The immaculate basis of the non-commutative symmetric functions

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Outline

1. Symmetric functions
2. Quasi-symmetric functions
3. Non-commutative symmetric functions
4. The immaculate basis
5. Applications

Symmetric Functions

Symmetric functions

Definition: Symmetric Function

A function $f = f(x_1, x_2, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ is symmetric if for every collection of positive integers $[\alpha_1, \ldots, \alpha_k]$ the coefficient of $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ is equal to the coefficient of $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ for all $i_1, i_2, \ldots, i_k$.

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Graded Hopf algebra

Sym is a graded Hopf algebra, graded by the degree of the polynomial. The dimension of the graded component of Sym_n is the number of partitions of n.

Monomial symmetric functions

Definition: Monomial Symmetric Function

The monomial symmetric function indexed by λ is:

$$m_\lambda = \sum_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where α runs over all distinct rearrangements of λ.

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Definition: Homogeneous symmetric functions

Let $h_i = \sum_{\lambda \vdash i} m_\lambda$. The complete homogenous functions are:

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_m}.$$

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Definition: Scalar product

The scalar product on Sym_n is defined by $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$.

Schur functions

The most important basis of Sym is the basis of Schur functions. The Schur basis s_λ is defined by any of the following:

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The Jacobi-Trudi rule:

$$s_\lambda := \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+\ell-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_\ell-\ell+1} & h_{\lambda_\ell-\ell+2} & \cdots & h_{\lambda_\ell} \end{bmatrix} = \det \left| h_{\lambda_i+j-i} \right|_{1\le i,j\le \ell}$$

where we use the convention that $h_0 = 1$ and $h_{-m} = 0$ for $m > 0$.

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where we use the convention that $h_0 = 1$ and $h_{-m} = 0$ for $m > 0$.

Example: $\lambda = (2, 1)$

$$s_{2,1} = \begin{vmatrix} h_2 & h_3 \\ h_0 & h_1 \end{vmatrix} = h_2 h_1 - h_3$$

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The Pieri rule:

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the sum over all μ for which μ/λ is a horizontal strip of size i

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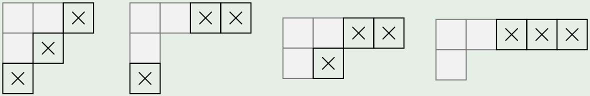
Example: $\lambda = (2,1)$ and $i = 3$

$$s_{2,1} h_3 = s_{3,2,1} + s_{4,1,1} + s_{4,2} + s_{5,1}$$



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Semi-standard tableaux:

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Example: $\lambda = (2,1)$ in three variables

$$s_{2,1} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + 2 x_1 x_2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

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Structure coefficients (the Littlewood Richardson rule):

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu,$$

$c^\nu_{\lambda,\mu}$ is the Yamanouchi tableaux of shape ν/λ and content μ.

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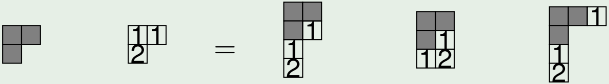

$$s_{21} \quad s_{21} = s_{2211} + s_{222} + s_{3111}$$

$$+ \; 2s_{321} + s_{33} + s_{411} + s_{42}$$

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Representation Theory of S_n

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Via the Frobenius transformation, the Schur functions correspond to the irreducible representations of the symmetric group.

Representation Theory of GL_n

The Schur functions are the characters of the irreducible representations of the general linear group.

Quasi-symmetric functions

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Definition: Quasi-symmetric function

A function $f = f(x_1, x_2, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ is quasi-symmetric if for every collection of positive integers $(\alpha_1, \ldots, \alpha_k)$, the coefficient of $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ is equal to the coefficient of $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ for all $i_1 < i_2 < \cdots < i_k$.

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Hopf Algebra

QSym is a graded Hopf algebra, graded by the degree of the polynomial. The dimension of the graded component QSym_n is the number of compositions of n.

Monomial quasi-symmetric functions

Definition: Monomial quasi-symmetric function

The monomial quasi-symmetric function indexed by a composition α is defined as

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_m} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_m}^{\alpha_m}.$$

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Positive expansion of monomials

It is clear that

$$m_\lambda = \sum_\alpha M_\alpha,$$

the sum over all compositions which re-arrange to form the partition λ. This implies that Sym is a subalgebra of QSym.

Fundamental quasi-symmetric functions

Definition: Fundamental basis

The fundamental basis of QSym is defined by:

$$F_\alpha = \sum_{\beta \leq \alpha} M_\beta.$$

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Positive expansion of Schur functions

The Schur functions expand positively in the Fundamental basis nicely:

$$s_\lambda = \sum_{T \in SYT(\lambda)} F_{D(T)}$$

Non-commutative symmetric functions

Non-commutative symmetric functions

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NSym is the algebra generated by the non-commuting elements H_i of degree i, i.e. $\mathrm{NSym} = \mathbb{Q}\langle H_1, H_2, \ldots \rangle$.

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Complete homogenous functions

A basis for NSym_n are the non-commutative complete homogenous functions

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Sym is a quotient of NSym

$\chi : \mathrm{NSym} \to \mathrm{Sym}$ is defined by $\chi(H_\alpha) = h_{\lambda(\alpha)}$.

Representation theory behind NSym

Dual Hopf algebras

NSym and QSym are in fact dual Hopf algebras. There is a scalar product between the two algebras by viewing M_α and H_β as dual bases:

$$\langle M_\alpha, H_\beta \rangle = \delta_{\alpha,\beta}$$

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$$\langle M_\alpha, H_\beta \rangle = \delta_{\alpha,\beta}$$

Representation theory

NSym and QSym are isomorphic to $K_0(H_n(0))$ and $G_0(H_n(0))$ respectively, where $H_n(0)$ denotes the 0-Hecke algebra.

The immaculate basis

Immaculate tableaux

Definition: Immaculate tableaux

An *immaculate tableau* of shape α and content β is a labeling of the boxes of the diagram of α by positive integers in such a way that:

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2. the sequence of entries in each row, from left to right, is weakly increasing;
3. the sequence of entries in the *first* column, from top to bottom, is increasing.

Example: $\alpha = [4, 2, 3]$ and $\beta = [3, 1, 2, 3]$

1	1	1	3
2	3		
4	4	4	

1	1	1	3
2	4		
3	4	4	

1	1	1	4
2	3		
3	4	4	

1	1	1	4
2	4		
3	3	4	

1	1	1	2
3	3		
4	4	4	

The immaculate basis

The immaculate basis of NSym can be defined by any of the following:

Jacobi-Trudi type rule:

$$\mathfrak{S}_\alpha := \det \begin{bmatrix} H_{\alpha_1} & H_{\alpha_1+1} & \cdots & H_{\alpha_1+\ell-1} \\ H_{\alpha_2-1} & H_{\alpha_2} & \cdots & H_{\alpha_2+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{\alpha_\ell-\ell+1} & H_{\alpha_\ell-\ell+2} & \cdots & H_{\alpha_\ell} \end{bmatrix}$$

$$= \sum_{\sigma\in S_\ell} (-1)^\sigma H_{\alpha_1+\sigma_1-1,\alpha_2+\sigma_2-2,\ldots,\alpha_\ell+\sigma_\ell-\ell}$$

where $H_0 = 1$ and $H_{-m} = 0$ for $m > 0$.

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where $H_0 = 1$ and $H_{-m} = 0$ for $m > 0$.

Example: $\alpha = (1,2)$

$$\mathfrak{S}_{1,2} = \begin{vmatrix} H_1 & H_2 \\ H_1 & H_2 \end{vmatrix} = H_1 H_2 - H_2 H_1$$

The immaculate basis

The immaculate basis of NSym can be defined by any of the following:

Pieri type rule:

$$\mathfrak{S}_\alpha H_i = \sum_{\beta} \mathfrak{S}_\beta,$$

the sum over all β for which $\alpha_i \leq \beta_i$ and $len(\beta) \leq len(\alpha) + 1$.

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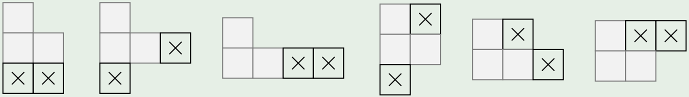
Example: $\alpha = (1, 2)$ and $i = 2$

$$\mathfrak{S}_{1,2} H_2 = \mathfrak{S}_{1,2,2} + \mathfrak{S}_{1,3,1} + \mathfrak{S}_{1,4} + \mathfrak{S}_{2,2,1} + \mathfrak{S}_{2,3} + \mathfrak{S}_{3,2}$$



The immaculate basis

The immaculate basis of NSym can be defined by any of the following:

Littlewood Richardson type rule:

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_\beta C^\beta_{\alpha,\lambda} \mathfrak{S}_\beta,$$

$C^\beta_{\alpha,\lambda}$ is the number of Yamanouchi tableau of shape β/α and content λ.

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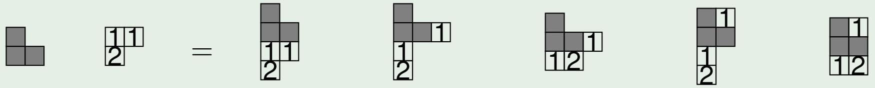
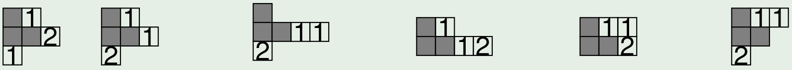
Example: $\alpha = (1,2)$, $\lambda = (2,1)$



$$\mathfrak{S}_{12} \quad \mathfrak{S}_{21} = \mathfrak{S}_{1221} + \mathfrak{S}_{1311} + \mathfrak{S}_{132} + \mathfrak{S}_{2211} + \mathfrak{S}_{222}$$



$$+ \; 2\mathfrak{S}_{231} + \mathfrak{S}_{141} + \mathfrak{S}_{24} + \mathfrak{S}_{33} + \mathfrak{S}_{321}$$

The immaculate basis

The immaculate basis of NSym can be defined by any of the following:

A tableaux expansion:

$$\mathfrak{S}^*_\alpha = \sum_{T \in IT(\alpha)} x^T,$$

the sum over immaculate tableaux, where $\mathfrak{S}^*_\alpha$ is the basis of QSym which is dual to the immaculate basis.

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Example: $\alpha = (2,2)$ in three variables

$$\mathfrak{S}^*_{22} = x_1^2x_2^2 + x_1x_2^3 + x_1^2x_2x_3 + 2x_1x_2^2x_3 + x_1^2x_3^2 + 2x_1x_2x_3^2 + x_2^2x_3^2 + x_1x_3^3 + x_2x_3^3$$

$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}$ $\begin{array}{|c|c|}\hline 1&2\\\hline 2&2\\\hline\end{array}$ $\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}$ $\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 2&2\\\hline\end{array}$ $\begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}$ $\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 2&3\\\hline\end{array}$ $\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 3&3\\\hline\end{array}$ $\begin{array}{|c|c|}\hline 2&3\\\hline 3&3\\\hline\end{array}$

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Example: $\alpha = [2, 1, 3]$

$$\mathfrak{S}^*_{213} = F_{1122} + F_{1131} + F_{114} + F_{123} + F_{213}$$

1 4	1 5	1 6	1 3	1 2
2	2	2	2	3
3 5 6	3 4 6	3 4 5	4 5 6	4 5 6

Applications

A new computation of Littlewood-Richardson coefficients

Relationship amongst coefficients

For compositions α, β, ν and a partition λ such that $\ell(\nu) \leq \ell(\alpha)$,

$$C^{\beta}_{\alpha,\lambda} = C^{\beta+\nu}_{\alpha+\nu,\lambda}$$

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Example: $\lambda = (2,1)$ and $\alpha = (1,1)$

$$\mathfrak{S}_{1,1}\mathfrak{S}_{2,1} = \mathfrak{S}_{1,1,2,1} + \mathfrak{S}_{1,2,1,1} + \mathfrak{S}_{1,2,2} + \mathfrak{S}_{1,3,1} + \mathfrak{S}_{2,1,1,1} + \mathfrak{S}_{2,1,2}$$

$$+2\mathfrak{S}_{2,2,1} + \mathfrak{S}_{2,3} + \mathfrak{S}_{3,1,1} + \mathfrak{S}_{3,2}.$$

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$$+2\mathfrak{S}_{2,2,1} + \mathfrak{S}_{2,3} + \mathfrak{S}_{3,1,1} + \mathfrak{S}_{3,2}.$$

Therefore:

$$s_{3,2}s_{2,1} = s_{3,2,2,1} + s_{3,3,1,1} + s_{3,3,2} + \color{red}{s_{3,4,1}} + s_{4,2,1,1} + s_{4,2,2}$$

$$+2s_{4,3,1} + s_{4,4} + s_{5,2,1} + s_{5,3}$$

Modules for the immaculate basis

Representation Theory of 0-Hecke algebra

The irreducible representations of $H_n(0)$ are all one dimensional. Under the identification of QSym with $G_0(H_n(0))$, an irreducible indexed by α corresponds to the fundamental quasi-symmetric function F_α.

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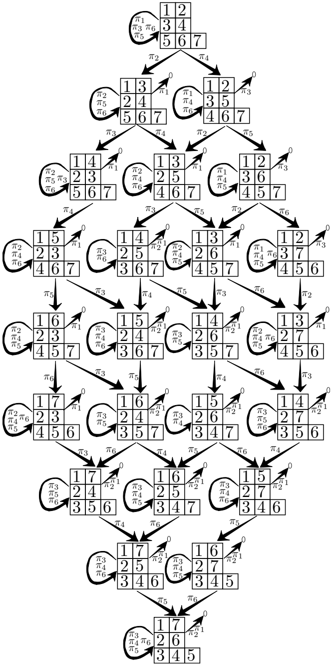
Construction of modules

There exists a combinatorially defined, indecomposable, finite dimensional, representation $\mathcal{V}_\alpha$ of $H_n(0)$ whose characteristic is $\mathfrak{S}^*_\alpha$.



The hook length formula by example

Proposition: Hook length formula

If $\alpha \models n$, the number of standard immaculate tableaux of shape α is equal to

$$\frac{n!}{\prod_{c \in \alpha} h_\alpha(c)}$$

where $c \in \alpha$ indicates $c = (i, j)$ with $1 \leq i \leq \ell(\alpha)$ and $1 \leq j \leq \alpha_i$.

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Example: $\alpha = [4, 2, 3]$

9	3	2	1
5	1		
3	2	1	

The number of standard immaculate tableaux of shape $[4, 2, 3]$ is equal to

$$\frac{9!}{9 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 224 \ .$$

THANKS!
Slides to be available on the web.
Immaculate basis and their duals available in Sage!
Papers: 1208:5191, 1304:1224 , 1305:4700
FPSAC Abstract: 1303:4801