A t-generalization for Schubert Representatives of the Affine Grassmannian

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Abstract. We introduce two families of symmetric functions with an extra parameter t that specialize to Schubert representatives for cohomology and homology of the affine Grassmannian when $t = 1$. The families are defined by a statistic on combinatorial objects associated to the type-A affine Weyl group and their transition matrix with Hall-Littlewood polynomials is t-positive. We conjecture that one family is the set of k-atoms.

Nous présentons deux familles de fonctions symétriques dépendant d'un paramètre t et dont les spécialisations à $t = 1$ correspondent aux classes de Schubert dans la cohomologie et l'homologie des variétés Grassmanniennes affines. Les familles sont définies par des statistiques sur certains objets combinatoires associés au groupe de Weyl affine de type A et leurs matrices de transition dans la base des polynômes de Hall-Littlewood sont t-positives. Nons conjecturons qu'une de ces familles correspond aux k-atomes.

Keywords: k-Schur functions, Pieri rule, Bruhat order, Hall-Littlewood polynomials

1 Introduction

Affine Schubert calculus is a generalization of classical Schubert calculus where the Grassmannian is replaced by infinite-dimensional spaces Gr_G known as affine Grassmannians. As with Schubert calculus, topics under the umbrella of affine Schubert calculus are vast but now, it is the combinatorics of a family of polynomials called k-Schur functions that underpins the theory.

The theory of k-Schur functions came out of a study of symmetric functions over $\mathbb{Q}(q,t)$ called Macdonald polynomials. Macdonald polynomials posses remarkable properties whose proofs inspired deep work in many areas One aspect that has been intensely studied from a combinatorial, representation theoretic, and algebraic geometric perspective is the Macdonald/Schur transition matrix. In particular, in the late 1980's, Macdonald conjectured [Mac88] that the coefficients in the expansion

$$H_\mu[X;q,t] = \sum_\lambda K_{\lambda,\mu}(q,t)\, s_\lambda \tag{1}$$

are positive sums of monomials in q and t; that is, $K_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$. These coefficients have since been a matter of great interest. For starters, they generalize the *Kostka-Foulkes polynomials*. These are given

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by $K_{\lambda,\mu}(0,t)$ and they appear in many contexts such as Hall-Littlewood polynomials [Gre55], affine Kazhdan-Lusztig theory [Lus81], and affine tensor product multiplicities [NY97]. Moreover, Kostka-Foulkes polynomials encode the dimensions of certain bigraded S_n-modules [GP92]. They were beautifully characterized by Lascoux and Schützenberger [LS78] by associating a statistic (non-negative integer) called *charge* to each tableau T so that

$$K_{\lambda,\mu}(0,t) = \sum_{\substack{weight(T)=\mu \\ \text{shape}(T)=\lambda}} t^{\text{charge}(T)}\,. \tag{2}$$

Despite having such concrete results for the $q=0$ case, it was a big effort even to establish polynomiality for general $K_{\lambda,\mu}(q,t)$ [GR96, GT96, Kno97, LV98, KN96, Sah96] and the geometry of Hilbert schemes was eventually needed to prove positivity [Hai01]. A formula in the spirit of (2) still remains a mystery.

In one study of Macdonald polynomials, Lapointe, Lascoux, and Morse found computational evidence for a family of new bases

$$\{A_\mu^{(k)}[X;t]\}_{\mu_1\le k} \tag{3}$$

for subspaces

$$\Lambda_t^{(k)} = \text{span}\{H_\lambda[X;q,t]\}_{\lambda_1\le k}$$

in a filtration $\Lambda_t^{(1)} \subseteq \Lambda_t^{(2)} \subseteq \cdots \subseteq \Lambda_t^{(\infty)}$ of Λ. Conjecturally, the star feature of each basis was the property that Macdonald polynomials expand positively in terms of it, giving a remarkable factorization for the Macdonald/Schur transition matrices over $\mathbb{N}[q,t]$. To be precise, for any fixed integer $k>0$ and each $\lambda\in\mathcal{P}^k$ (a partition where $\lambda_1\le k$),

$$H_\lambda[X;q,t] = \sum_{\mu\in\mathcal{P}^k} K_{\mu,\lambda}^{(k)}(q,t)\,A_\mu^{(k)}[X;t] \quad \text{where} \quad K_{\mu,\lambda}^{(k)}(q,t)\in\mathbb{N}[q,t]\,. \tag{4}$$

It was conjectured in [LLM03] that for all $k>0$, $\{A_\mu^{(k)}[X;t]\}_{\mu_1\le k}$ exists and forms a basis for $\Lambda_t^{(k)}$, and that for $k\ge|\mu|$, $A_\mu^{(k)}[X;t]=s_\mu$. These conjectures and the decomposition (4) strengthen Macdonald's conjecture.

A construction for $A_\mu^{(k)}[X;t]$ is given in [LLM03], but it is so intricate that these conjectures remain unproven. However, pursuant investigations of these bases led to various conjecturally equivalent characterizations. One such family of polynomials $\{s_\lambda^{(k)}\}$ was introduced in [LM05] and conjectured to be the $t=1$ case of $A_\lambda^{(k)}[X;t]$. It has since been proven that the $s_\lambda^{(k)}$ refine the very aspects of Schur functions that make them so fundamental and wide-reaching and they are now called *k-Schur functions*.

The role of k-Schur functions in affine Schubert calculus emerged over a number of years. The springboard was a realization that the combinatorial backbone of k-Schur theory lies in the setting of the affine Weyl group. The k-Schur functions are tied to Pieri rules, tableaux, Young's lattice, sieved q-binomial identities, and Cauchy identities that are naturally described in terms of posets of elements in $\tilde{A}^k$. For example, $K_{\lambda,\mu}^{(k)}(1,1)$ is the number of reduced expressions for an element in $\tilde{A}^k$. The combinatorial exploration fused into a geometric one when the k-Schur functions were connected to the quantum cohomology of Grassmannians. Quantum cohomology originated in string theory and symplectic geometry. It has had a great impact on algebraic geometry and is intimately tied to the *Gromov-Witten invariants*. These invariants appear in the study of subtle enumerative questions such as: how many degree d plane curves of genus g contain r generic points? Lapointe and Morse [LM08] showed that each Gromov-Witten invariant for the quantum cohomology of Grassmannians exactly equals a k-Schur coefficient in the product of k-Schur functions in Λ. A basis of *dual (or affine) k-Schur functions* was also introduced in [LM08] and Lam proved [Lam08] that the Schubert bases for cohomology and homology of the affine Grassmannian $\mathrm{Gr}_{SL_{k+1}}$ are given by the dual k-Schur functions and the k-Schur functions, respectively.

Our motivation here is that the k-Schur functions $s_\lambda^{(k)}$ are parameterless and the t is needed to connect with theories outside of geometry. Unfortunately, the characterizations for generic t lack in mechanism for proofs. We introduce a new family of functions that reduce to $\{s_\lambda^{(k)}\}$ when $t = 1$. Our definition uses a combinatorial object called affine Bruhat counter-tableaux (ABC's), whose weight generating functions are the dual k-Schur functions [DM12]. We associate a statistic (a non-negative integer) to each ABC called the k-charge. From this, we use the polynomials

$$K_{\lambda,\mu}^{(k)}(t) = \sum_{\substack{\mathrm{shape}(A)=\mathfrak{c}(\lambda)\\ \mathrm{weight}(A)=\mu}} t^{k\text{-charge}(A)} \tag{5}$$

to define a t-generalization of $s_\lambda^{(k)}$. In particular, we show that the matrix $(K_{\lambda,\mu}^{(k)}(t))_{\{\lambda,\mu\in\mathcal{P}^k\}}$ is unitriangular and taking the inverse of this matrix to be $\tilde{K}^{(k)}$, a basis for Λ_t^k is given by

$$s_\lambda^{(k)}[X;t] = \sum_\mu \tilde{K}_{\lambda,\mu}^{(k)}(t)\, H_\mu[X;t]\,,$$

for all λ with $\lambda_1 \leq k$. We prove that $s_\lambda^{(k)}[X;t]$ reduce to k-Schur functions when $t = 1$. When $k = \infty$, these are Schur functions, and thus (5) gives a new description for the Kostka-Foulkes polynomials. Naturally, we conjecture that these functions are the $A_\lambda^{(k)}[X;t]$.

2 Related work

A refinement of the plactic monoid to a structure on k-tableaux that can be applied to combinatorial problems involving k-Schur functions is partially given in [LLMS12] by a bijection compatible with the RSK-bijection. A deeper understanding of this intricate bijection is underway. Towards this effort, Lapointe and Pinto [LP] have recently shown that a statistic on k-tableaux is compatible with the bijection. There are now several statistics (on k-tableaux, elements of the affine symmetric group, and on ABC's) whose charge generating functions are the same. The ABC's can be used to find the image of certain elements under this bijection and we are working to put the ABC's in a context that simplifies the bijection.

3 Background

We identify each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ with its Ferrers shape (having λ_i lattice squares in the i^{th} row, from the bottom to top). For partitions λ and μ, we say λ contains μ, denoted $\mu \subseteq \lambda$, if $\lambda_i \geq \mu_i$. A skew shape is a pair of partitions λ, μ such that $\mu \subseteq \lambda$, denoted λ/μ.

A *semistandard tableau* T is a filling of a Ferrers shape λ with positive integers that weakly decrease along rows and strictly increase up the columns. The *weight* of a semistandard tableau is the composition $(\mu_i)_{i\in\mathbb{N}}$, where μ_i is the number of cells containing i. For a partition λ and composition μ, let $SSYT(\lambda, \mu)$ be the set of semistandard tableaux of shape λ and weight μ.

The *hook-length* of a cell (i, j) of any partition is the number of cells to the right of (i, j) in row i plus the number of cells above (i, j) in column j plus 1. A p*-core* is a partition that does not contain any cell with hook-length p. The p-degree of a p-core λ, $deg^p(\lambda)$, is the number of cells in λ whose hook-length is smaller than p. Hereafter we work with a fixed integer $k > 0$ and all cores (resp. residues) are $k+1$-cores (resp. $k+1$-residues) and deg^{k+1} will simply be written as deg. We let $\mathcal{P}^k$ denote the set of all partitions λ with $\lambda_1 \leq k$. We also let $\mathcal{C}^{k+1}$ denote the set of all $k+1$-cores. We use a bijection given in [LM05] $\mathfrak{c} : \mathcal{P}^k \to \mathcal{C}^{k+1}$.

For $n \geq 0$, an n*-ribbon* R is a skew diagram λ/μ consisting of n rookwise connected cells such that there is no 2×2 shape contained in R. We refer to the southeasternmost cell of a ribbon as its *head*, and the northweasternmost cell of a ribbon as its *tail*.

A *ribbon tableau* T of shape λ/μ is a chain of partitions

$$\mu = \mu^0 \subset \mu^1 \subset \cdots \subset \mu^r = \lambda$$

such that each μ^i/μ^{i-1} is a tiling of ribbons filled with a positive integer. A *ribbon counter-tableau* A of shape λ/μ is a ribbon tableau such that each skew shape μ^i/μ^{i-1} is filled with the same positive integer $r - i + 1$. We set the cell (i, j) of a ribbon counter-tableau to be the cell in row i, column j, where row one is the topmost row and column one is the leftmost column. For more on partitions and tableaux see [Mac95], [Sta99], [Ber09].

4 Schubert representatives for $H^*(\mathrm{Gr}_{SL_{k+1}})$ and $H_*(\mathrm{Gr}_{SL_{k+1}})$

Despite the many characterizations for the Schubert representatives for the cohomology and homology of the infinite dimensional affine Grassmannian spaces for SL_{k+1} (e.g. [LM05, LM08, Lam06, LLMS10, DM12, AB12]), none have been shown to be the $t = 1$ case of functions conjectured to give a positive Macdonald expansion (4). Our goal is to present functions with a t parameter which reduce to the k-Schur functions as formulated in [DM12] when $t = 1$. The formulation is given in terms of a combinatorial structure called ABC's.

Recall that the strong (Bruhat) order on the affine Weyl group $\tilde{A}^k$ can be instead realized on $k+1$-cores by the covering relation:

$$\rho \lessdot_B \gamma \iff \rho \subseteq \gamma \text{ and } deg(\gamma) = deg(\rho) + 1.$$

An important fact about strong covers is useful in our study.

Lemma 1 *[LLMS10] Let $\rho \lessdot_B \gamma$ be cores. Then*

1. *Each connected component of ρ/γ is a ribbon.*
2. *The components are translates of each other and their heads have the same residue.*

A specific subset of ribbon counter-tableaux are those where each ribbon is of height one. An ABC will be defined as such ribbon counter-tableaux where the skew shapes are a certain strip defined in terms of strong order.

Definition 2 *For $0 < \ell \leq k$ and $k+1$-cores λ and ν, the skew shape $(k + \lambda_1, \lambda)/\nu$ is a* bottom strong $(k - \ell)$-strip *if there is a saturated chain of cores*

$$\nu = \nu^0 \lessdot_B \nu^1 \lessdot_B \cdots \lessdot_B \nu^{k-\ell} = (k + \lambda_1, \lambda)\,,$$

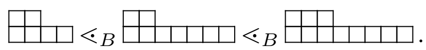
where

1. $(k+\lambda_1, \lambda)/\nu$ *is a horizontal strip*

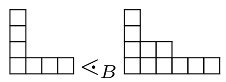
2. *The bottom rightmost cell of ν^i is also a cell in ν^i/ν^{i-1}, for $1 \leq i \leq k-\ell$.*

It turns out that if a skew shape is a bottom strong strip then there is a unique chain meeting the conditions described in Definition 2.

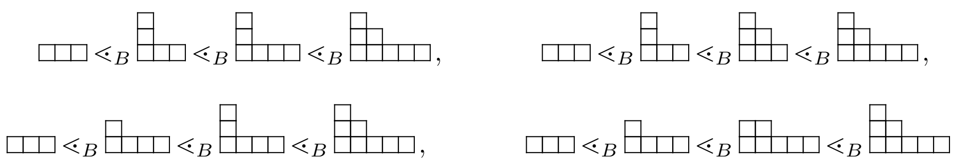
Example 3 *The skew shape $(8,3)/(4,2)$ of 6-cores is a bottom strong 2-strip as there is the saturated chain*



Example 4 *The skew shape $(6,3,1,1)/(4,1,1,1)$ of 4-cores is a bottom strong 1-strip as there is the saturated chain*



Example 5 *There are 4 saturated chains of 4-cores in the strong order from (3) to $(5,2,1)$,*



Since none of these give a bottom strong strip, $(5,2,1)/(3)$ is not a bottom strong strip.

Remark 6 *Bottom strong $(k-\ell)$-strips are a distinguished subset of strong strips in [LLMS10] that define the Pieri rule for the cohomology of the affine Grassmannian.*

The iteration of bottom strong strips leads to the definition of an ABC. First let us set some notation. Given a ribbon counter-tableau A, let $A^{(x)}$ denote the subtableau made up of the rows of A weakly higher than row x. Let $A_{>i}$ denote the restriction of A to letters strictly larger than i where empty cells in a skew are considered to contain ∞. With this in hand, we are now ready to define the ABC's.

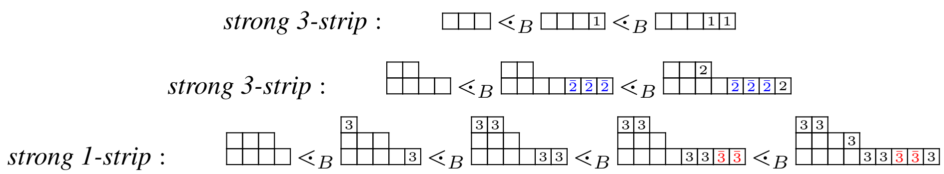
Definition 7 *For a composition α whose entries are not larger than k, a skew ribbon counter-tableau A is an affine Bruhat counter-tableau (or ABC) of k-weight α if*

$$(k+\lambda_1^{(i-1)}, \lambda^{(i-1)})/\lambda^{(i)} \ \text{ is a bottom strong } \alpha_i\text{-strip for all } 1 \leq i \leq \ell(\alpha)\,,$$

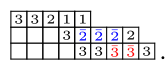
where $\lambda^{(x)} = \mathrm{shape}(A_{>x}^{(x)})$. We define the inner shape of A to be $\lambda^{(\ell(\alpha))}$.

The easiest method to construct an ABC of k-weight α is iteratively, from the empty shape $\lambda^{(0)}$, using Definition 2 to successively add bottom strong strips that are a tiling of $(k+\lambda_1^{(i-1)}, \lambda^{(i-1)})/\lambda^{(i)}$ with α_i-ribbons at each step.

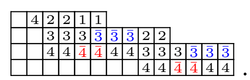
Example 8 *With $k = 5$, we construct an ABC of 5-weight $(3,3,1)$ by*



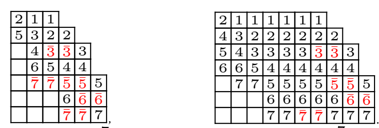
The black letters are ribbons of size one, red letters make a ribbon of size two and blue letters make a ribbon of size 3 (or for those without color the ribbons are depicted with a bar). This can be more compactly represented as



Example 9 *An example of an ABC of 6-weight $(4,4,2,1)$ with inner shape $(8,2,2,1) = \mathfrak{c}(6,2,2,1)$ is*



Example 10 *Two examples of ABC's of k-weight $(1,1,1,1,1,1,1) = (1^7)$ are*



3-*weight* (1^7) 7-*weight* (1^7)

The weight generating functions of the ABC's turn out to be the dual k-Schur functions.

Theorem 11 *[DM12] For any $\lambda \in \mathcal{C}^{k+1}$, the dual k-Schur function can be defined by*

$$\mathfrak{S}_\lambda^{(k)} = \sum_A x^A$$

where the sum is over all affine Bruhat counter-tableaux of inner shape λ, and $x^A = x^{k\text{-}weight(A)}$.

These are symmetric functions, implying that

$$\mathfrak{S}_\lambda^{(k)} = \sum_{\mu:\mu_1 \le k} K_{\lambda,\mu}^{(k)}\, m_\mu\,, \tag{6}$$

where $K_{\lambda,\mu}^{(k)}$ is the number of affine Bruhat counter-tableaux of inner shape λ and k-weight μ. Then, using the Hall-inner product defined by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}\,,$$

we arrive at a characterization for k-Schur functions.

$$h_\mu = \sum_\lambda K_{\lambda,\mu}^{(k)}\, s_\lambda^{(k)}. \tag{7}$$

5 Kostka-Foulkes polynomials

Our goal is to introduce polynomials $s_\lambda^{(k)}[X;t]$ that reduce to $s_\lambda^{(k)}[X]$ when $t=1$ using (4) as an inspiration. Our approach is to introduce a statistic on ABC's. When $k = deg(\lambda)$, both $\mathfrak{S}_\lambda^{(k)}$ and $s_\lambda^{(k)}$ are simply the Schur function s_λ. One advantage of using ABC combinatorics in the theory of k-Schur functions is that known results concerning Schur functions can be reinterpreted in the ABC framework with k large and this can shed light on the smaller k cases. With this in mind, we consider a reformulation for the Kostka-Foulkes polynomials in terms of ABC's. Our results enable us to give a characterization for symmetric polynomials in an extra parameter t that reduce to $\mathfrak{S}_\lambda^{(k)}$ and $s_\lambda^{(k)}$ when $t=1$.

Let us start by recalling the Hall-Littlewood polynomials $\{H_\lambda[X;t]\}_\lambda$. These are a basis for Λ over the polynomial ring $\mathbb{Z}[t]$, which reduces to the homogeneous basis when the parameter $t=1$. These often are denoted by $\{Q'_\lambda[X;t]\}$ in the literature ([Mac95]). Hall-Littlewood polynomials arise and can be defined in various contexts such as the Hall Algebra, the character theory of finite linear groups, projective and modular representations of symmetric groups, and algebraic geometry. We define them here via a tableaux Schur expansion due to Lascoux and Schützenberger [LS78].

The key notion is the *charge* statistic on semistandard tableaux. This is given by defining charge on words and then defining the charge of a tableau to be the charge of its reading word. For our purposes, it is sufficient to define charge only on words whose evaluation is a partition. We begin by defining the charge of a word with weight $(1,1,\ldots,1)$, or a *permutation*. If w is a permutation of length n, then the charge of w is given by $\sum_{i=1}^n c_i(w)$ where $c_1(w)=0$ and $c_i(w)$ is defined recursively as

$$c_i(w) = c_{i-1}(w) + \chi\left(i \text{ appears to the right of } i-1 \text{ in } w\right).$$

Here we have used the notation that when P is a proposition, $\chi(P)$ is equal to 1 if P is true and 0 if P is false.

Example 12 *The charge,* $ch(3,5,1,4,2) = 0+1+1+2+2 = 6$.

We will now describe the decomposition of a word with partition evaluation into *charge subwords*, each of which are permutations. The charge of a word will then be defined as the sum of the charge of its charge subwords. To find the first charge subword $w^{(1)}$ of a word w, we begin at the *right* of w (i.e. at the last element of w) and move leftward through the word, marking the first 1 that we see. After marking a 1, we continue to travel to the left, now marking the first 2 that we see. If we reach the beginning of the word, we loop back to the end. We continue in this manner, marking successively larger elements, until we have marked the largest letter in w, at which point we stop. The subword of w consisting of the marked elements (with relative order preserved) is the first charge subword. We then remove the marked elements from w to obtain a word w'. The process continues iteratively, with the second charge subword being the first charge subword of w', and so on.

Example 13 *Given* $w = (5,2,3,4,4,1,1,1,2,2,3)$, *the first charge subword of* w *are the bold elements in* $(\mathbf{5},\mathbf{2},3,4,\mathbf{4},1,1,\mathbf{1},2,2,\mathbf{3})$. *If we remove the bold letters, the second charge subword is given by the bold elements in* $(\mathbf{3},\mathbf{4},1,\mathbf{1},2,\mathbf{2})$. *It is now easy to see that the third and final charge subword is* $(\mathbf{1},\mathbf{2})$. *Thus we get that* $ch(w) = ch(5,2,4,1,3) + ch(3,4,1,2) + ch(1,2) = 8$. *Since* w *is the reading word of the tableau* $T = \begin{array}{|c|c|c|c|c|c|c|}\hline 5 \\ \cline{1-1} 2 & 3 & 4 & 4 \\ \cline{1-4} 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ \hline\end{array}$ *we find that the* $ch(T) = 8$.

Equipped with the definition of charge, Hall-Littlewood polynomials are then defined by

$$H_\mu[X;t] = \sum_\lambda K_{\lambda,\mu}(t)\, s_\lambda\,, \tag{8}$$

where $K_{\lambda,\mu}(t) = K_{\lambda,\mu}(0,t)$ from (2).

Our first order of business to reformulate Kostka-Foulkes polynomials is to describe the reading word of an ABC. To do so, we first modify a given ABC by lengthening the row sizes.

Definition 14 *From a given ABC A of partition k-weight μ, the extension of A, $ext(A)$, is the counter-tableau constructed from A by adding k cells with letter i to each row i, where the first $\mu_i - s_i + r_i + 1$ added cells form a ribbon for s_i the sum of the size of the ribbons filled with the letter i in row i and r_i the number of such ribbons.*

Example 15 *Consider the following extension of an ABC with 5-weight $(3,3,3,1)$.*

$$A = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\square & 4 & 2 & 1 & 1 \\
\square & \square & 4 & 3 & \bar{2} & \bar{2} & \bar{2} & 2 \\
\square & \square & \square & \bar{4} & \bar{4} & 4 & \bar{3} & \bar{3} & 3 \\
\square & \square & \square & \square & \square & \square & 4 & 4 & \bar{4} & \bar{4} & 4 \\
\hline
\end{array} \Longrightarrow ext(A) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\square & 4 & 2 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & \bar{1} & 1 \\
\square & \square & 4 & 3 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 \\
\square & \square & \square & \bar{4} & \bar{4} & 4 & \bar{3} & \bar{3} & 3 & \bar{3} & \bar{3} & \bar{3} & 3 & 3 \\
\square & \square & \square & \square & \square & \square & 4 & 4 & \bar{4} & \bar{4} & 4 & 4 & 4 & 4 & 4 & 4 \\
\hline
\end{array}$$

5.1 Reading word of standard ABC's

As with tableaux, we first define the reading word of a standard ABC (one of k-weight 1^n) and use this to describe the general reading word. Standard ABC's have a much more predictable structure than the general case. Namely, a standard ABC A has only ribbons of size 1 or 2. In fact, if a row i in A has an i-ribbon of size 2, then $\mu_i - s_i + r_i = 1$. Otherwise $\mu_i - s_i + r_i + 1 = 2$. Thus, each row i of $ext(A)$ has a unique i-ribbon of size 2.

Our construction of the word of an ABC A considers only a subset of the cells in $ext(A)$. Namely,

$$V_A = \{(i, c_i) \in ext(A) : (i, c_i) \text{ is any cell in a } i\text{-ribbon of row } i \text{ that is not its tail}\}. \tag{9}$$

For standard A of k-weight 1^n, V_A is simply a set of n ribbon heads; the one in each row i of $ext(A)$ that contains i. Using V_A, we define the reading word on this standard ABC A.

Definition 16 *For a given ABC A of k-weight 1^n, iteratively construct the reading word $w(A)$ by inserting letter i directly right of letter j where $j < i$ is the largest index such that $c_j < c_i$ and $(j, c_j) \in V_A$. If there is no such j then i is placed at the beginning.*

Example 17 *Recall the ABC from example 10 of 3-weight (1^7) is*

$$A = \begin{array}{|c|c|c|c|c|c|}
\hline
2 & 1 & 1 \\
5 & 3 & 2 & 2 \\
\square & 4 & \bar{3} & \bar{3} & 3 \\
\square & 6 & 5 & 4 & 4 \\
\square & \bar{7} & \bar{7} & \bar{5} & \bar{5} & 5 \\
\square & \square & \square & 6 & \bar{6} & \bar{6} \\
\square & \square & \square & \bar{7} & \bar{7} & 7 \\
\hline
\end{array} \quad \Longrightarrow \quad ext(A) = \begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
2 & 1 & 1 & \bar{1} & \bar{1} & 1 \\
5 & 3 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
\square & 4 & \bar{3} & \bar{3} & 3 & 3 & 3 & 3 \\
\square & 6 & 5 & 4 & 4 & \bar{4} & \bar{4} & 4 \\
\square & \bar{7} & \bar{7} & \bar{5} & \bar{5} & 5 & 5 & 5 & 5 \\
\square & \square & \square & 6 & \bar{6} & \bar{6} & 6 & 6 & 6 \\
\square & \square & \square & \bar{7} & \bar{7} & 7 & 7 & 7 & 7 \\
\hline
\end{array}$$

From $ext(A)$, we see that $V_A = \{(1,5),(2,6),(3,4),(4,7),(5,5),(6,6),(7,5)\}$. From V_A, we have the iterative construction of the reading word of A as $(1) \to (1,2) \to (3,1,2) \to (3,1,2,4) \to (3,5,1,2,4) \to (3,5,6,1,2,4) \to (3,7,5,6,1,2,4)$. This tells us that $w(A) = (3,7,5,6,1,2,4)$.

5.2 Reading words of ABC

Equipped with a method to obtain the reading word (permutation) of standard ABC's, we now define a way to construct a sequence of permutations from any ABC with partition k-weight.

Definition 18 *Let A be an ABC of partition k-weight μ. For $r = 1, 2, \ldots, \mu_1$, starting with $r = 1$, we iteratively construct sets E^r_A from $ext(A)$ as follows; put $(1, k + \mu_1 + 2 - r) \in E^r_A$, and let $(i, c_i) \in E^r_A$ if and only if $(i-1, c_{i-1}) \in E^r_A$ and*

$$(c_{i-1} - c_i + k) \; mod \; (k+1) = min\{(c_{i-1} - x + k) \; mod \; (k+1) | (i, x) \in V_A \setminus \cup_{p=1}^{r-1} E^p_A\}.$$

Example 19 *Recall the ABC A from example 15 of 5-weight $(3, 3, 3, 1)$. From its $ext(A)$, we see that*

$$V_A = \{(1, 7), (1, 8), (1, 9), (2, 6), (2, 7), (2, 10), (3, 8), (3, 11), (3, 12), (4, 10)\}.$$

We iteratively construct the sets E^r_A for each $r = 1, 2, 3$, using $ext(A)$ and V_A. For $r = 1$, begin by setting $E^1_A = \{(1, 9)\}$. Next, we see that $(2, 7) \in E^1_A$, because $(1, 9) \in E^1_A$ and

$$1 = min\{2 = (9 - 6 + 5) \; mod \; 6, 1 = (9 - 7 + 5) \; mod \; 6, 4 = (9 - 10 + 5) \; mod \; 6\}.$$

So the next iteration gives us that $E^1_A = \{(1, 10), (2, 7)\}$. Next, we see that $(3, 12) \in E^1_A$, because $(2, 7) \in E^1_A$ and $0 = min\{4 = (7 - 8 + 5) \; mod \; 6, 1 = (7 - 11 + 5) \; mod \; 6, 0 = (7 - 12 + 5) \; mod \; 6\}$. So the next iteration gives us that $E^1_A = \{(1, 10), (2, 7), (3, 12)\}$. Finally since the $(4, 10)$ is the only element in V_A from the fourth row of A, then we see that $E^1_A = \{(1, 9), (2, 7), (3, 12), (4, 10)\}$.

For $r = 2$, to construct E^2_A, we begin by setting $E^2_A = \{(1, 8\}$, and repeat what we did to construct E^1_A, except this time we only consider elements from the set $V_A \setminus E^1_A = \{(1, 7), (1, 8), (2, 6), (2, 10), (3, 8), (3, 11)\}$. This gives us $E^2_A = \{(1, 8), (2, 6), (3, 11)\}$.

Finally for $r = 3$, to construct E^3_A, we begin by setting $E^3_A = \{(1, 7)\}$, and we only consider elements from the set $V_A \setminus (E^1_A \cup E^2_A) = \{(1, 7), (2, 10), (3, 8)\}$, which immediately gives us $E^3_A = \{(1, 7), (2, 10), (3, 8)\}$.

Using each set E^r_A, we construct a sequence of reading word w_r for $1 \leq r \leq \mu_1$.

Definition 20 *Given an ABC A of partition k-weight μ, for $1 \leq r \leq \mu_1$, the r^{th} reading word of A, $w_r(A)$, is constructed using the same procedure in definition 16, where V_A is replaced by E^r_A.*

Example 21 *If we consider the ABC A from example 15, then we know from example 19 that $E^1_A = \{(1, 9), (2, 7), (3, 12), (4, 10)\}$, $E^2_A = \{(1, 8), (2, 6), (3, 11)\}$, $E^3_A = \{(1, 7), (2, 10), (3, 8)\}$. This tells us from definition 20 that $w_1(A) = (2, 1, 4, 3)$, $w_2(A) = (2, 1, 3)$ and $w_3(A) = (3, 1, 2)$.*

For partitions λ, μ with $|\lambda| = |\mu| = n$, an ABC A is of n-weight μ and inner shape λ, has a charge statistic associated to it.

Definition 22 *Suppose λ and μ are partitions with $|\lambda| = |\mu| = n$. For any ABC A of n-weight μ and inner shape λ, the charge of A is $ch(A) = \sum_{r=1}^{\mu_1} ch(w_r(A))$.*

Example 23 *The ABC A from example 15 has the reading words $w_1(A) = (2, 1, 4, 3)$, $w_2(A) = (2, 1, 3)$ and $w_3(A) = (3, 1, 2)$. as described in example 18. Hence, we have that the charge of A is $ch(A) = ch((2, 1, 4, 3)) + ch((2, 1, 3)) + ch((3, 1, 2)) = 2 + 1 + 2 = 5$.*

There is a direct connection between reading words of semi-standard Young tableaux and a certain set of ABC's.

Theorem 24 *Suppose λ and μ are partitions with $|\lambda| = |\mu| = n$. If the set*

$$ABC(\lambda, \mu) = \{A |\ A \text{ is an } ABC \text{ of } n\text{-weight } \mu \text{ and inner shape } \mathfrak{c}(\lambda)\}\,,$$

then there is a bijection between the sets $ABC(\lambda, \mu)$ and $SSYT(\lambda, \mu)$, which is charge preserving.

From this theorem, we have the following corollary that gives the Kostka-Foulkes polynomials in the spirit of ABC's.

Corollary 25 *For partitions λ and μ, the Kostka-Foulkes polynomial*

$$K_{\lambda,\mu}(t) = \sum_{A \in ABC(\lambda,\mu)} t^{ch(A)}.$$

6 *k*-charge and *k*-Schur functions

We now look towards generalizing Corollary 25 by considering ABC's of any partition k-weight. What is needed is an extra concept of an *offset* of a given ABC. Any r-ribbon of an ABC is an *offset* if there is a lower r-ribbon filled with the same letter as R whose head has the same residue as the head of R.

Definition 26 *For any ABC A of partition k-weight μ, we set*

$$\textit{off}^{\,k}(A) = \sum_{R\text{: offset in } A} (\textit{size}(R) - 1)$$

Definition 27 *Let A be an ABC of partition k-weight μ and inner shape λ, and $w_1(A), \ldots, w_{\mu_1}(A)$ be the sequence of reading words that result from definition 18. Then, the k-charge of A*

$$ch^k(A) = \sum_{r=1}^{\mu_1} ch(w_r(A)) - \textit{off}^{\,k}(A) - \beta(A)$$

where $\beta(A)$ is the number of cells in λ whose hook-length exceeds k.

Example 28 *Consider the ABC of 3-weight 1^5* $A = \begin{array}{|c|c|c|c|c|c|} \hline 3 & 1 & 1 & & & \\ \hline & 2 & \bar{2} & \bar{2} & & \\ \hline & 4 & 3 & 3 & & \\ \hline & \bar{5} & \bar{5} & 4 & 4 & \\ \hline & & & 5 & \bar{5} & \bar{5} \\ \hline \end{array}$ *. Here we see that A has only one offset* $\boxed{\bar{5}\,\bar{5}}$ *in the second row from the bottom. The only reading word for this A is $w_1(A) = (2, 5, 1, 3, 4)$. So we get $ch^3(A) = ch((2, 5, 1, 3, 4)) - 1 - 1 = 5 - 1 - 1 = 3$.*

Definition 29 *For any λ, $\mu \in \mathcal{P}^k$, we let*

$$K^{(k)}_{\lambda,\mu}(t) = \sum_{\substack{A\text{: } ABC \text{ of } k\text{-weight } \mu, \\ \text{inner shape } \mathfrak{c}(\lambda)}} t^{ch^k(A)}.$$

Note that when $k \geq |\lambda|$, the polynomials $K^{(k)}_{\lambda,\mu}(t)$ are the Kostka-Foulkes polynomials of 2. Definition 29 generalizes the Kostka-Foulkes polynomials, and it also helps us to define a new set of symmetric functions with parameter t. To see this, we only need the following claim.

Lemma 30 *The matrix*
$$\left[K^{(k)}_{\lambda,\mu}(t)\right]_{\{\lambda,\mu\in\mathcal{P}^k\}}$$
is unitriangular.

Taking the inverse of the matrix in theorem 30, we form a basis for the subring $\Lambda^{(k)}_t$ of the ring Λ.

Definition 31 *For $\lambda\in\mathcal{P}^k$, the k-Schur function with parameter t is*
$$s^{(k)}_\lambda[X;t]=\sum_\mu \tilde{K}^{(k)}_{\lambda,\mu}(t)H_\mu[X;t],$$
where $\tilde{K}^{(k)}_{\lambda,\mu}(t)$ are entries in the inverse of the matrix $\left[K^{(k)}_{\lambda,\mu}(t)\right]_{\{\lambda,\mu\in\mathcal{P}^k\}}$.

These new symmetric functions exhibit properties which connect them to the k-Schur and the Schur functions.

Property 32 *As λ ranges over partitions in $\mathcal{P}^k$, $s^{(k)}_\lambda[X;t]$ forms a basis for the subring $\Lambda^{(k)}_t$, $s^{(k)}_\lambda[X;1]=s^{(k)}_\lambda$, and $s^{(\infty)}_\lambda[X;1]=s_\lambda$.*

Finally we make the following conjecture which ties the functions in Definition 31 to those described in [LLM03].

Conjecture 33 *For $\mu\in\mathcal{P}^k$, $s^{(k)}_\mu[X;t]=A^{(k)}_\mu[X;t]$.*

References

[AB12] S. Assaf and S. Billey. Affine dual equivalence and k-Schur functions. preprint, 2012.

[Ber09] F. Bergeron. *Algebraic Combinatorics and Coinvariant Spaces*. A. K. Peters/CRC Press, 2009.

[DM12] A. Dalal and J. Morse. The ABC's of the affine Grassmannian and Hall-Littlewood polynomials. *DMTCS Proceedings*, 2012.

[GP92] A. M. Garsia and C. Procesi. On certain graded S_n-modules and the q-Kostka polynomials. *Adv. Math*, 87:82–138, 1992.

[GR96] A. M. Garsia and J. Remmel. Plethystic formulas and positivity for q,t-Kostka coefficients. *Mathematical essays in honor of Gian-Carlo Rota (Cambridge, MA*, pages 245–262, 1996.

[Gre55] J. A. Green. The characters of the finite general linear groups. *Trans Amer Math Soc*, 80:442–407, 1955.

[GT96] A. M. Garsia and G. Tesler. Plethystic formulas for Macdonald q,t-Kostka coefficients. *Adv Math,* , 123:144–222, 1996.

[Hai01] M. Haiman. Hilbert schemes, polygraphs, and the Macdonald positivity conjecture. *J. Am Math. Soc.*, 14:941–1006, 2001.

[KN96] A. N. Kirillov and M. Noumi. q-difference raising operators for Macdonald polynomials and the integrality of transition coefficients. *Algebraic methods and q-special functions (Montreal, QC*, pages 227–243, 1996.

[Kno97] F. Knop. Integrality of two variable Kostka functions. *J. Reine Agnew*, 482:177–189, 1997.

[Lam06] T. Lam. Affine Stanley symmetric functions. *Amer. J of Math*, 128(6):1553–1586, 2006.

[Lam08] T. Lam. Schubert polynomials for the affine Grassmannian. *J. Amer. Math Soc*, 21(1):259–281, 2008.

[LLM03] L. Lapointe, A. Lascoux, and J. Morse. Tableau atoms and a new Macdonald positivity conjecture. *Duke Math J*, 116(1):103–146, 2003.

[LLMS10] T. Lam, L. Lapointe, J. Morse, and M. Shimozono. Affine insertion and Pieri rules for the affine Grassmannian. *Memoirs of the AMS*, 208(977), 2010.

[LLMS12] T. Lam, L. Lapointe, J. Morse, and M. Shimozono. *The poset of k-shapes and branching of k-Schur functions*. to appear in Memoirs of the AMS, 2012.

[LM05] L. Lapointe and J. Morse. Tableaux on $k+1$-cores, reduced words for affine permutations, and k-Schur function expansions. *J Combin Theory Ser*, 112(1):44–81, 2005.

[LM08] L. Lapointe and J. Morse. Quantum cohomology and the k-Schur basis. *Trans Amer Math Soc,* , 360(4):2021–2040, 2008.

[LP] L. Lapointe and M. E. Pinto. Private communication. Private communication with Authors.

[LS78] A. Lascoux and M.-P. Schützenberger. Sur une conjecture de H.O. Foulkes. *C.R. Acad. Sc. Paris*, 294:323–324, 1978.

[Lus81] G. Lusztig. Singularities, character formulas, and a q-analog of weight multiplicities. *Analysis and topology on singular spaces, II, III (Luminy*, 101-102:208–229, 1981.

[LV98] L. Lapointe and L. Vinet. A short proof of the integrality of the Macdonald (q,t)-Kostka coefficients. *Duke Math J*, 91:205–214, 1998.

[Mac88] I. G. Macdonald. *A new class of symmetric functions*. Séminaire Lotharingien de Combinatoire, **B20a** 41pp, 1988.

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Clarendon Press, Oxford, 2nd edition, 1995.

[NY97] A. Nakayashiki and Y. Yamada. Kostka polynomials and energy functions in solvable lattice models. *Selecta Math*, 3(4):547–599, 1997.

[Sah96] S. Sahi. Interpolation, integrality, and a generalization of Macdonald polynomials. *Internat Math. Res. Notices*, pages 457–471, 1996.

[Sta99] R. Stanley. *Enumerative Combinatorics*. Cambridge, Vol 2, 1999.