (w_i + w_j) \in \{0, 1\}$.

In Figure 1 below the set brackets and commas have been omitted.

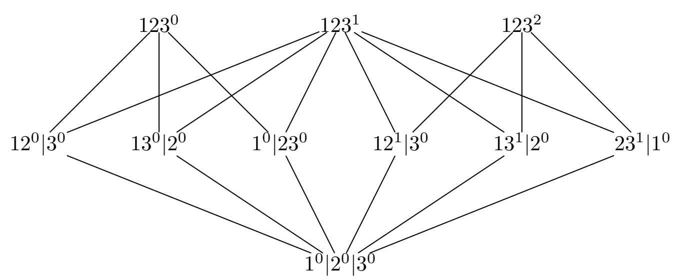


Fig. 1: Weighted partition poset for $n = 3$

We remark that Π_n^w resembles, but is not the same as, a Rees product poset introduced by Björner and Welker [4] and studied in [15].

The poset Π_n^w has a minimum element $\hat{0} = 1^0|2^0|\dots|n^0$ and n maximal elements $[n]^0,\ [n]^1, \dots, [n]^{n-1}$. Note that for all i, the maximal intervals $[\hat{0}, [n]^i]$ and $[\hat{0}, [n]^{n-1-i}]$ are isomorphic to each other, and the two maximal intervals $[\hat{0}, [n]^0]$ and $[\hat{0}, [n]^{n-1}]$ are isomorphic to Π_n.

In this paper[(i)] we prove that the augmented weighted partition poset $\widehat{\Pi_n^w} := \Pi_n^w \cup \{\hat{1}\}$ is EL-shellable by providing an interesting weighted analog of the Björner-Stanley EL-labeling of Π_n. In fact our labeling restricts to the Björner-Stanley EL-labeling on the intervals $[\hat{0}, [n]^0]$ and $[\hat{0}, [n]^{n-1}]$. A consequence of shellability is that $\widehat{\Pi_n^w}$ is Cohen-Macaulay, which implies a result of Dotsenko and Khoroshkin [7], obtained through operad theory, that all maximal intervals $[\hat{0}, [n]^i]$ of Π_n^w are Cohen-Macaulay. (Two prior attempts [6, 19] to establish Cohen-Macaulayness of $[\hat{0}, [n]^i]$ are discussed in Remark 2.3.) The falling chains of our EL-labeling also provide a generalization of the Lyndon basis for cohomology of Π_n.

It follows from an operad theoretic result of Vallette [20] and the Cohen-Macaulayness of each maximal interval $[\hat{0}, [n]^i]$ that the following $\mathfrak{S}_n$-module isomorphism holds:

$$\bigoplus_{i=0}^{n-1} \tilde{H}_{n-3}((\hat{0}, [n]^i)) \simeq_{\mathfrak{S}_n} \mathcal{L}ie_2(n) \otimes \operatorname{sgn}_n, \tag{1.2}$$

where $\mathcal{L}ie_2(n)$ is the representation of $\mathfrak{S}_n$ on the multilinear component of the free Lie algebra on n generators with two compatible brackets (defined in Section 3.2) and $\tilde{H}_{n-3}((\hat{0}, [n]^i))$ is the reduced simplicial homology of the order complex of the open interval $(\hat{0}, [n]^i)$. A graded version of this isomorphism, which can also be proved using Vallette's technique, implies (1.1). In Section 3.3 we reveal the connection between these graded modules by providing an explicit bijection between generating sets of $\bigoplus_{i=0}^{n-1} \tilde{H}^{n-3}((\hat{0}, [n]^i))$ and $\mathcal{L}ie_2(n)$, which generalizes the bijection that Wachs [21] used to prove (1.1).

In [13] Liu proves a conjecture of Feigin that $\dim \mathcal{L}ie_2(n) = n^{n-1}$ by constructing a combinatorial basis for $\mathcal{L}ie_2(n)$ indexed by rooted trees. (It is well-known that n^{n-1} is the number of rooted trees on node set $[n]$.) An operad theoretic proof of Feigin's conjecture was obtained by Dotsenko and Khoroshkin [6], but with a gap pointed out in [19] and corrected in [7].

Liu and Dotsenko/Khoroshkin obtain a more general graded version of Feigin's conjecture. In this paper we take a different path to proving the graded version. In Section 2.2 we compute the Möbius invariant of the maximal intervals of Π_n^w by exploiting the recursive nature of Π_n^w and applying the compositional formula. From our computation and the fact that $\widehat{\Pi_n^w}$ is Cohen-Macaulay we conclude that

$$\sum_{i=0}^{n-1} \dim \tilde{H}_{n-3}((\hat{0}, [n]^i)) t^i = \prod_{i=1}^{n-1} ((n-i) + it). \tag{1.3}$$

The Liu and Dotsenko/Khoroshkin result is a consequence of this and (1.2). Since, as was proved by Drake [8], the right hand side of (1.3) is the generating function for rooted trees on $[n]$ with i descents, it follows that $\dim \tilde{H}_{n-3}((\hat{0}, [n]^i))$ is equal to the number of rooted trees on $[n]$ with i descents. In Section 4 we construct a nice combinatorial basis for $\tilde{H}_{n-3}((\hat{0}, [n]^i))$ indexed by such rooted trees, which generalizes Björner's basis for $\tilde{H}_{n-3}(\bar{\Pi}_n)$. A generalization of the comb basis for cohomology of Π_n and the generalization of the Lyndon basis for cohomology of Π_n given by the falling chains of the EL-labeling of $[\hat{0}, [n]^i]$ are also presented in Section 4.

In the final section of this paper, we show that the characteristic polynomial of Π_n^w equals $(x-n)^{n-1}$ and present some consequences. Finally we mention some generalizations of what is presented here.

(i) i.e., the full version of this paper [11]

2 The topology of the poset of weighted partitions

2.1 EL-Shellability

We assume familiarity with basic terminology and results in poset topology; see [22].

For each $i \in [n]$, let $P_i := \{(i,j)^u | i < j \leq n+1,\ u \in \{0,1\}\}$. We partially order P_i by letting $(i,j)^u \leq (i,k)^v$ if $j \leq k$ and $u \leq v$. Note P_i is isomorphic to the direct product of the chain $i+1 < i+2 < \cdots < n+1$ and the chain $0 < 1$. Now define Λ_n to be the ordinal sum $\Lambda_n := P_1 \oplus P_2 \oplus \cdots \oplus P_n$.

The Hasse diagram of the poset Λ_n when $n = 3$ is given in Figure 2(b).

Theorem 2.1 *The labeling* $\lambda : \mathcal{E}(\widehat{\Pi_n^w}) \to \Lambda_n$ *defined by:*

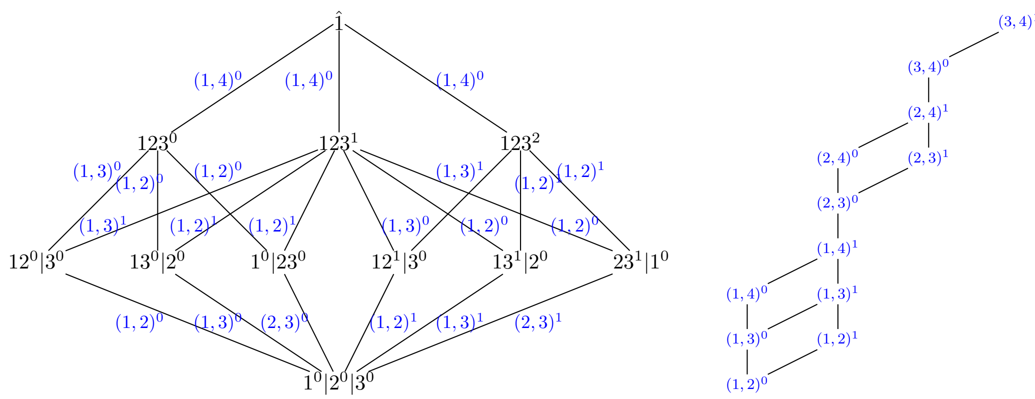
$$\lambda(A_1^{w_1}|A_2^{w_2}|\cdots|A_l^{w_l} \lessdot A_1^{w_1}|A_2^{w_2}|\cdots|(A_i \cup A_j)^{w_i+w_j+u}|\cdots|A_l^{w_l}) := (\min A_i, \min A_j)^u\,,$$

and

$$\tilde{\lambda}([n]^r \lessdot \hat{1}) = (1, n+1)^0,$$

where $\min A_i < \min A_j$, *is an EL-labeling of* Π_n^w.

The proof is given in the full version of this paper. In Figure 2(a) we give an example.



(a) Labeling λ (b) Λ_3

Fig. 2: EL-labeling of the poset $\widehat{\Pi_3^w}$

In [7] Dotsenko and Khoroshkin use operad theory to prove that all intervals of Π_n^w are Cohen-Macaulay. We have the following extension of their result.

Corollary 2.2 *The poset* $\widehat{\Pi_n^w}$ *is Cohen-Macaulay.*

Remark 2.3 In a prior attempt to establish Cohen-Macaulayness of each maximal interval $[\hat{0}, [n]^i]$ of Π_n^w, it is argued in [6] that the intervals are totally semimodular (and hence CL-shellable). In [19] it is noted that this is not the case and a proposed recursive atom ordering of each maximal interval $[\hat{0}, [n]^i]$ is given in order to establish CL-shellability. We note here that one of the requisite conditions in the definition of recursive atom ordering fails to hold when $n = 4$ and $i = 2$. Indeed, assuming (without loss of generality) that the first two atoms in the atom ordering of $[\hat{0}, [4]^2]$ given in [19] are 12^0 and 12^1 (the singleton blocks have been omitted), it is not difficult to see that the proposed atom ordering of the interval $[12^1, [4]^2]$ given in [19] will fail to satisfy the condition that the covers of 12^0 come first.

2.2 Möbius Invariant

Since each maximal interval of Π_n^w is Cohen-Macaulay, we can determine the dimension of the top (unique nonvanishing) (co)homology of the interval by the use of the Möbius invariant. We use the recursive definition of the Möbius function and the compositional formula to derive the following result.

Proposition 2.4 *For all* $n \geq 1$,

$$\sum_{i=0}^{n-1} \mu_{\Pi_n^w}(\hat{0}, [n]^i) t^i = (-1)^{n-1} \prod_{i=1}^{n-1} ((n-i) + it). \tag{2.1}$$

Consequently,

$$\sum_{i=0}^{n-1} \mu_{\Pi_n^w}(\hat{0}, [n]^i) = (-1)^{n-1} n^{n-1}.$$

Let T be a rooted tree on node set $[n]$. A *descent* of T is a node that is smaller than its parent. We denote by $\mathcal{T}_{n,i}$ the set of rooted trees on node set $[n]$ with exactly i descents. In [8] Drake proves that

$$\sum_{i=0}^{n-1} |\mathcal{T}_{n,i}| t^i = \prod_{i=1}^{n-1} ((n-i) + it). \tag{2.2}$$

The following result is a consequence of this and Proposition 2.4.

Corollary 2.5 *For all* $n \geq 1$ *and* $i \in \{0, 1, \ldots, n-1\}$,

$$\mu_{\Pi_n^w}(\hat{0}, [n]^i) = (-1)^{n-1} |\mathcal{T}_{n,i}|.$$

Recall that if P is an EL-shellable poset of length ℓ then $\Delta(\bar{P})$ has the homotopy type of a wedge of $|\mu(P)|$ spheres of dimension $\ell - 2$. The following result is therefore a consequence of Theorem 2.1 and Corollary 2.5.

Theorem 2.6 *For all* $n \geq 1$ *and* $i \in \{0, 1, \ldots, n-1\}$, *the simplicial complex* $\Delta((\hat{0}, [n]^i))$ *has the homotopy type of a wedge of* $|\mathcal{T}_{n,i}|$ *spheres of dimension* $n - 3$.

Corollary 2.7 ([6, 7]) *For all* $n \geq 1$ *and* $i \in \{0, 1, \ldots, n-1\}$,

$$\dim \tilde{H}^{n-3}((\hat{0}, [n]^i)) = |\mathcal{T}_{n,i}| \qquad \textit{and} \qquad \dim \bigoplus_{i=0}^{n-1} \tilde{H}^{n-3}((\hat{0}, [n]^i)) = n^{n-1}.$$

In the full version of the paper we also obtain the following result by using the same methods.

Theorem 2.8 *For all* $n \geq 1$, *the simplicial complex* $\Delta(\Pi_n^w \setminus \{\hat{0}\})$ *has the homotopy type of a wedge of* $(n-1)^{n-1}$ *spheres of dimension* $n - 2$.

3 Connection with the doubly bracketed free Lie algebra

3.1 The doubly bracketed free Lie algebra

Recall that a *Lie bracket* on a vector space V is a bilinear binary product $[\cdot,\cdot] : V \times V \to V$ such that for all $x, y, z \in V$,

$$[x,y] = -[y,x] \qquad \text{(Antisymmetry)} \tag{3.1}$$

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0 \qquad \text{(Jacobi Identity)} \tag{3.2}$$

The *free Lie algebra on* $[n]$ is the complex vector space generated by the elements of $[n]$ and all the possible bracketings involving these elements subject only to the relations (3.1) and (3.2). Let $\mathcal{L}ie(n)$ denote the *multilinear* component of the free Lie algebra on $[n]$, ie., the subspace generated by bracketings that contain each element of $[n]$ exactly once. For example $[[2,3],1]$ is an element of $\mathcal{L}ie(3)$, while $[[2,3],2]$ is not.

Now let V be a vector space equipped with two Lie brackets $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$. The brackets are said to be *compatible* if any linear combination of them is a Lie bracket. As pointed out in [6, 13], compatibility is equivalent to the condition that for all $x, y, z \in V$

$$[x,\langle y,z\rangle] + [z,\langle x,y\rangle] + [y,\langle z,x\rangle] + \langle x,[y,z]\rangle + \langle z,[x,y]\rangle + \langle y,[z,x]\rangle = 0 \qquad \text{(Mixed Jacobi)} \tag{3.3}$$

Let $\mathcal{L}ie_2(n)$ be the multilinear component of the free Lie algebra on $[n]$ with two compatible brackets $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$, that is, the multilinear component of the vector space generated by (mixed) bracketings of elements of $[n]$ subject only to the five relations given by (3.1) and (3.2), for each bracket, and (3.3). We will call the bracketed words that generate $\mathcal{L}ie_2(n)$ *bracketed permutations*.

It will be convenient to refer to the bracket $[\cdot,\cdot]$ as the *blue* bracket and the bracket $\langle\cdot,\cdot\rangle$ as the *red* bracket. For each i, let $\mathcal{L}ie_2(n,i)$ be the subspace of $\mathcal{L}ie_2(n)$ generated by bracketed permutations with exactly i red brackets and $n-1-i$ blue brackets.

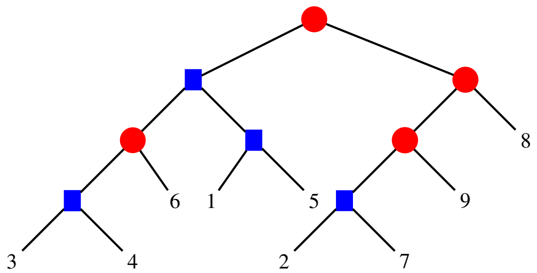
A permutation $\tau \in \mathfrak{S}_n$ acts on the bracketed permutations by replacing each letter i by $\tau(i)$. For example $(1,2)\ \langle[\langle 3,5\rangle,[2,4]],1\rangle = \langle[\langle 3,5\rangle,[1,4]],2\rangle$. Since this action respects the five relations, it induces a representation of $\mathfrak{S}_n$ on $\mathcal{L}ie_2(n)$. Since this action also preserves the number of blue and red brackets, we have the following decomposition into $\mathfrak{S}_n$-submodules: $\mathcal{L}ie_2(n) = \oplus_{i=0}^{n-1}\mathcal{L}ie_2(n,i)$. Note that $\mathcal{L}ie_2(n,i) \simeq_{\mathfrak{S}_n} \mathcal{L}ie_2(n,n-1-i)$ for all i, and that $\mathcal{L}ie_2(n,0) \simeq_{\mathfrak{S}_n} \mathcal{L}ie_2(n,n-1) \simeq_{\mathfrak{S}_n} \mathcal{L}ie(n)$.

A *bicolored binary tree* is a complete planar binary tree (i.e., every internal node has a left child and a right child) for which each internal node has been colored blue or red. For a bicolored binary tree T with n leaves and $\sigma \in \mathfrak{S}_n$, define the *labeled bicolored binary tree* (T,σ) to be the tree T whose jth leaf from left to right has been labeled $\sigma(j)$. We denote by $\mathcal{BT}_n$ the set of labeled bicolored binary trees with n leaves and by $\mathcal{BT}_{n,i}$ the set of labeled bicolored binary trees with n leaves and i red internal nodes. It will also be convenient to allow the leaf labeling σ to be a bijection from $[n]$ to any subset of $\mathbb{Z}^+$ of size n.

We can represent the bracketed permutations that generate $\mathcal{L}ie_2(n)$ with labeled bicolored binary trees. More precisely, let (T_1,σ_1) and (T_2,σ_2) be the respective left and right labeled subtrees of the root r of $(T,\sigma) \in \mathcal{BT}_n$. Then define recursively

$$[T,\sigma] = \begin{cases} [[T_1,\sigma_1],[T_2,\sigma_2]] & \text{if } r \text{ is blue and } n > 1 \\ \langle[T_1,\sigma_1],[T_2,\sigma_2]\rangle & \text{if } r \text{ is red and } n > 1 \\ \sigma(1) & \text{if } n = 1 \end{cases} \tag{3.4}$$

Clearly $(T,\sigma) \in \mathcal{BT}_{n,i}$ if and only if $[T,\sigma]$ is a bracketed permutation of $\mathcal{L}ie_2(n,i)$. See Figure 3.



$\langle[\langle[3,4],6\rangle,[1,5]],\langle\langle[2,7],9\rangle,8\rangle\rangle$

Fig. 3: Example of a tree $(T,[346152798]) \in \mathcal{BT}_{9,4}$ and $[T,[346152798]] \in \mathcal{L}ie_2(9,4)$

3.2 A generating set for $\tilde{H}^{n-3}((\hat{0},[n]^i))$

The top dimensional cohomology of a pure poset P, say of length ℓ, has a particularly simple description. Let $\mathcal{M}(P)$ denote the set of maximal chains of P and let $\mathcal{M}'(P)$ denote the set of chains of length $\ell-1$. We view the coboundary map δ as a map from the chain space of P to itself, which takes chains of length d to chains of length $d+1$ for all d. Since the image of δ on the top chain space (i.e. the space spanned by $\mathcal{M}(P)$) is 0, the kernel is the entire top chain space. Hence top cohomology is the quotient of the space spanned by $\mathcal{M}(P)$ by the image of the space spanned by $\mathcal{M}'(P)$. The image of $\mathcal{M}'(P)$ is what we call the coboundary relations. We thus have the following presentation of the top cohomology

$$\tilde{H}^{\ell}(P) = \langle \mathcal{M}(P) | \text{ coboundary relations}\rangle.$$

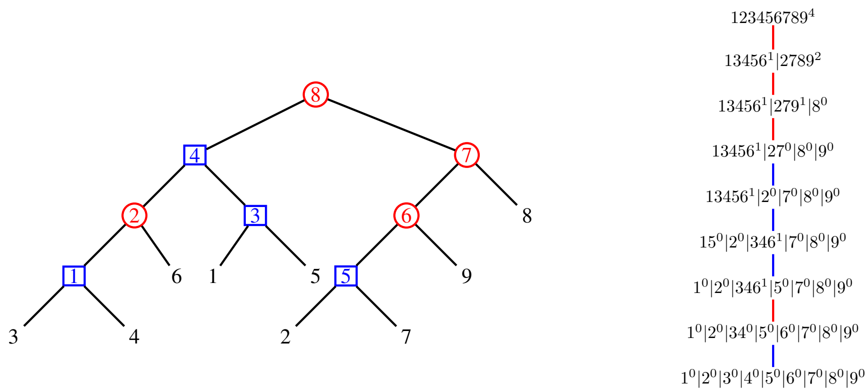
Recall that the *postorder listing* of the internal nodes of a binary tree T is defined recursively as follows: first list the internal nodes of the left subtree in postorder, then list the internal nodes of the right subtree in postorder, and finally list the root.

The postorder listing of the internal nodes of the binary tree of Figure 3 is illustrated in Figure 4(a) by labeling the internal nodes.

Definition 3.1 *For $(T,\sigma) \in \mathcal{BT}_n$, let x_k be the kth internal node in the postorder listing of the internal nodes of T. The chain $\mathrm{c}(T,\sigma) \in \mathcal{M}(\Pi_n^w)$ is the one whose rank k weighted partition π_k is obtained from the rank $k-1$ weighted partition π_{k-1} by merging the blocks L_k and R_k, where $k \geq 1$, L_k is the set of leaf labels in the left subtree of the node x_k, and R_k is the set of leaf labels in the right subtree of the node x_k. The weight attached to the new block $L_k \cup R_k$ is the sum of the weights of the blocks L_k and R_k in π_{k-1} plus u, where $u=0$ if x_k is colored blue and $u=1$ if x_k is colored red. See Figure 4.*

Not all maximal chains in $\mathcal{M}(\Pi_n^w)$ are of the form $\mathrm{c}(T,\sigma)$. It can be shown, however, that any maximal chain $c \in \mathcal{M}(\Pi_n^w)$ is cohomology equivalent to a chain of the form $\mathrm{c}(T,\sigma)$; more precisely, in cohomology $\bar{c} = \pm\bar{\mathrm{c}}(T,\sigma)$, where $\bar{c}$ is the chain obtained from c by removing the top and bottom elements.

Let $I(\Upsilon)$ denote the number of internal nodes of a labeled bicolored binary tree Υ. Let $\Upsilon_1 \overset{\mathrm{col}}{\wedge} \Upsilon_2$ denote a labeled bicolored binary tree whose left subtree is Υ_1, right subtree is Υ_2 and root is colored col, where col $\in \{\text{blue}, \text{ red}\}$. If Υ is a labeled bicolored binary tree then $\alpha(\Upsilon)\beta$ denotes a labeled bicolored binary tree with Υ as a subtree. The following result generalizes [21, Theorem 5.3].



(a) $(T,\sigma) \in \mathcal{BT}_9$ (b) $\mathrm{c}(T,\sigma)$

Fig. 4: Example of postorder labeling of the binary tree T of Figure 3 and the chain $\mathrm{c}(T,\sigma)$

Theorem 3.2 *The set* $\{\bar{\mathrm{c}}(T,\sigma) | (T,\sigma) \in \mathcal{BT}_{n,i}\}$ *is a generating set for* $\tilde{H}^{n-3}((\hat{0},[n]^i))$, *subject only to the relations*

- $\bar{\mathrm{c}}(\alpha(\Upsilon_1 \overset{\mathrm{col}}{\wedge} \Upsilon_2)\beta) = (-1)^{I(\Upsilon_1)I(\Upsilon_2)}\bar{\mathrm{c}}(\alpha(\Upsilon_2 \overset{\mathrm{col}}{\wedge} \Upsilon_1)\beta)$ *where* $\mathrm{col} \in \{blue, red\}$.

- $\bar{\mathrm{c}}(\alpha(\Upsilon_1 \overset{\mathrm{col}}{\wedge} (\Upsilon_2 \overset{\mathrm{col}}{\wedge} \Upsilon_3))\beta) + (-1)^{I(\Upsilon_3)}\bar{\mathrm{c}}(\alpha((\Upsilon_1 \overset{\mathrm{col}}{\wedge} \Upsilon_2) \overset{\mathrm{col}}{\wedge} \Upsilon_3)\beta)$
  $+ (-1)^{I(\Upsilon_1)I(\Upsilon_2)}\bar{\mathrm{c}}(\alpha(\Upsilon_2 \overset{\mathrm{col}}{\wedge} (\Upsilon_1 \overset{\mathrm{col}}{\wedge} \Upsilon_3))\beta) = 0,$ *where* $\mathrm{col} \in \{blue, red\}$.

- $\bar{\mathrm{c}}(\alpha(\Upsilon_1 \overset{\mathrm{red}}{\wedge} (\Upsilon_2 \overset{\mathrm{blue}}{\wedge} \Upsilon_3))\beta) + (-1)^{I(\Upsilon_3)}\bar{\mathrm{c}}(\alpha((\Upsilon_1 \overset{\mathrm{blue}}{\wedge} \Upsilon_2) \overset{\mathrm{red}}{\wedge} \Upsilon_3)\beta)$
  $+ (-1)^{I(\Upsilon_1)I(\Upsilon_2)}\bar{\mathrm{c}}(\alpha(\Upsilon_2 \overset{\mathrm{red}}{\wedge} (\Upsilon_1 \overset{\mathrm{blue}}{\wedge} \Upsilon_3))\beta) + \bar{\mathrm{c}}(\alpha(\Upsilon_1 \overset{\mathrm{blue}}{\wedge} (\Upsilon_2 \overset{\mathrm{red}}{\wedge} \Upsilon_3))\beta)$
  $+ (-1)^{I(\Upsilon_3)}\bar{\mathrm{c}}(\alpha((\Upsilon_1 \overset{\mathrm{red}}{\wedge} \Upsilon_2) \overset{\mathrm{blue}}{\wedge} \Upsilon_3)\beta) + (-1)^{I(\Upsilon_1)I(\Upsilon_2)}\bar{\mathrm{c}}(\alpha(\Upsilon_2 \overset{\mathrm{blue}}{\wedge} (\Upsilon_1 \overset{\mathrm{red}}{\wedge} \Upsilon_3))\beta) = 0$

The proof is given in the full version of this paper. The bicolored comb basis of Theorem 4.1 and the formula for the Möbius function given in Proposition 2.4 play a key role in showing that these relations generate all the relations.

3.3 The isomorphism

Define the sign of a bicolored binary tree T recursively by $\mathrm{sgn}(T) = 1$ if $I(T) = 0$ and $\mathrm{sgn}(T_1 \overset{\mathrm{col}}{\wedge} T_2) = (-1)^{I(T_2)}\,\mathrm{sgn}(T_1)\,\mathrm{sgn}(T_2)$ otherwise.

Theorem 3.3 *For each* $i \in \{0,1,\ldots,n-1\}$, *there is an* $\mathfrak{S}_n$*-module isomorphism* $\phi : \mathcal{L}ie_2(n,i) \to \tilde{H}^{n-3}((\hat{0},[n]^i)) \otimes \mathrm{sgn}_n$ *determined by*

$$\phi([T,\sigma]) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(T)\bar{\mathrm{c}}(T,\sigma),$$

for all $(T,\sigma) \in \mathcal{BT}_{n,i}$.

The map ϕ maps generators to generators and clearly respects the $\mathfrak{S}_n$ action. In the full version of the paper we prove that the map ϕ is a well-defined $\mathfrak{S}_n$-module isomorphism by showing that the relations for $\mathcal{L}ie_2(n)$ given in (3.1), (3.2), (3.3) map to the relations in Theorem 3.2.

4 Combinatorial bases

4.1 The bicolored comb basis and Lyndon basis for cohomology

In this section we present generalizations of both the comb basis and the Lyndon basis for cohomology of Π_n (see [21]). Define a *normalized tree* to be a labeled bicolored binary tree in which the leftmost leaf of each subtree has the smallest label in the subtree. A *bicolored comb* is a normalized tree in which each internal node with an internal right child is colored red and the right child is colored blue. Note that if a bicolored comb is monochromatic then the right child of every internal node is a leaf. Hence the monochromatic ones are the usual combs in the sense of [21].

Given a normalized tree, we can extend the leaf labeling to the internal nodes by labeling each internal node with the smallest leaf label in its right subtree. A *bicolored Lyndon tree* is a normalized tree in which each internal node that has an internal left child with a smaller label is colored blue and the left child is colored red. It is easy to see that the monochromatic ones are the classical Lyndon trees.

Let $\mathsf{Comb}^2_{n,i} \subseteq \mathcal{BT}_{n,i}$ be the set of all bicolored combs with label set $[n]$ and i red internal nodes and let $\mathsf{Lyn}^2_{n,i} \subseteq \mathcal{BT}_{n,i}$ be the set of all bicolored Lyndon trees with label set $[n]$ and i red internal nodes.

Theorem 4.1 *The sets* $\mathcal{C}_{n,i} := \{\bar{c}(T,\sigma) | (T,\sigma) \in \mathsf{Comb}^2_{n,i}\}$ *and* $\mathcal{L}_{n,i} := \{\bar{c}(T,\sigma) | (T,\sigma) \in \mathsf{Lyn}^2_{n,i}\}$ *are bases for* $\tilde{H}^{n-3}((\hat{0},[n]^i))$.

The proof for $\mathcal{C}_{n,i}$ is obtained by "straightening" via the relations in Theorem 3.2 and counting bicolored combs. The proof for $\mathcal{L}_{n,i}$ follows from the observation that $\mathcal{L}_{n,i}$ is the falling chain basis coming from the EL-labeling of $[\hat{0},[n]^i]$ given in Section 2.1.

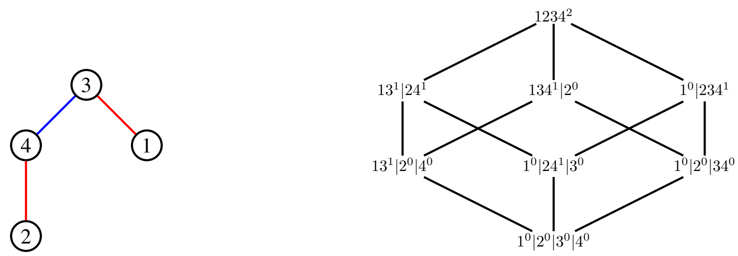
It is a consequence of Theorem 3.3 then that the sets $\{[T,\sigma] | (T,\sigma) \in \mathsf{Comb}^2_{n,i}\}$ and $\{[T,\sigma] | (T,\sigma) \in \mathsf{Lyn}^2_{n,i}\}$ are bases for $\mathcal{L}ie_2(n,i)$. The bicolored comb basis for $\mathcal{L}ie_2(n,i)$ was first obtained in [6]. The bicolored Lyndon basis for $\mathcal{L}ie_2(n,i)$ is not the same as a bicolored Lyndon basis obtained in [13].

It follows from Theorem 4.1 that $|\mathsf{Comb}^2_{n,i}| = |\mathsf{Lyn}^2_{n,i}|$. In [9] an explicit color preserving bijection between the bicolored combs and bicolored Lyndon trees is given. These trees also provide a combinatorial interpretation of γ-positivity of the tree analog of the Eulerian polynomials, $\sum_{T\in\mathcal{T}_n} t^{\mathrm{des}(T)}$ (see [9]).

4.2 The tree basis for homology

We now present a generalization of an NBC basis of Björner for the homology of Π_n (see [3, Proposition 2.2]). Let T be a rooted tree on node set $[n]$. For each subset A of the edge set $E(T)$ of T, let T_A be the subgraph of T with node set $[n]$ and edge set A. Clearly T_A is a forest on $[n]$ consisting of rooted trees $T_1, T_2, \ldots, T_k$.

Consider the weighted partition $\pi(T_A) := \{N(T_i)^{\mathrm{des}(T_i)} \mid i \in [k]\}$, where $N(T_i)$ is the node set of the tree T_i and $\mathrm{des}(T_i)$ is the number of descents of T_i. We define Π_T to be the induced subposet of Π^w_n on the set $\{\pi(T_A) \mid A \subseteq E(T)\}$. The poset Π_T is clearly isomorphic to the boolean algebra $\mathcal{B}_{n-1}$. Hence $\Delta(\overline{\Pi_T})$ is the barycentric subdivision of the boundary of the $(n-2)$-simplex. We let ρ_T denote a fundamental cycle of the spherical complex $\Delta(\overline{\Pi_T})$. See Figure 5 for an example of Π_T.



(a) T (b) Π_T

Fig. 5: Example of a tree T with two descents (in red) and the corresponding poset Π_T

Theorem 4.2 *Let $\mathcal{T}_{n,i}$ be the set of rooted trees on node set $[n]$ with i descents. Then $\{\rho_T \mid T \in \mathcal{T}_{n,i}\}$ is a basis for $\tilde{H}_{n-3}((\hat{0}, [n]^i))$.*

To prove this theorem we only need to show that the set $\{\rho_T \mid T \in \mathcal{T}_{n,i}\}$ is linearly independent since we know from Corollary 2.7 that $\dim \tilde{H}_{n-3}((\hat{0}, [n]^i)) = |\mathcal{T}_{n,i}|$. We prove linear independence (in the full version of the paper) by showing that the the fundamental cycles in $\{\rho_T \mid T \in \mathcal{T}_{n,i}\}$ have a triangular incidence relationship with a basis of maximal chains corresponding (under Theorem 3.3) to a bicolored version of the Lyndon basis for $\mathcal{L}ie_2(n,i)$ proposed by Liu [13].

5 Other combinatorial and algebraic properties

5.1 The characteristic polynomial and rank generating polynomial

Recall that the characterstic polynomial of Π_n factors nicely. In the full version of this paper we prove that the same is true for Π_n^w.

Theorem 5.1 *For all $n \geq 1$, the characteristic polynomial of Π_n^w is given by*

$$\chi_{\Pi_n^w}(x) := \sum_{\alpha \in \Pi_n^w} \mu_{\Pi_n^w}(\hat{0}, \alpha) x^{n-1-\rho(\alpha)} = (x-n)^{n-1},$$

where $\rho(\alpha)$ is the rank of α, and the rank generating function is given by

$$\mathcal{F}(\Pi_n^w, x) := \sum_{\alpha \in \Pi_n^w} x^{\rho(\alpha)} = \sum_{k=0}^{n-1} \binom{n}{k} (n-k)^k x^k.$$

Recall that for a poset P of length r the *Whitney number of the first kind* $w_k(P)$ is the coefficient of x^{r-k} in the characteristic polynomial $\chi_P(x)$ and the *Whitney number of the second kind* $W_k(P)$ is the coefficient of x^k in the rank generating function $\mathcal{F}(P,x)$; see [18]. It follows from Theorem 5.1 that

$$w_k(\Pi_n^w) = (-1)^k \binom{n-1}{k} n^k \quad \text{and} \quad W_k(\Pi_n^w) = \binom{n}{k} (n-k)^k.$$

A pure bounded poset P, with rank function ρ, is said to be *uniform* if there is a family of posets $\{P_i\}$ such that for all $x \in P$ with $\rho(\hat{1}_P) - \rho(x) = i$, the intervals $[x, \hat{1}]$ and P_i are isomorphic. It is easy to see that Π_n and $\widehat{\Pi}_n^w$ are uniform.

Proposition 5.2 ([18, Exercise 3.130(a)]) *Let P be a uniform poset of length r. Then the matrices $[w_{i-j}(P_i)]_{0 \leq i,j \leq r}$ and $[W_{i-j}(P_i)]_{0 \leq i,j \leq r}$ are inverses of each other.*

From this and the uniformity of $\widehat{\Pi_n^w}$, we have the following corollary of Theorem 5.1.

Corollary 5.3 *The matrices $[(-1)^{i-j}\binom{i-1}{j-1} i^{i-j}]_{1 \leq i,j \leq n}$ and $[\binom{i}{j} j^{i-j}]_{1 \leq i,j \leq n}$ are inverses of each other.*

This result is not new and an equivalent dual version was already obtained by Sagan in [14], also by using essentially Proposition 5.2, but with a completely different poset. So we can consider this to be a new proof of that result.

5.2 Further work

In [10] one of the authors discusses some of the results presented here in greater generality. The author considers partitions with blocks weighted by k-tuples of nonnegative integers that sum to one less the cardinality of the block, and shows that the resulting poset of weighted partitions is EL-shellable. Then he finds the dimensions of the top cohomology of the maximal intervals and gives generalizations of the Lyndon basis and comb basis for these modules. The analogue of the explicit sign twisted isomorphism from the direct sum of the cohomology of the maximal intervals to the multilinear component of the free Lie algebra then can be constructed, this time with k linearly compatible brackets.

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