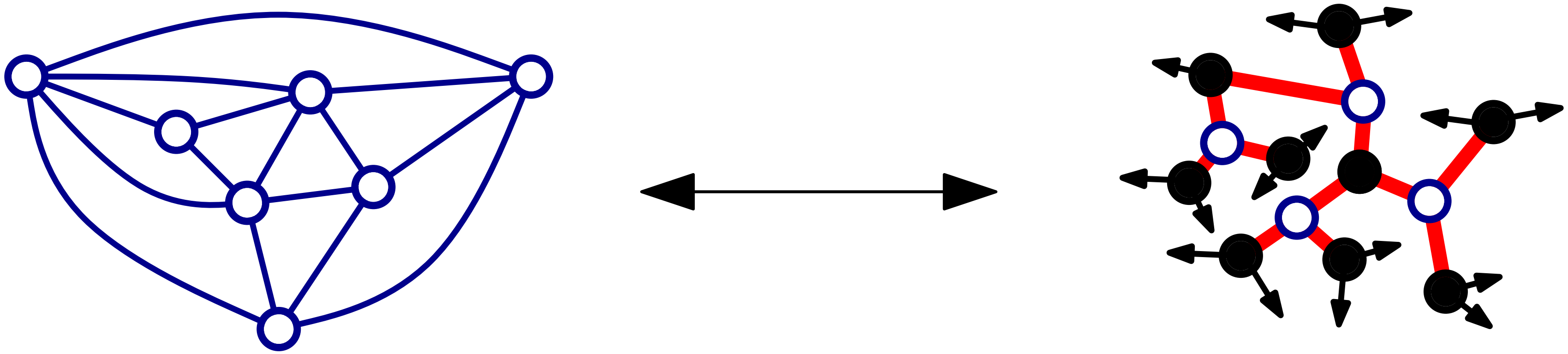
# Unified bijective framework for planar maps

Olivier Bernardi (Brandeis University)

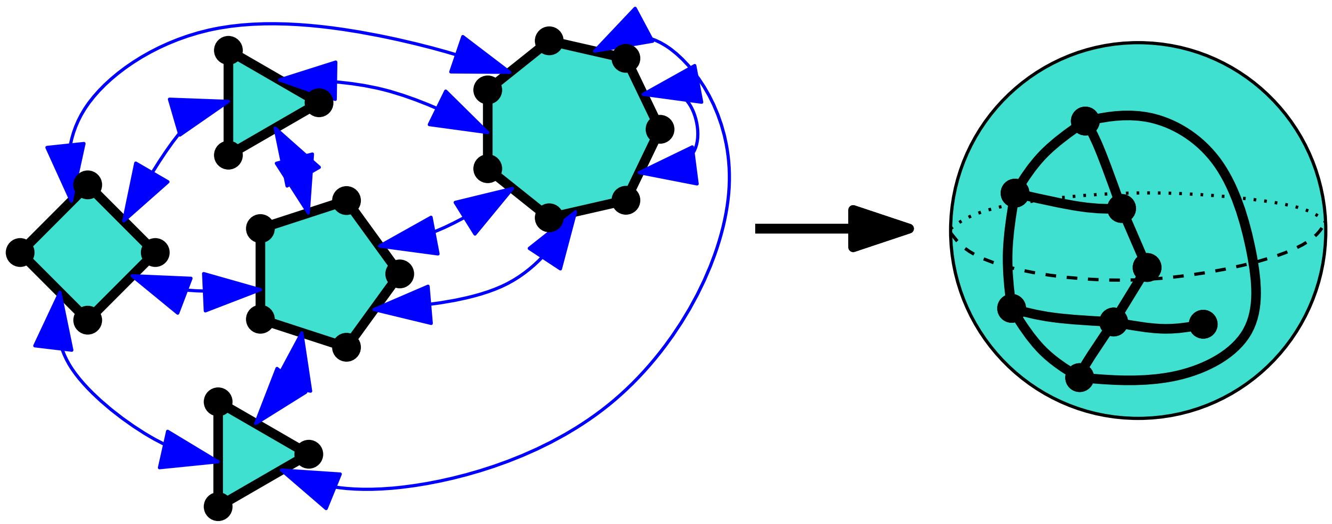
Joint work with Éric Fusy (CNRS/LIX)

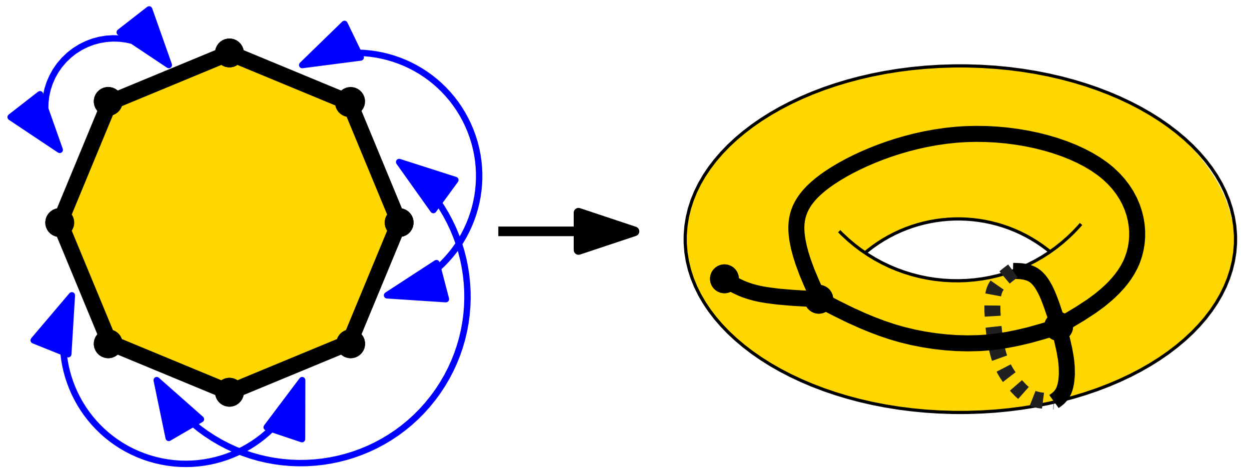


FPSAC, Paris 2013

## Maps

**Definition:** A **map** is a gluing of polygons (pairing of the edges) forming a connected surface without boundary.





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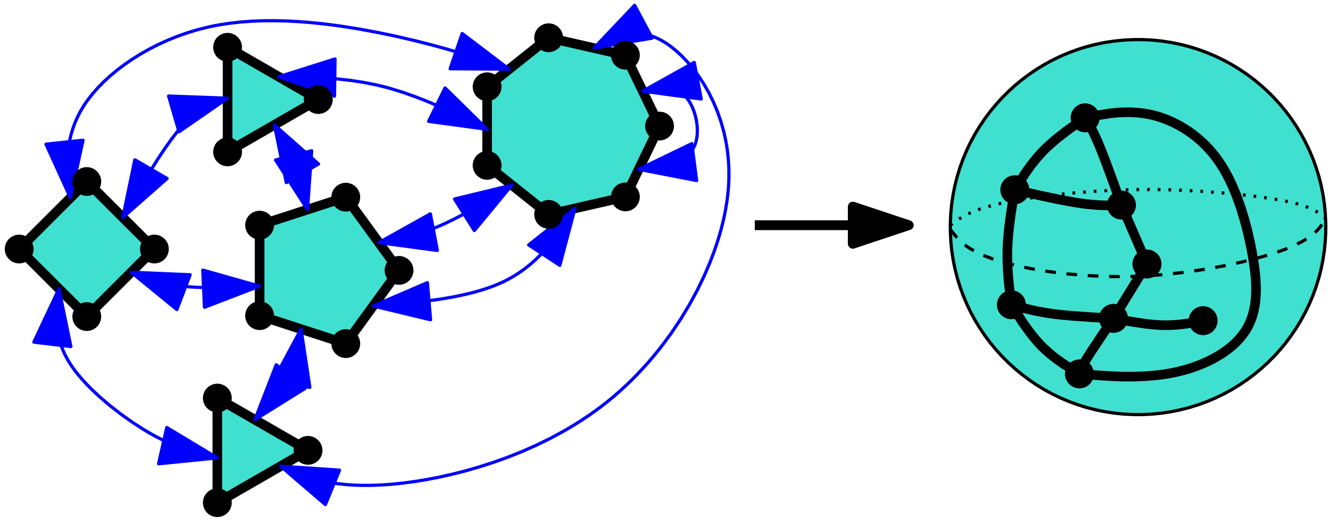
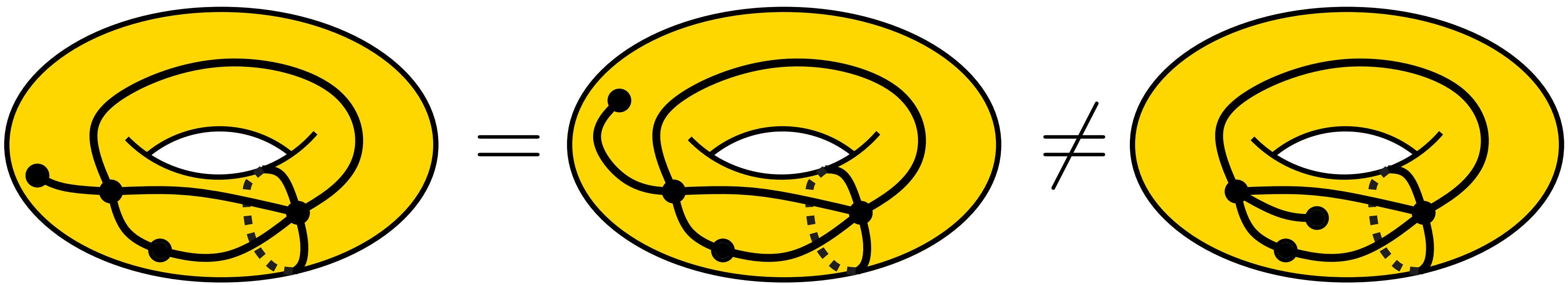
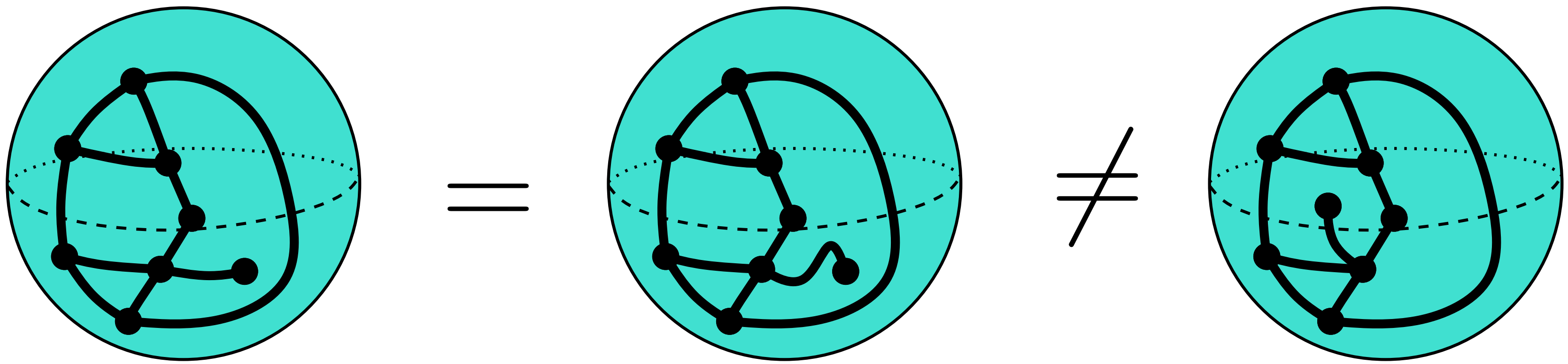


**Equivalent definition:** A **map** is a cellular embedding of a connected graph in a surface, considered up to homeomorphism.



## Planar maps.

A **planar map** is a map on the sphere.



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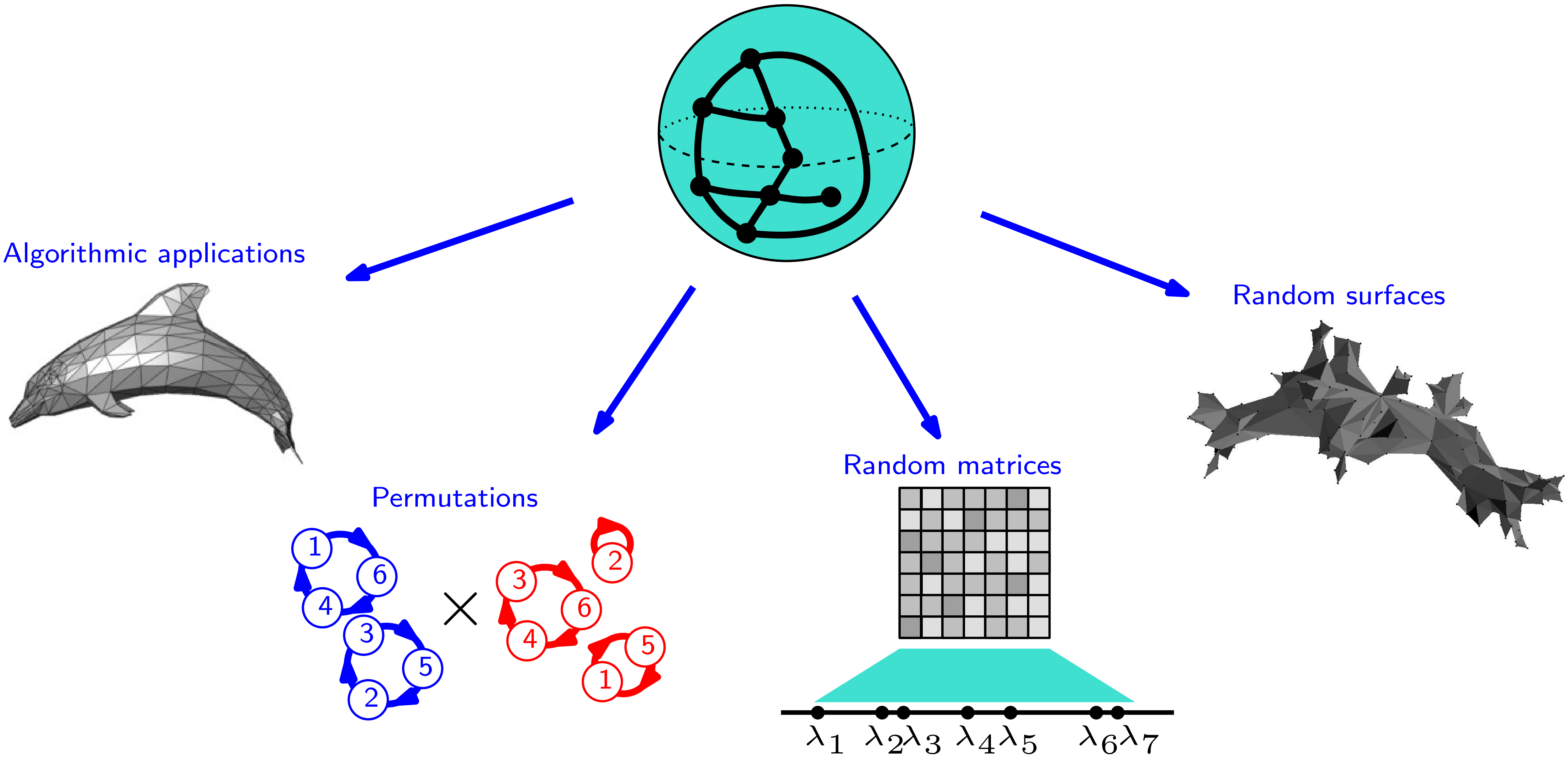
A **planar map** is a map on the sphere.

A planar triangulation.

# Motivations



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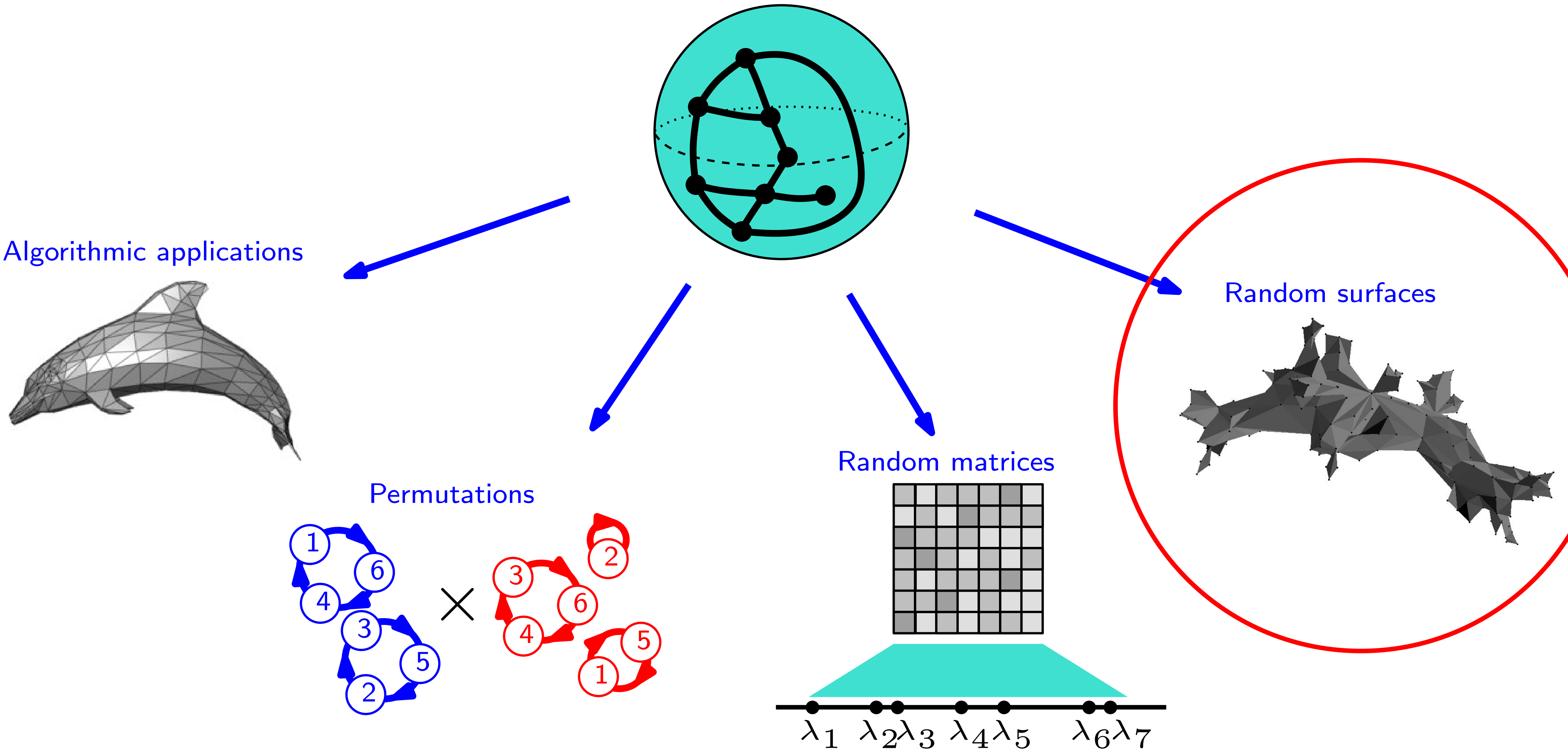


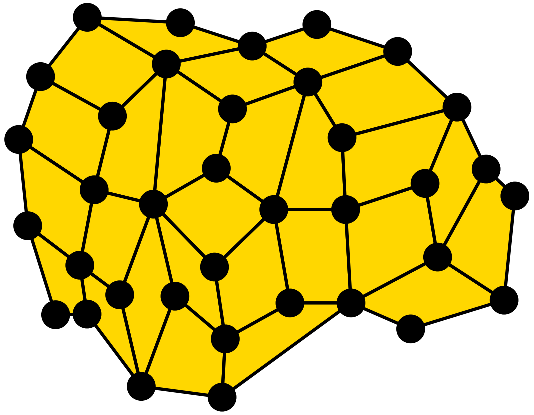
## Random surfaces

Random surfaces are important in physics (2D Quantum gravity)

*"There are methods and formulae in science, which serve as master-keys to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling sums over random surfaces. These sums replace the old-fashioned (and extremely useful) sums over random paths."*
A.M. Polyakov, Moscow, 1981

# Random surfaces

Consider the **random quadrangulation** obtained by choosing uniformly among all ways of gluing $n = 1000$ squares to form the sphere.



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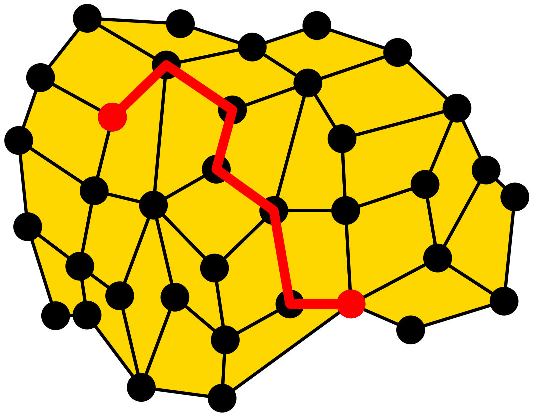

It is a **random metric space**: $(V_n, d_n)$.
Hope to define a random surface by taking limit when

- number of squares $\to \infty$,
- size of squares $\to 0$.

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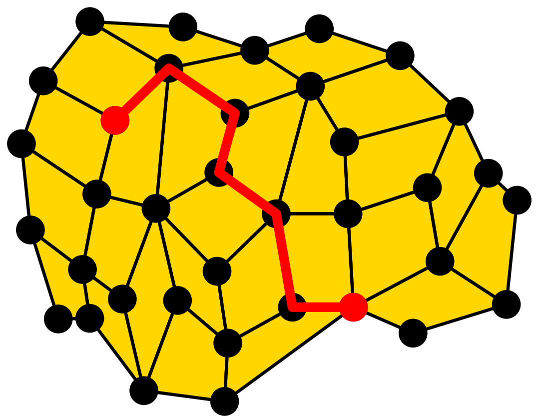

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- number of squares $\to \infty$,
- size of squares $\to 0$.

**Theorem [Chassaing, Schaeffer 03]** The distance between two random points of the random quadrangulations is $C_n\, n^{1/4}$, where the random variable $C_n$ converges in distribution toward a know law (ISE).

## Random surfaces

Let $M_n = (V_n, \frac{d_n}{n^{1/4}})$ be the random metric space corresponding to a rescaled uniformly random quadrangulations with $n$ squares.

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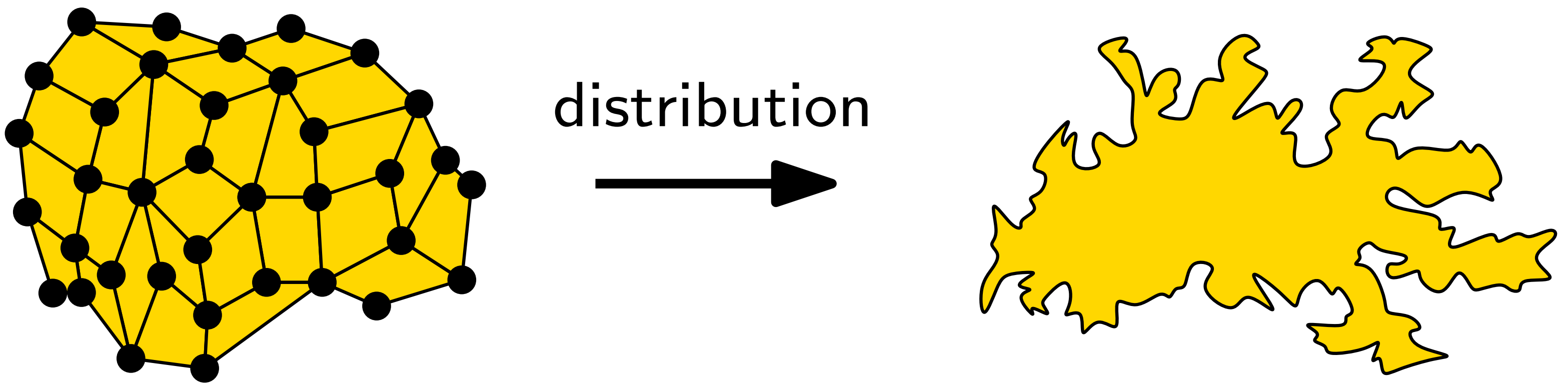
**Theorem.** [Le Gall 2007 + Miermont/Le Gall 2012]
The sequence $(M_n)$ converges in distribution (in the Gromov Hausdorff topology) toward a random metric space, which
- is **homeomorphic to the sphere**
- has **Hausdorff dimension 4**.
(+ "explicit" description of the space).

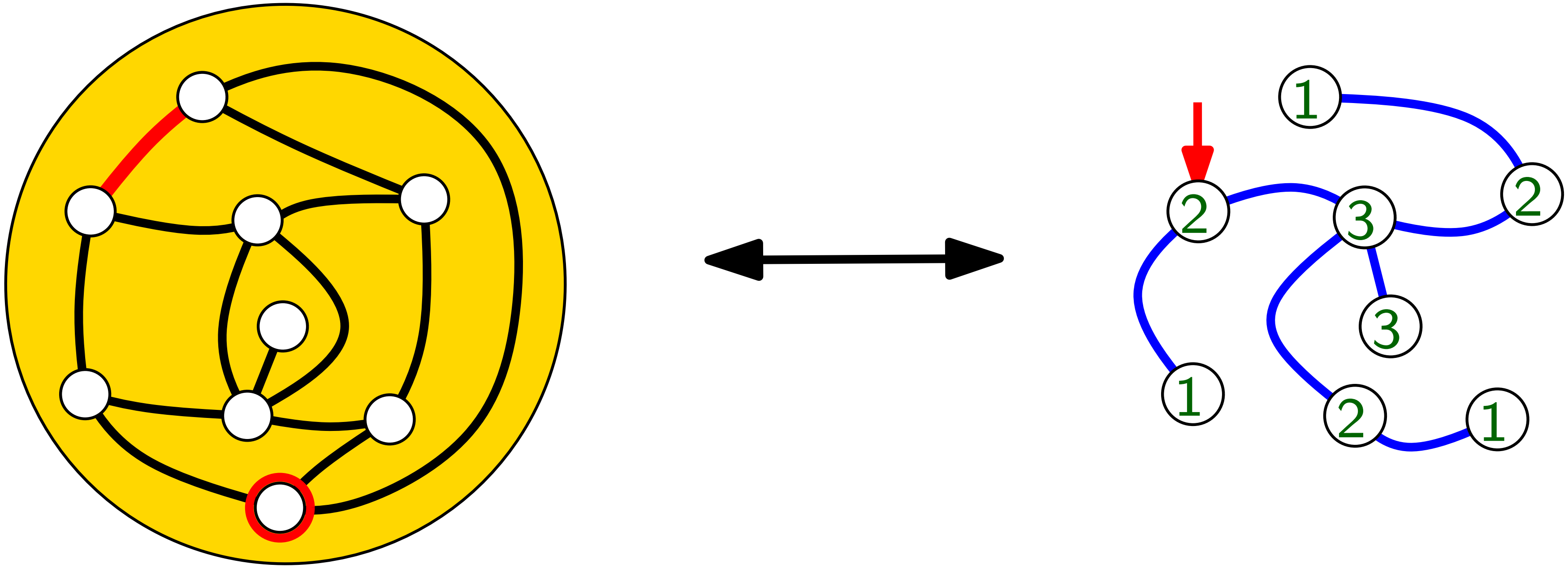



**Related work.** Bouttier, Di Francesco, Chassaing, Guitter, Le Gall, Marckert, Miermont, Mokkadem, Paulin, Schaeffer, Weill . . .

# Key tool

**Bijection** between quadrangulations and trees:

[Cori, Vauquelin 81, Schaeffer 98]




Quadrangulation with $n$ faces
+ marked vertex + marked edge

Rooted plane tree with $n$ edges
+vertex labels changing by $-1$, $0$, $1$
along edges and such that $\min = 1$.

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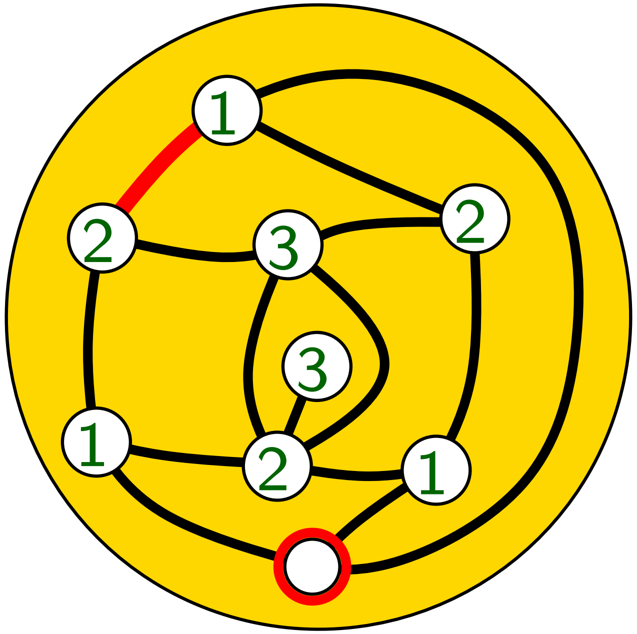


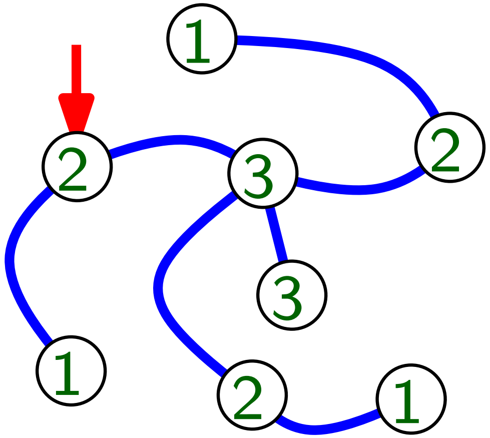


Quadrangulation with $n$ faces
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Rooted plane tree with $n$ edges
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along edges and such that $\min = 1$.

Vertex at distance $d$
from marked vertex

Vertex labeled $d$

# A master bijection for planar maps

# Counting formulas

## Example of counting formulas for rooted plane trees :

Binary trees ($n$ nodes)

$$\frac{1}{n+1}\binom{2n}{n}$$

$k$-ary trees ($n$ nodes)

$$\frac{1}{kn-n+1}\binom{kn}{n}$$

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## Example of counting formulas for **rooted planar maps** [Tutte 60's]:

Loopless triangulations
($2n$ triangles)

$$\frac{2^n}{(n+1)(2n+1)}\binom{3n}{n}$$

Simple triangulations
($2n$ triangles)

$$\frac{1}{n(2n-1)}\binom{4n-2}{n-1}$$

General quadrangulations
($2n$ squares)

$$\frac{2\cdot 3^n}{(n+1)(n+2)}\binom{2n}{n}$$

Simple quadrangulations
($2n$ squares)

$$\frac{2}{n(n+1)}\binom{3n}{n-1}$$

## A few bijections

- Triangulations ($2n$ faces)

Loopless: $\frac{2^n}{(n+1)(2n+1)}\binom{3n}{n}$

[**Poulalhon, Schaeffer 02**, Bernardi 07]

Simple: $\frac{1}{n(2n-1)}\binom{4n-2}{n-1}$

[Poulalhon, Schaeffer 06 **Fusy, Poulalhon, Schaeffer 08**]

- Quadrangulations ($n$ faces)

General: $\frac{2\cdot 3^n}{(n+1)(n+2)}\binom{2n}{n}$

[Schaeffer 97, **Schaeffer 98**]

Simple: $\frac{2}{n(n+1)}\binom{3n}{n-1}$

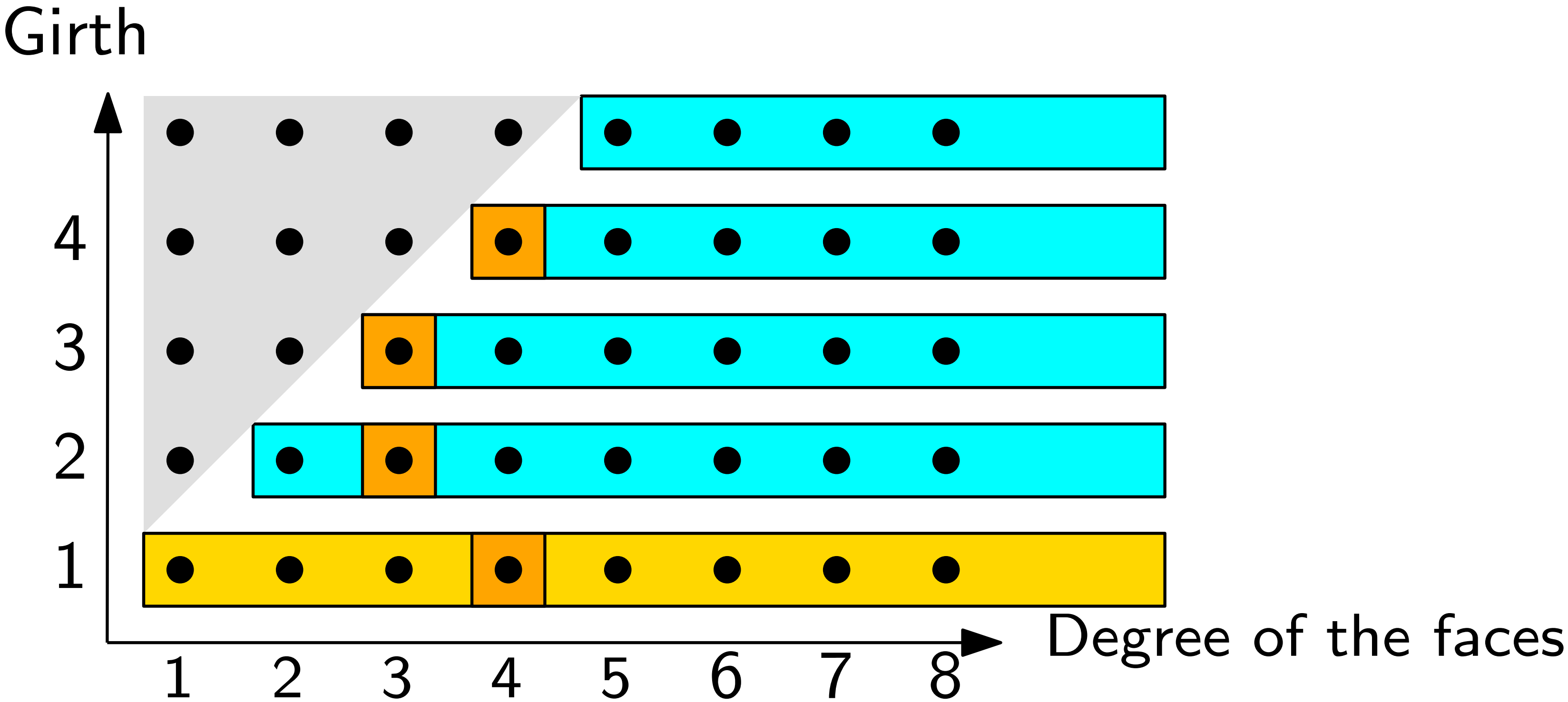
[**Schaeffer 98**, Fusy 07]

- Bipartite maps ($n_i$ faces of degree $2i$)

$$\frac{2\cdot(\sum i\,n_i)!}{(2+\sum(i-1)n_i)!}\prod_i \frac{1}{n_i!}\binom{2i-1}{i}^{n_i}$$

[Schaeffer 97, **Bouttier, Di Francesco, Guitter 04**]

# A few bijections



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Find a **master bijection** for planar maps
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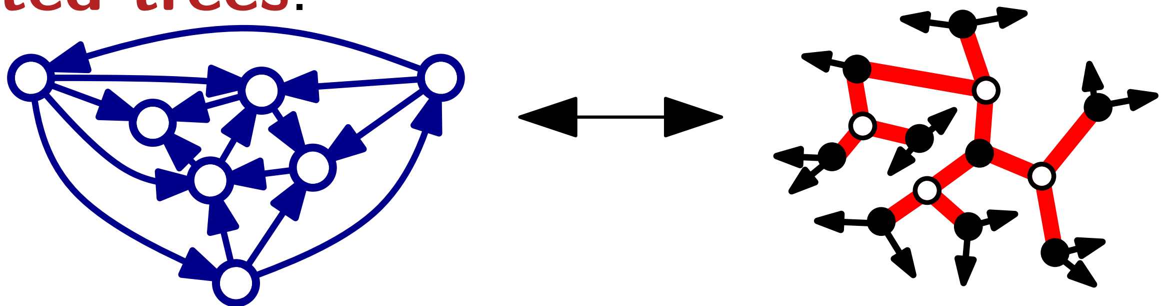
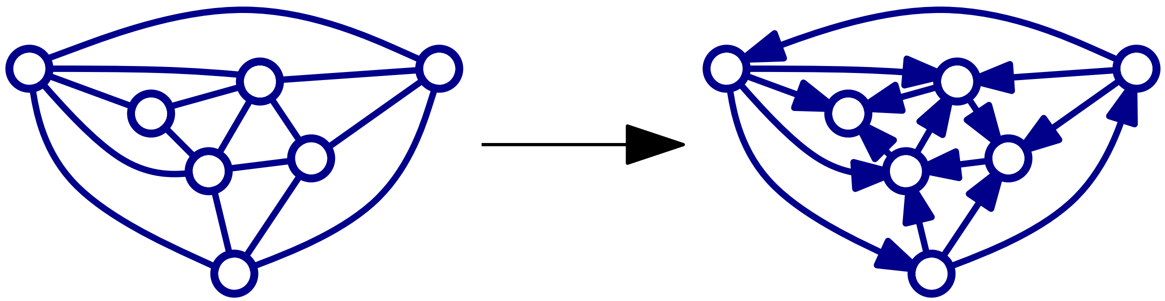
**Strategy:**

**1.** Define a **master bijection** between a class of **oriented maps** and a class of **decorated trees**.



**2.** Define **canonical orientations** for maps in any class defined by degree and girth constraints.



**Goal:**

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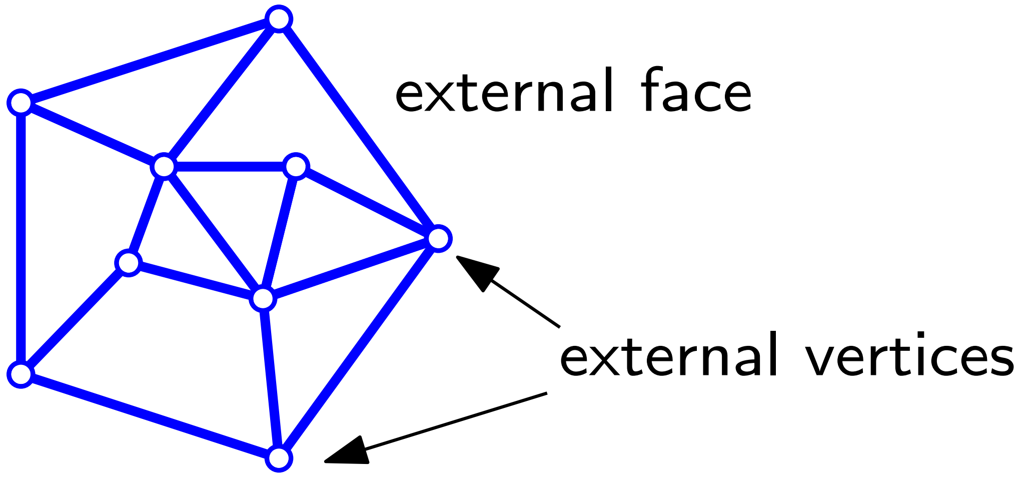
**Alternative strategies:**

- Bijection of the blue type [Albenque, Poulalhon 13+]
- Recursive decomposition by slices [Bouttier, Guitter 13+]

# Oriented maps

A **plane map** is a planar map with a distinguished "external face".



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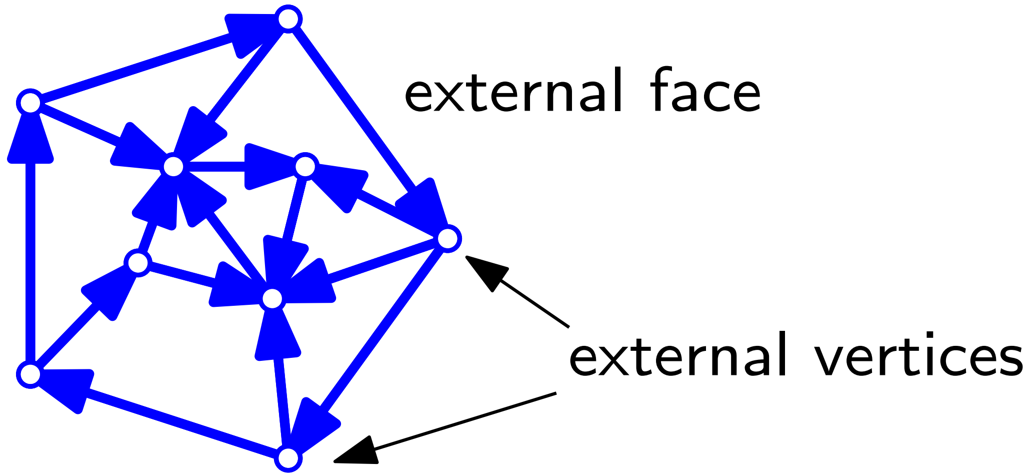
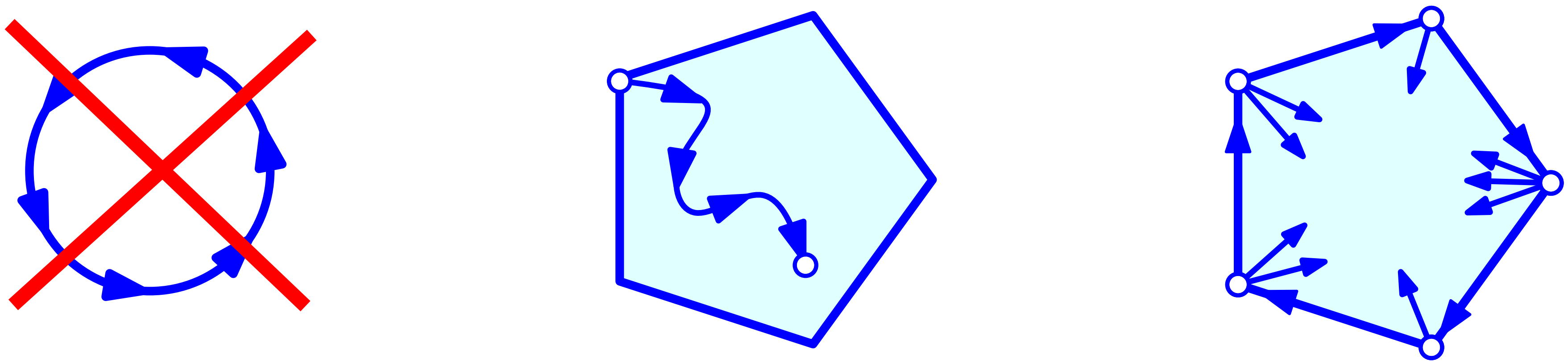


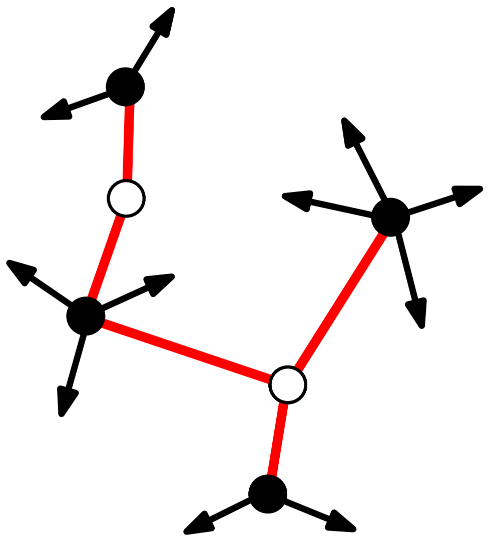

Let $\mathcal{O}$ be the set of oriented plane maps such that:

- there is no counterclockwise directed cycle (**minimal**),
- internal vertices can be reached from external vertices (**accessible**),
- external vertices have indegree 1.

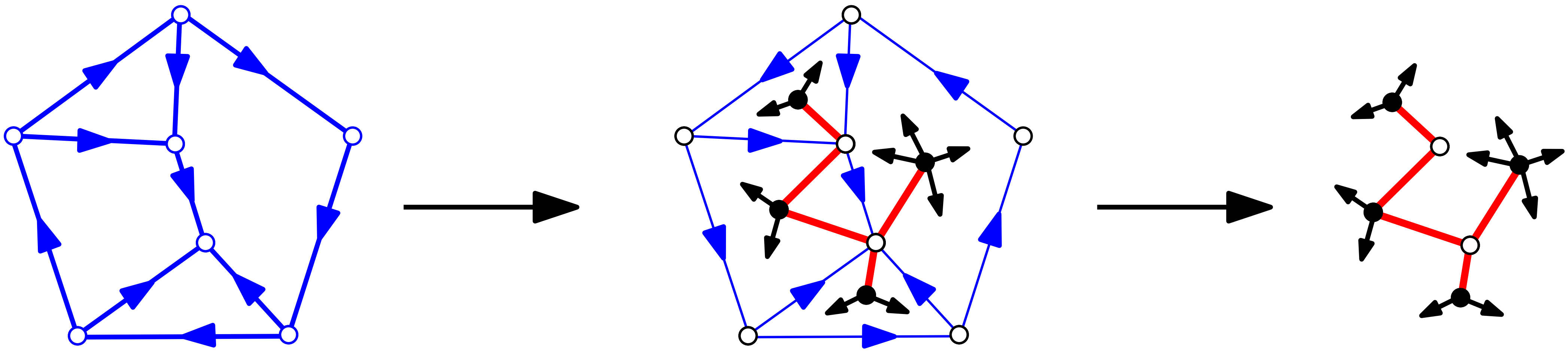


# Mobiles

A **mobile** is a plane tree with vertices properly colored in black and white, together with buds (arrows) incident only to black vertices.



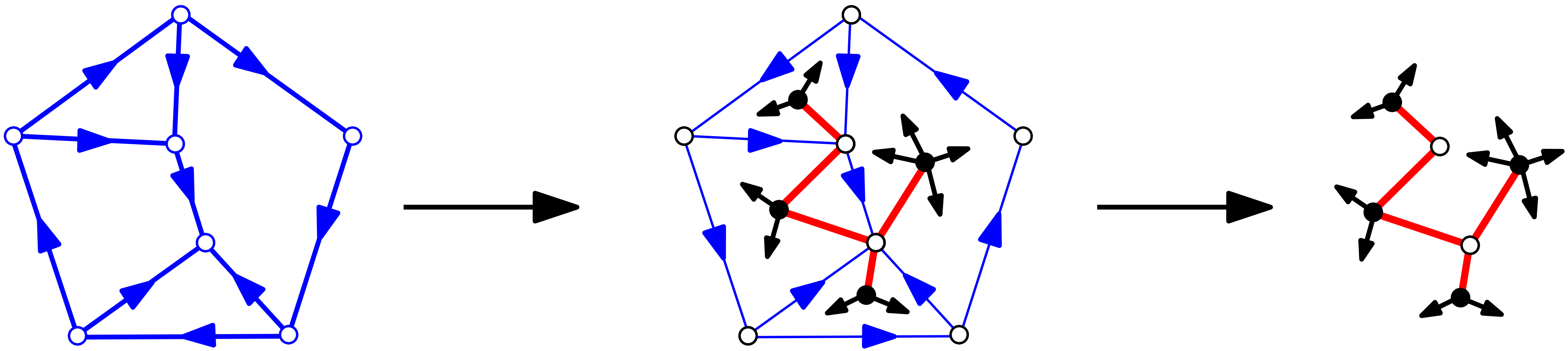
# Master bijection



Mapping $\Phi$ for an oriented map in $\mathcal{O}$:

- Return the external edges.
- Place a black vertex in each internal face.
  Draw an edge/bud for each clockwise/counterclockwise edge.
- Erase the map.

# Master bijection

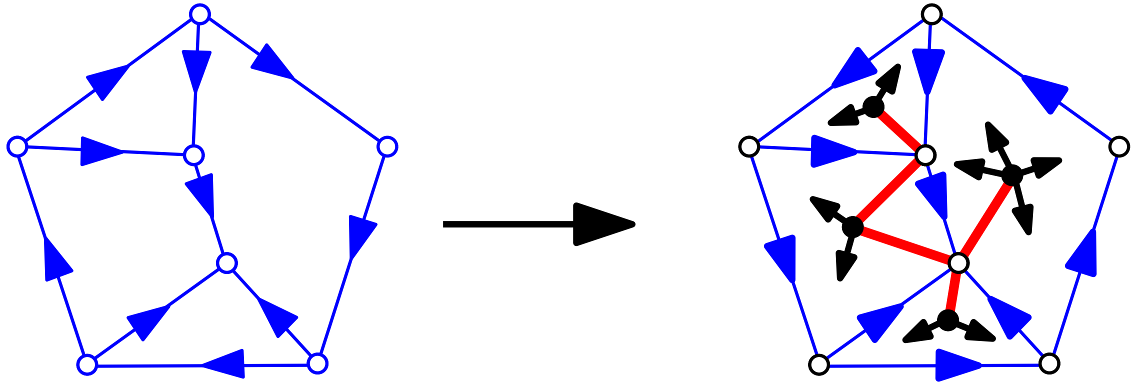


**Theorem [B.,Fusy]:** The mapping $\Phi$ is a bijection between the set $\mathcal{O}$ of oriented maps and the set of mobiles with more buds than edges.

Moreover, indegree of internal vertices $\longleftrightarrow$ degree of white vertices
degree of internal faces $\longleftrightarrow$ degree of black vertices
degree of external face $\longleftrightarrow$ #buds - #edges

# Master bijection: elements of proof

**Fact 1.** Local operation in the faces produces a tree
$\longleftrightarrow$ Orientation is minimal and accessible.



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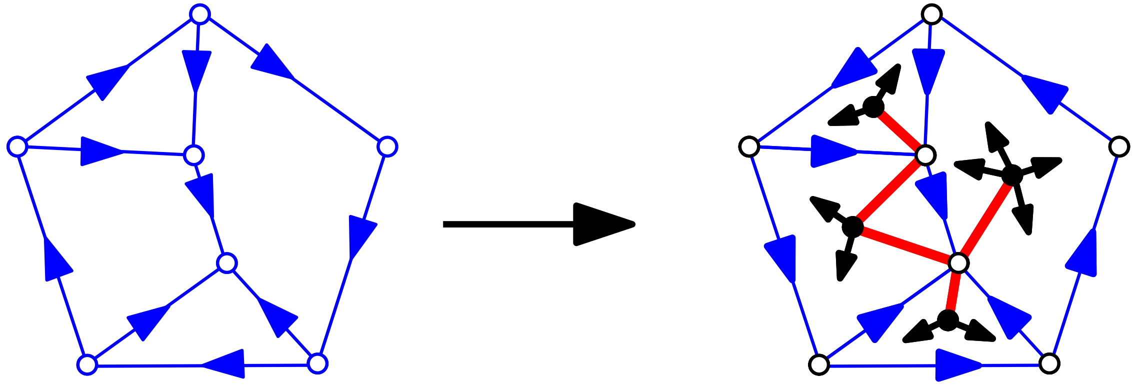
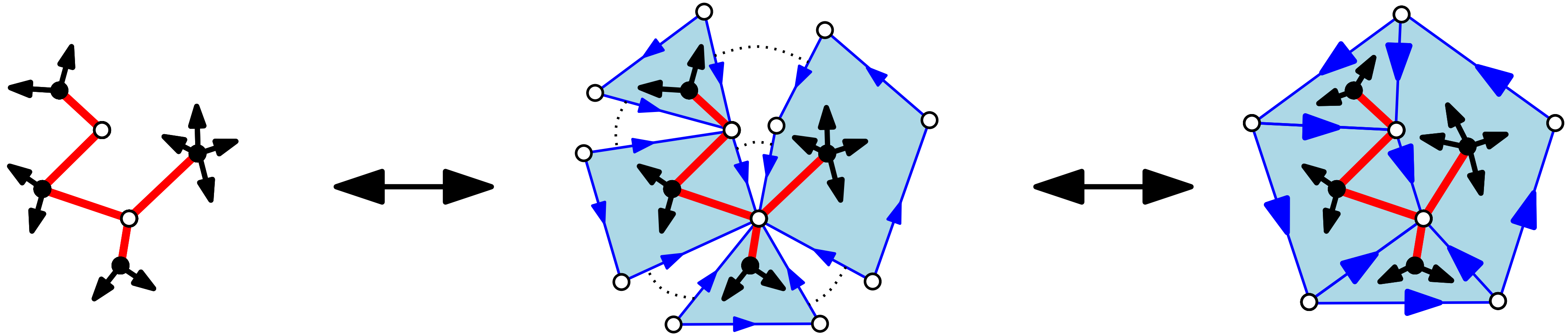


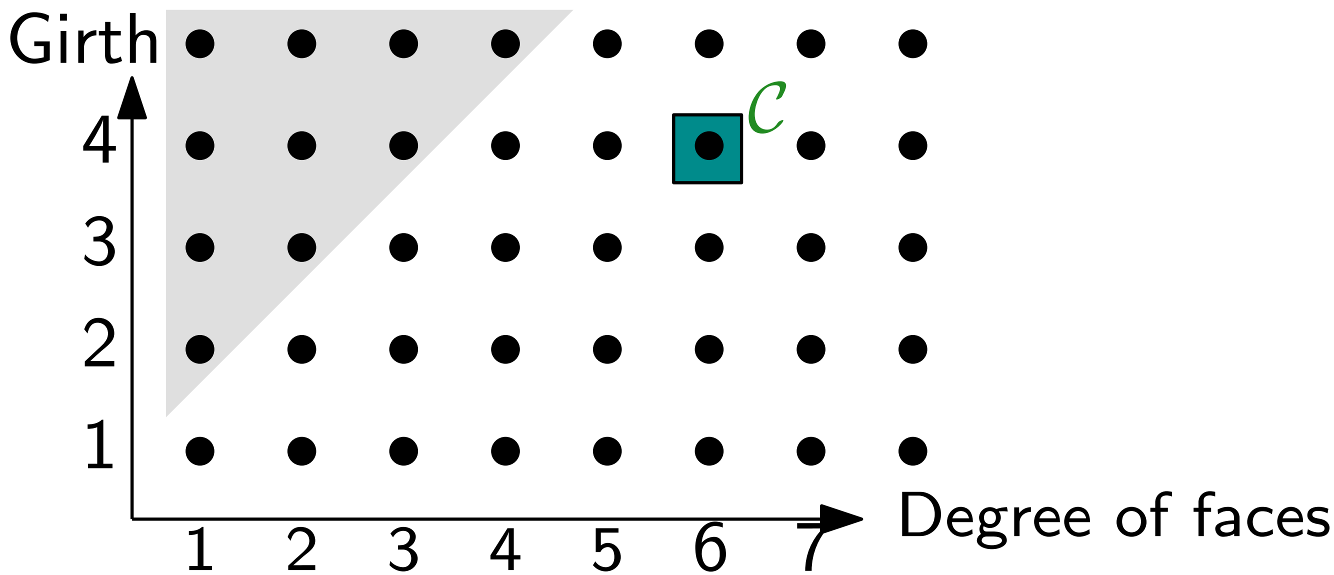
**Fact 2.** The mobile captures all the info about the oriented map.



# Canonical orientations

**Goal:**

$\mathcal{C}$= class of maps defined by girth constraints and degree constraints.
We want to define a canonical orientation in $\mathcal{O}$ for each map in $\mathcal{C}$



## How to define a canonical orientation?

We consider a plane map $M$ and want to define an orientation in $\mathcal{O}$ (orientations which are minimal + accessible + external indegree 1).

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**Fact 1**: Let $\alpha$ be a function from the vertices of $M$ to $\mathbb{N}$.
If there is an orientation of $M$ with indegree $\alpha(v)$ for each vertex $v$, then there is unique minimal one.

$\Rightarrow$ Orientations in $\mathcal{O}$ can be defined by specifying the indegree $\alpha(v)$.

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$\Rightarrow$ Orientations in $\mathcal{O}$ can be defined by specifying the indegree $\alpha(v)$.

**Fact 2**: An orientation with indegree $\alpha(v)$ exists (and is accessible) if and only if

- $\sum_{v \in V} \alpha(v) = |E|$
- $\forall U \subset V$, $\sum_{v \in U} \alpha(v) \geq |E_U|$ (strict if there is an external vertex $\notin U$).

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We consider a plane map $M$ and want to define an orientation in $\mathcal{O}$ (orientations which are minimal + accessible + external indegree 1).

**Conclusion**: For a map $G$, one can define an orientation in $\mathcal{O}$ by specifying an indegree function $\alpha$ such that:

- $\sum_{v\in V} \alpha(v) = |E|$,
- $\forall\, U \subset V, \quad \sum_{v\in U} \alpha(v) \geq |E_U|$ (strict if an external vertex $\notin U$),
- $\alpha(v) = 1$ for every external vertex $v$.

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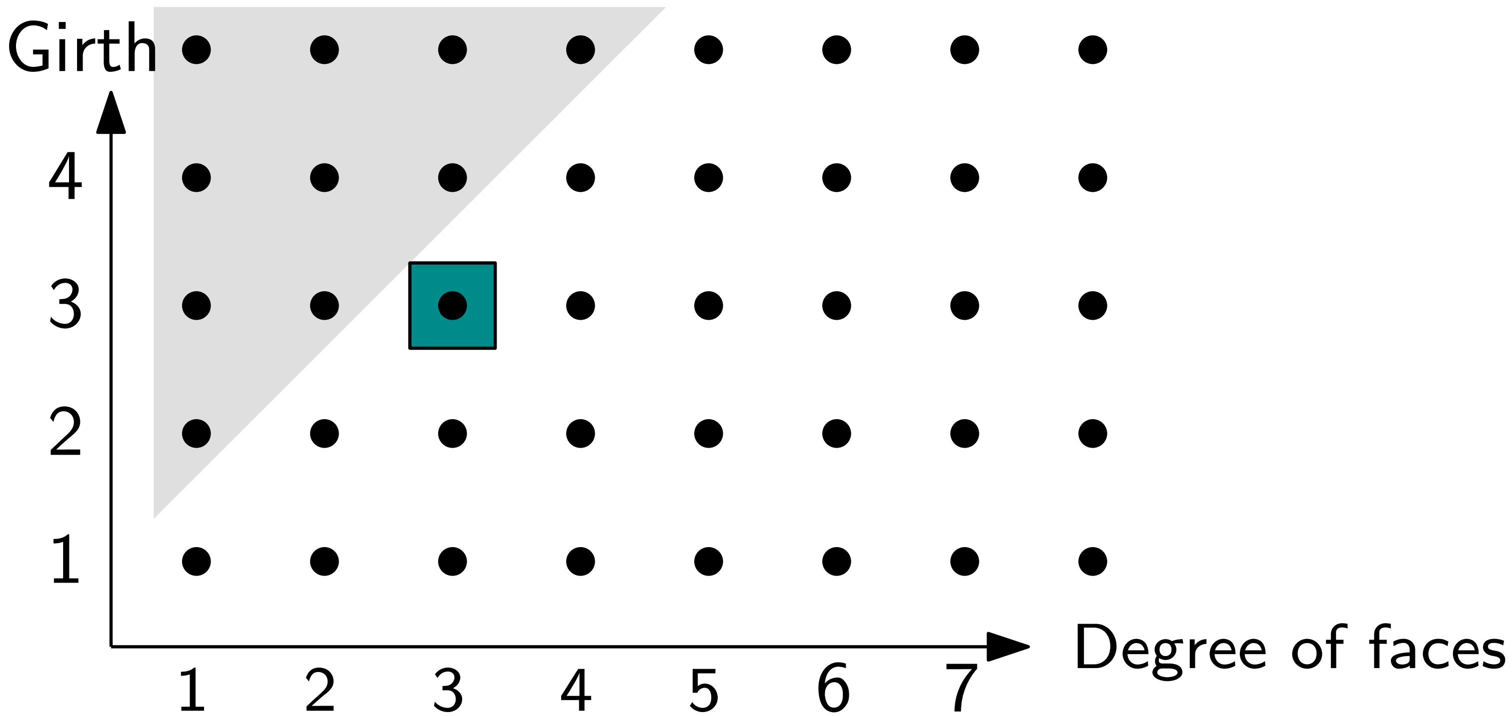
**Remark**: Specifying indegrees is also convenient for master bijection:

indegrees of internal vertices $\longleftrightarrow$ degrees of white vertices.

# Example: Simple triangulations

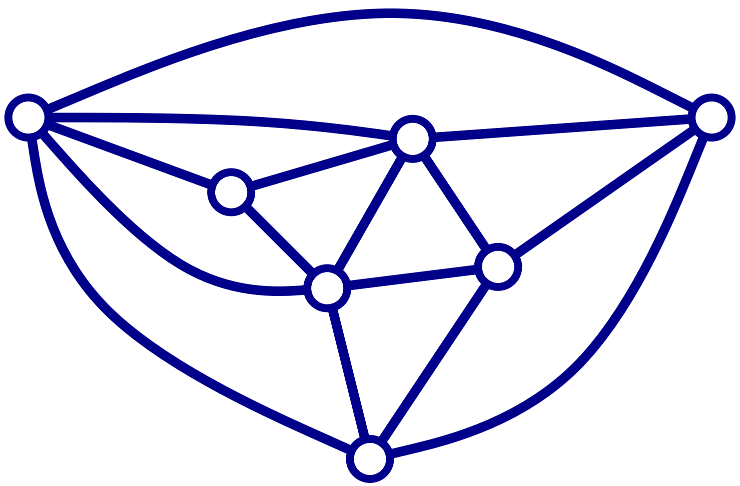


# Example: Simple triangulations

**Fact:** A triangulation with $n$ internal vertices has $3n$ internal edges.

**Proof:** The numbers $v$, $e$, $f$ of vertices edges and faces satisfy:

- Incidence relation: $3f = 2e$.
- Euler relation: $v - e + f = 2$. □

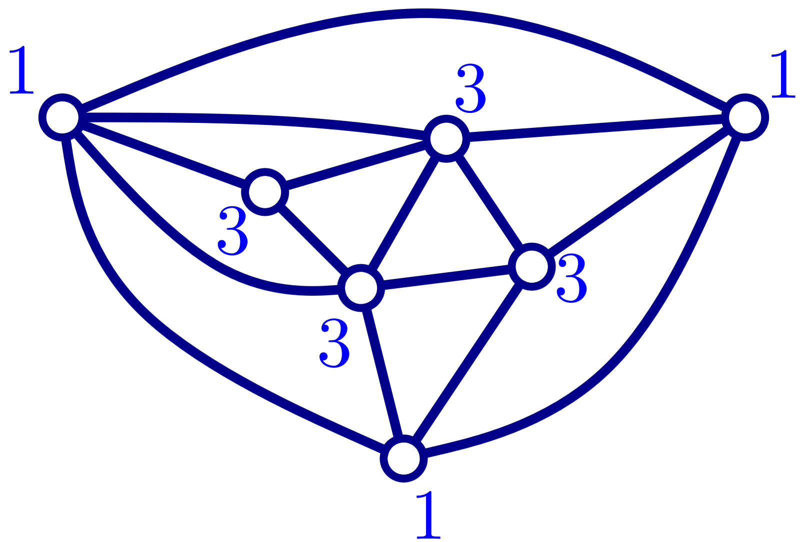


## Example: Simple triangulations

**Fact:** A triangulation with $n$ internal vertices has $3n$ internal edges.

**Natural candidate for indegree function:**

$$\alpha : \ v \mapsto \begin{vmatrix} 3 \text{ if } v \text{ internal} \\ 1 \text{ if } v \text{ external} \end{vmatrix}.$$

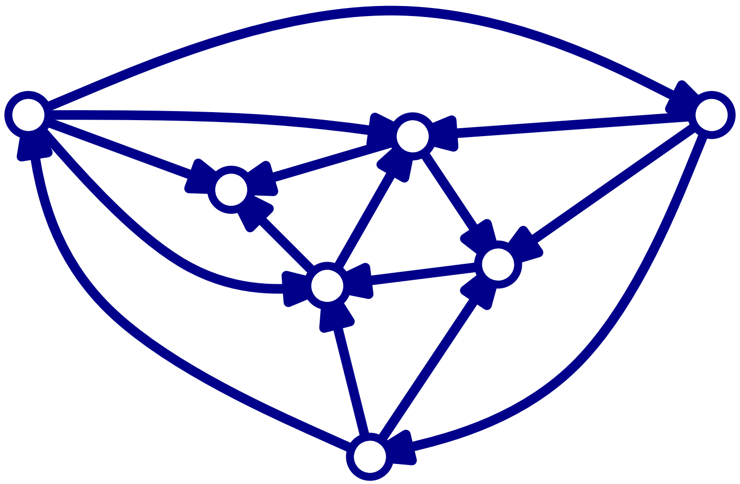


## Example: Simple triangulations

**Thm.**[Schnyder 89] A triangulation admits an orientation with indegree function $\alpha$ if and only if it is simple.

**New proof:** Euler relation + the incidence relation $\Rightarrow$ $\alpha$ satisfies:

- $\sum_{v\in V} \alpha(v) = |E|$,
- $\forall U \subset V$, $\sum_{u\in U} \alpha(u) \geq |E_U|$ (strict if an external vertex $\notin U$),
- $\alpha(v) = 1$ for every external vertex $v$. $\square$



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$\Rightarrow$ The class of simple triangulations is identified with the class of oriented maps in $\mathcal{O}$ such that

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**Thm [recovering FuPoSc08]:** The master bijection $\Phi$ induces a bijection between simple triangulations and mobiles such that

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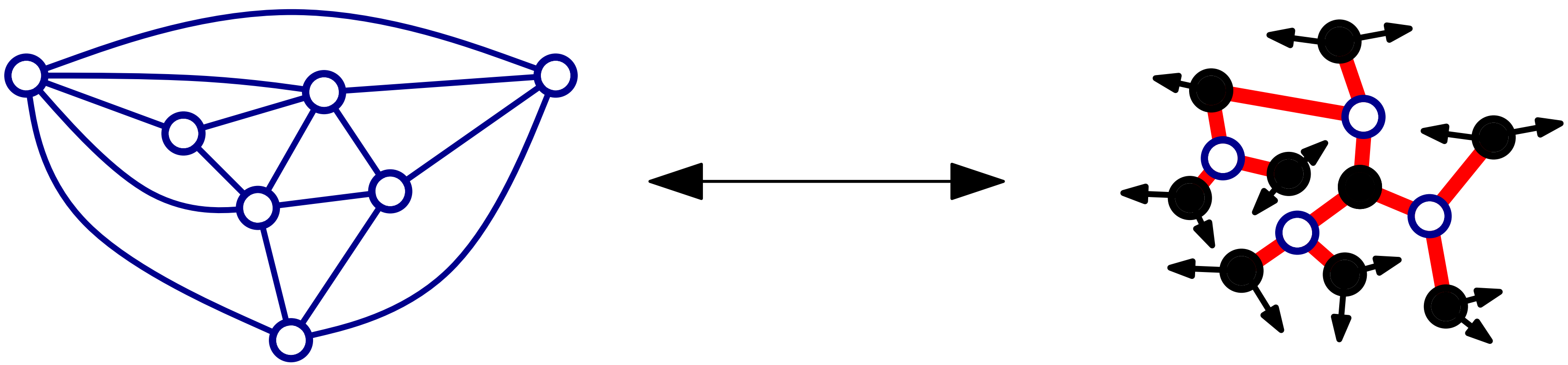

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**Corollary:** The number of rooted simple triangulations with $2n$ faces is $\frac{1}{n(2n-1)}\binom{4n-2}{n-1}$.

# More classes of maps

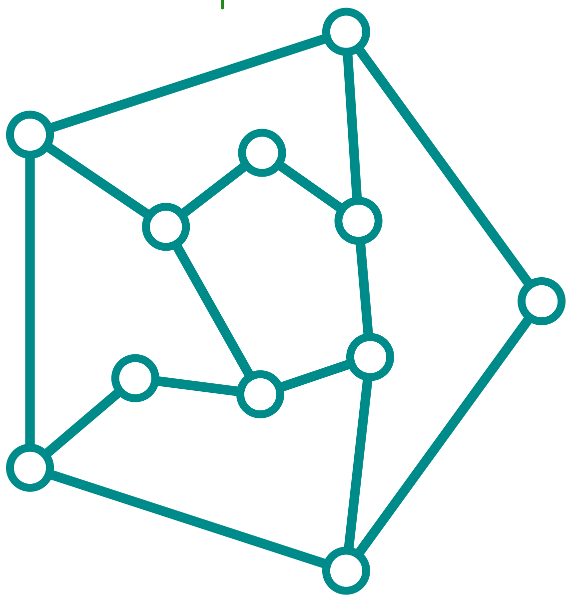
# Orientations for $d$-angulations of girth $d$

**Fact:** A $d$-angulation with $(d-2)n$ internal vertices has $dn$ internal edges.

**Natural candidate for indegree function:**

$$\alpha : v \mapsto \begin{cases} d/(d-2) \text{ if } v \text{ internal} \\ 1 \text{ if } v \text{ external} \end{cases} \quad \ldots$$

$d = 5$



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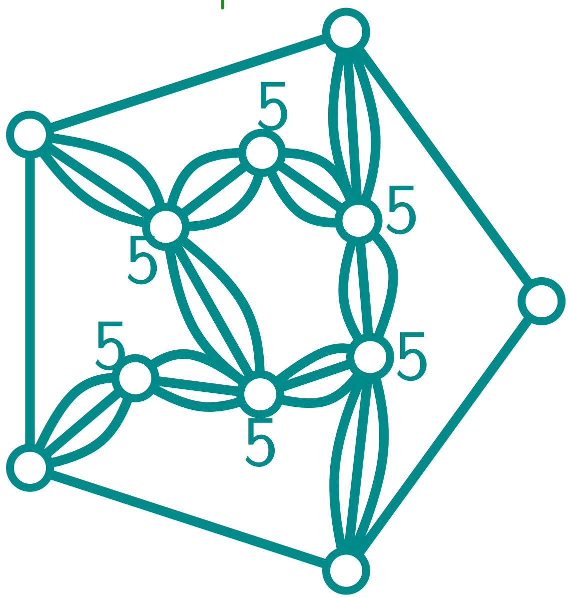



**Idea:** We can look for an orientation of $(d-2)G$ with indegree function

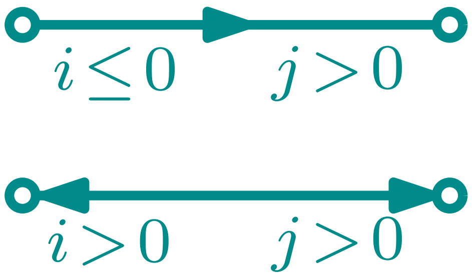
$$\alpha: \; v \mapsto \begin{cases} d \text{ if } v \text{ internal} \\ 1 \text{ if } v \text{ external} \end{cases} .$$

## Orientations for $d$-angulations of girth $d$

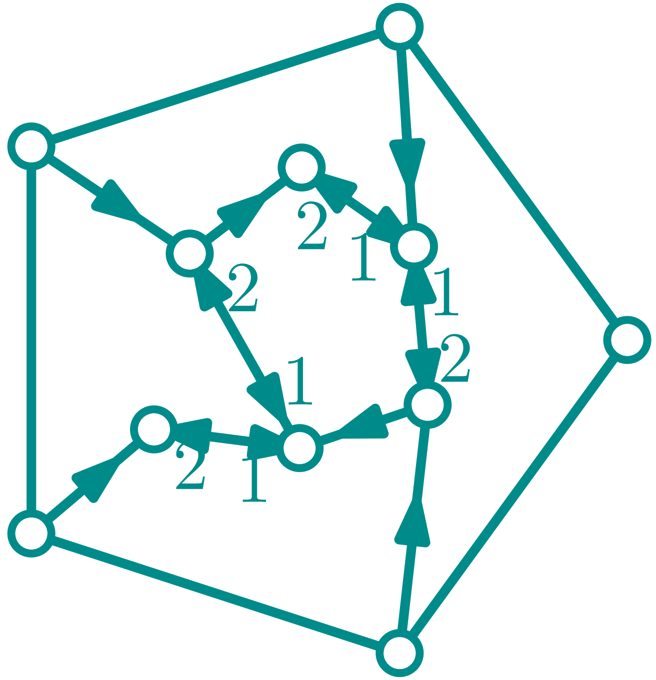
**Thm [B., Fusy]:** Let $G$ be a $d$-angulation.

$G$ has girth $d \longleftrightarrow G$ admits a weighted orientation with

- weight $d-2$ per edges.
- ingoing weight $d$ per internal vertex,
- ingoing weight $1$ per external vertex.




$$i+j = d-2$$



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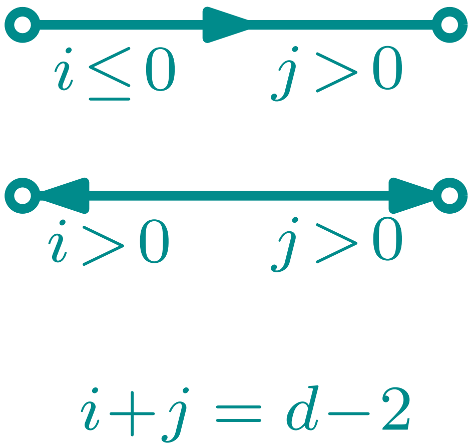
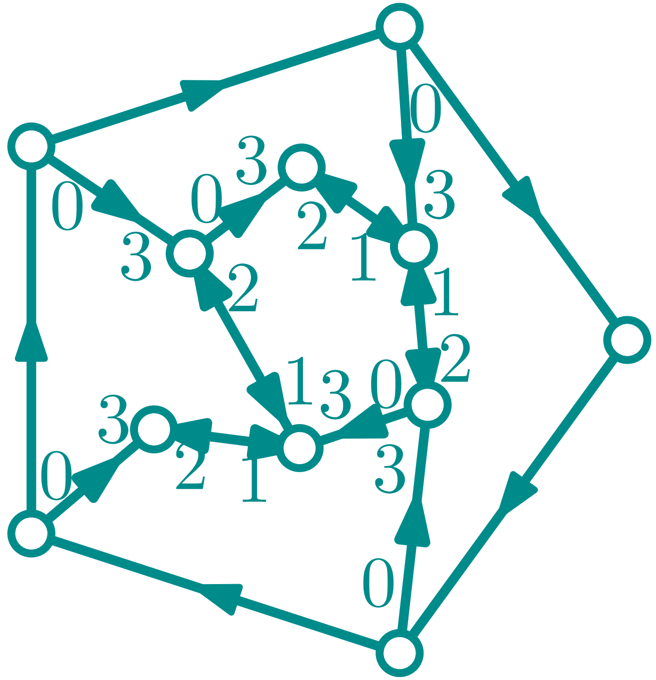
- ingoing weight $1$ per external vertex.

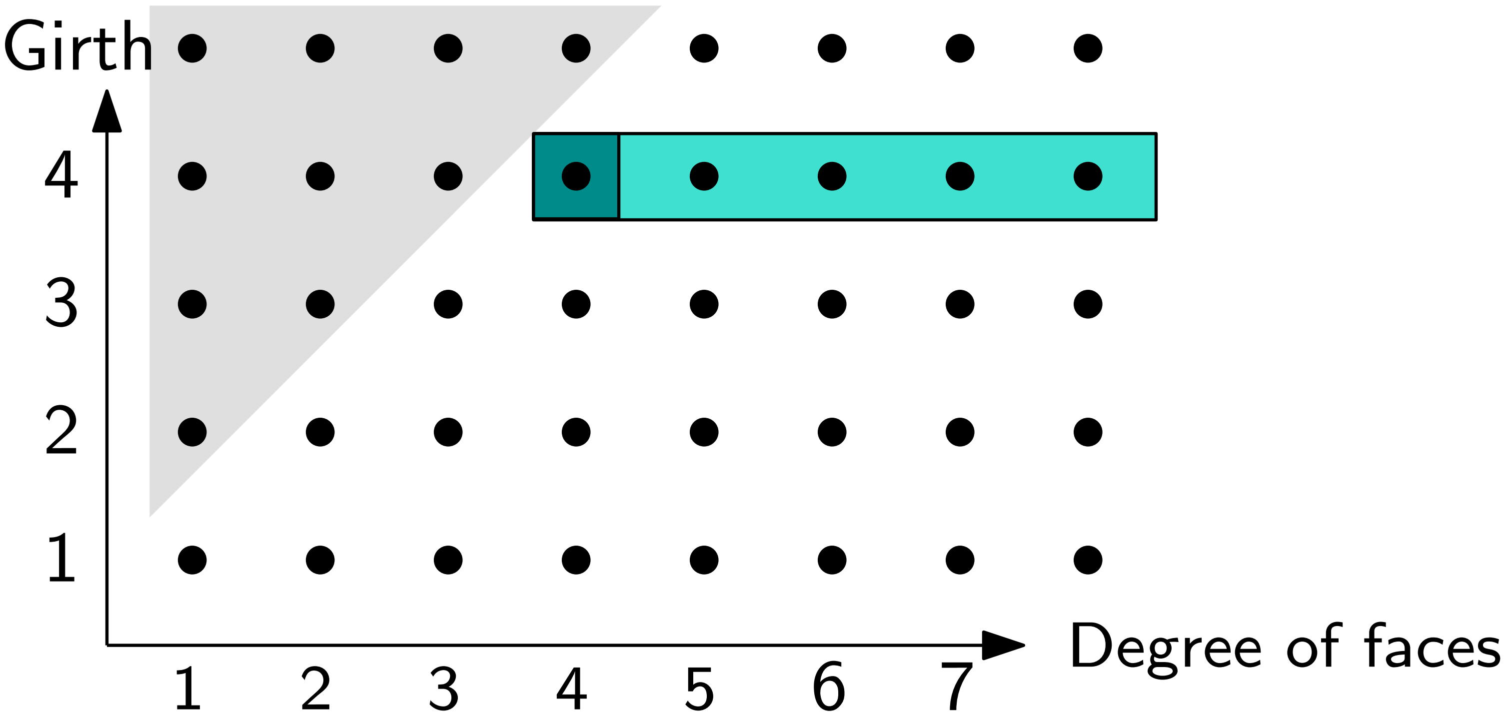
Moreover, $G$ admits a unique such orientation in $\mathcal{O}$ in this case.



**Proof:** Use the Euler relation + incidence relation as before. $\square$

## Orientations for maps of girth $d$



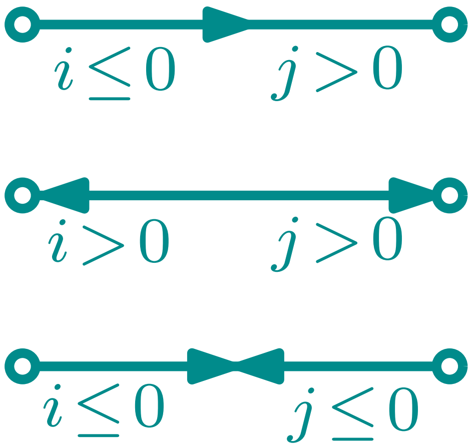
## Orientations for maps of girth $d$

**Thm [B., Fusy]:** Let $G$ be a map.

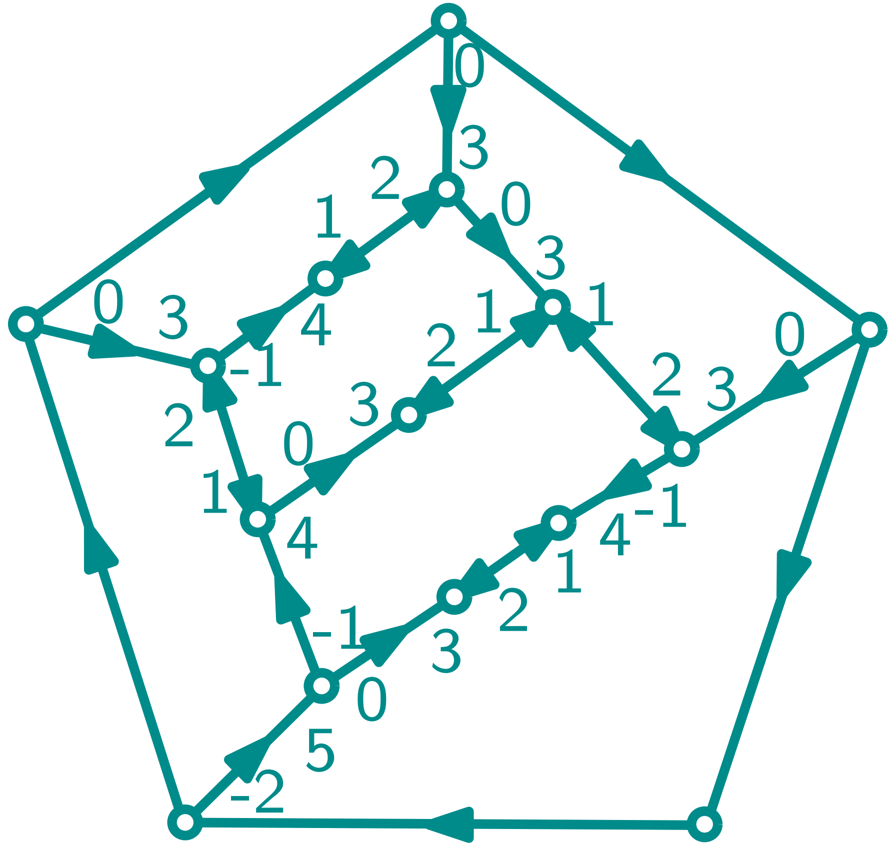
$G$ has girth $d \longleftrightarrow G$ admits a weighted bi-orientation with

- weight $d-2$ per edges.
- ingoing weight $d$ per internal vertex,
- ingoing weight $1$ per external vertex,
- outgoing weight $d - \deg$ per faces.

Moreover, $G$ admits a unique such orientation in $\mathcal{O}$ in this case.



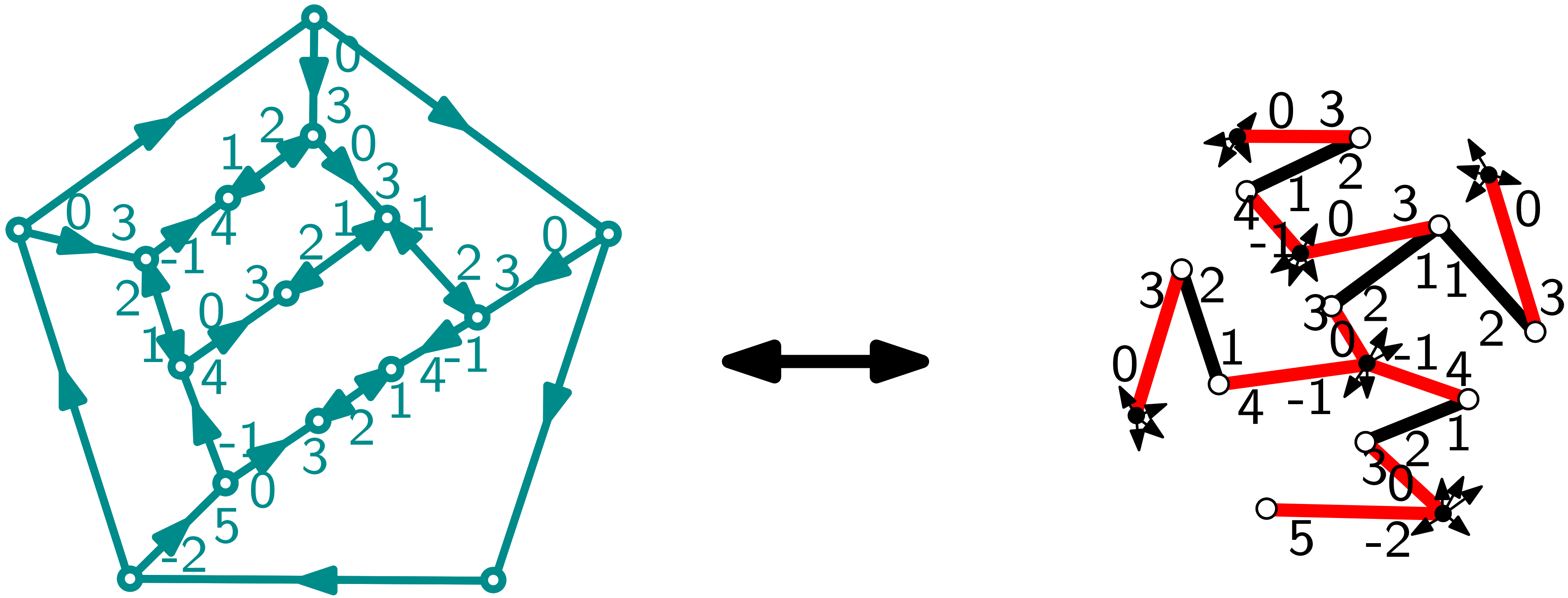

$i+j = d-2$



# Master bijection for weighted bi-orientation

**Theorem [B., Fusy]** There is a bijection between weighted bi-oriented plane maps in $\mathcal{O}$ and weighted mobiles. Moreover,

| | | |
|---|---|---|
| weight of internal edges | $\longleftrightarrow$ | weight of edges |
| ingoing weight of internal vertices | $\longleftrightarrow$ | weight of white vertices |
| degree of internal faces | $\longleftrightarrow$ | degree of black vertices |
| outgoing weight of internal faces | $\longleftrightarrow$ | weight of black vertices. |

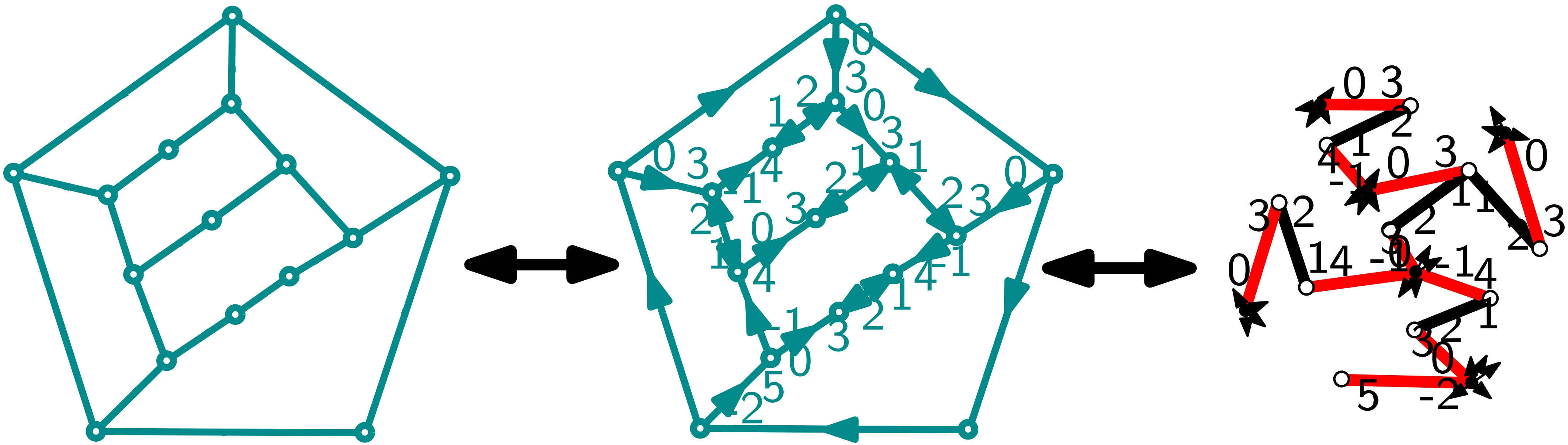


## Canonical orientations + Master bijection

**Thm [B., Fusy]:** There is a bijection between maps of girth $d$ and weighted mobiles such that

- edges have weight $d-2$,
- white vertices have weight $d$,
- black vertices have weight $d - \text{deg}$.

Moreover, faces of degree $d \longleftrightarrow$ black vertices of degree $d$.



## Counting

**Thm[B., Fusy]: $d$-angulations of girth $d$.**

The generating function $F_d(x) = \sum_{\substack{\text{d-angulation} \\ \text{of girth d}}} x^{\#\text{faces}}$ is given by

$$F_d(x) = W_{d-2} - \sum_{i=0}^{d-3} W_i W_{d-2-i}, \quad \text{and} \quad F'_d(x) = (1 + W_0)^d,$$

where $W_0, W_1, \ldots, W_{d-2}$ are defined by:

- $\forall j < d-2, \quad W_j = \sum_r \sum_{\substack{i_1,\ldots,i_r>0 \\ i_1+\cdots+i_r=j+2}} W_{i_1} \cdots W_{i_r},$
- $W_{d-2} = x(1 + W_0)^{d-1}.$

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- $W_{d-2} = x(1 + W_0)^{d-1}$.

**Example d=5:**

$W_0 = W_1^2 + W_2$

$W_1 = W_1^3 + 2W_1W_2 + W_3$

$W_2 = W_1^4 + 3W_1^2W_2 + 2W_1W_3 + W_2^2$

$W_3 = x(1 + W_0)^4$

## Counting

**Thm[B., Fusy]: Maps of girth $d$ (having outer degree $d$).**

The generating function $F_d(x_d, x_{d+1}, ..) = \sum_{\text{maps of girth d}} \prod_i x_i^{\#\text{faces of deg } i}$

is given by $F_d = W_{d-2} - \sum_{j=-2}^{d-3} W_j W_{d-2-j}$

where $\forall j \in [-2..d-3]$, $\quad W_j = \sum_r \sum_{\substack{i_1,\ldots,i_r>0 \\ i_1+\cdots+i_r=j+2}} W_{i_1} \cdots W_{i_r}$,

and $\forall j \in [d-2..d]$, $\quad W_j = [u^{j+1}] \sum_i x_i \, (u + uW_0 + W_{-1} + u^{-1})^{i-1}$.

Extends case $d = 1$ [Bouttier, Di Francesco, Guitter 02]

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**Corollaries:** If the set of admissible face degrees is finite, then

- Algebraic generating function.
- Asymptotic number of maps: $\sim c\, n^{-5/2} \rho^n$.

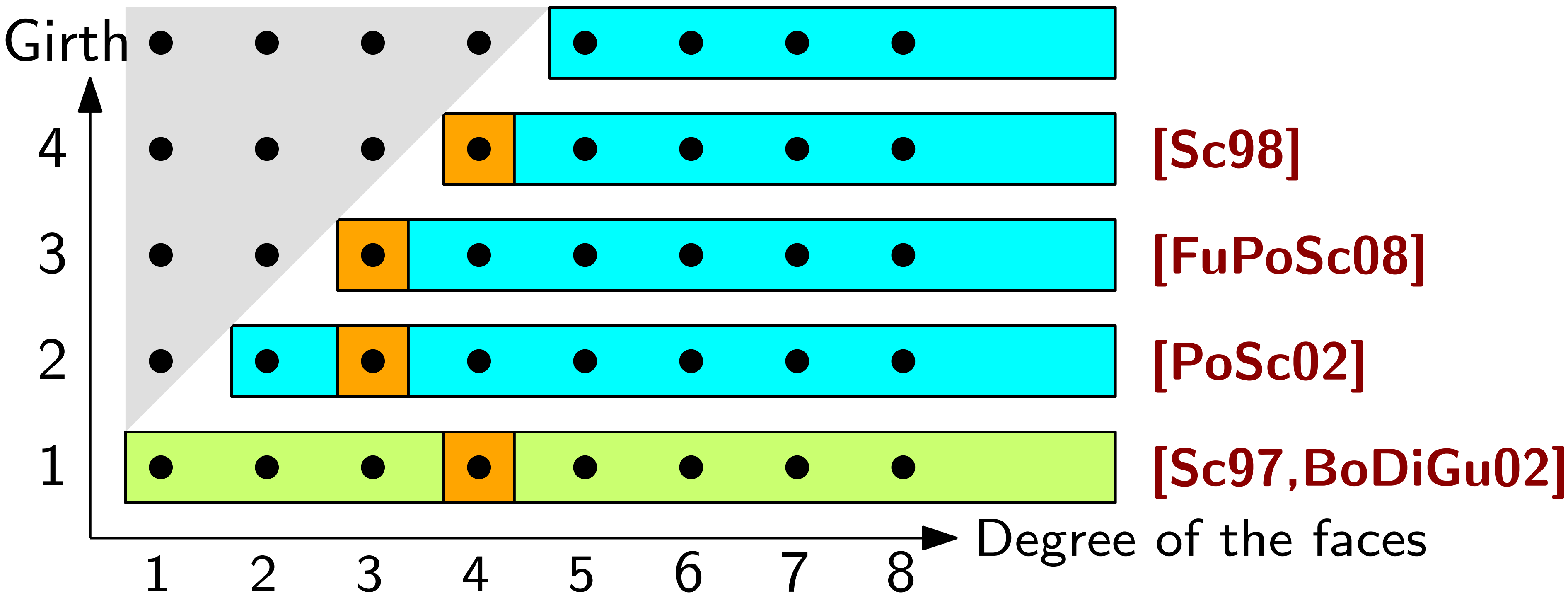
Extends case $d=1$ [Bender, Canfield 94]

# Additional results and questions

## Bijections for planar maps

Master bijection approach covers

- Classes of maps defined by girth constraint + degree constraints.



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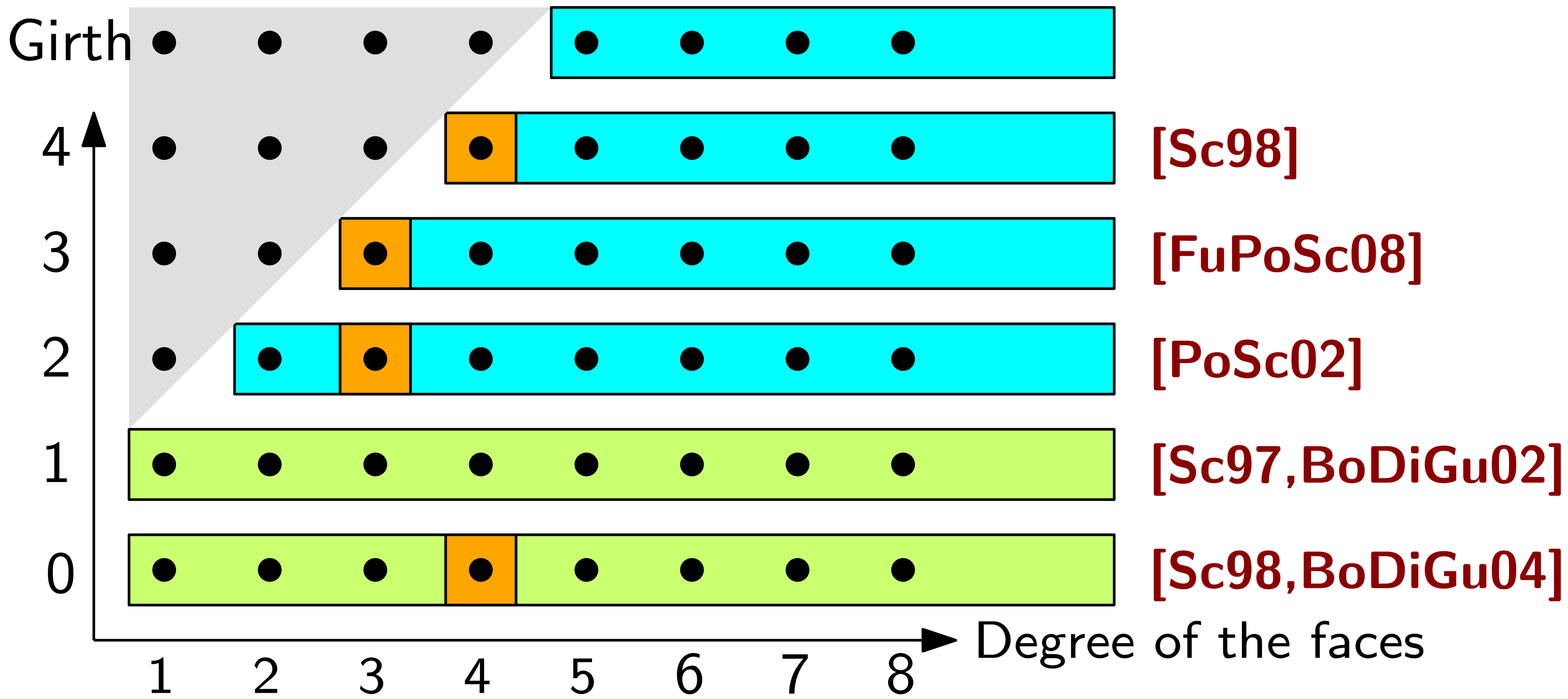

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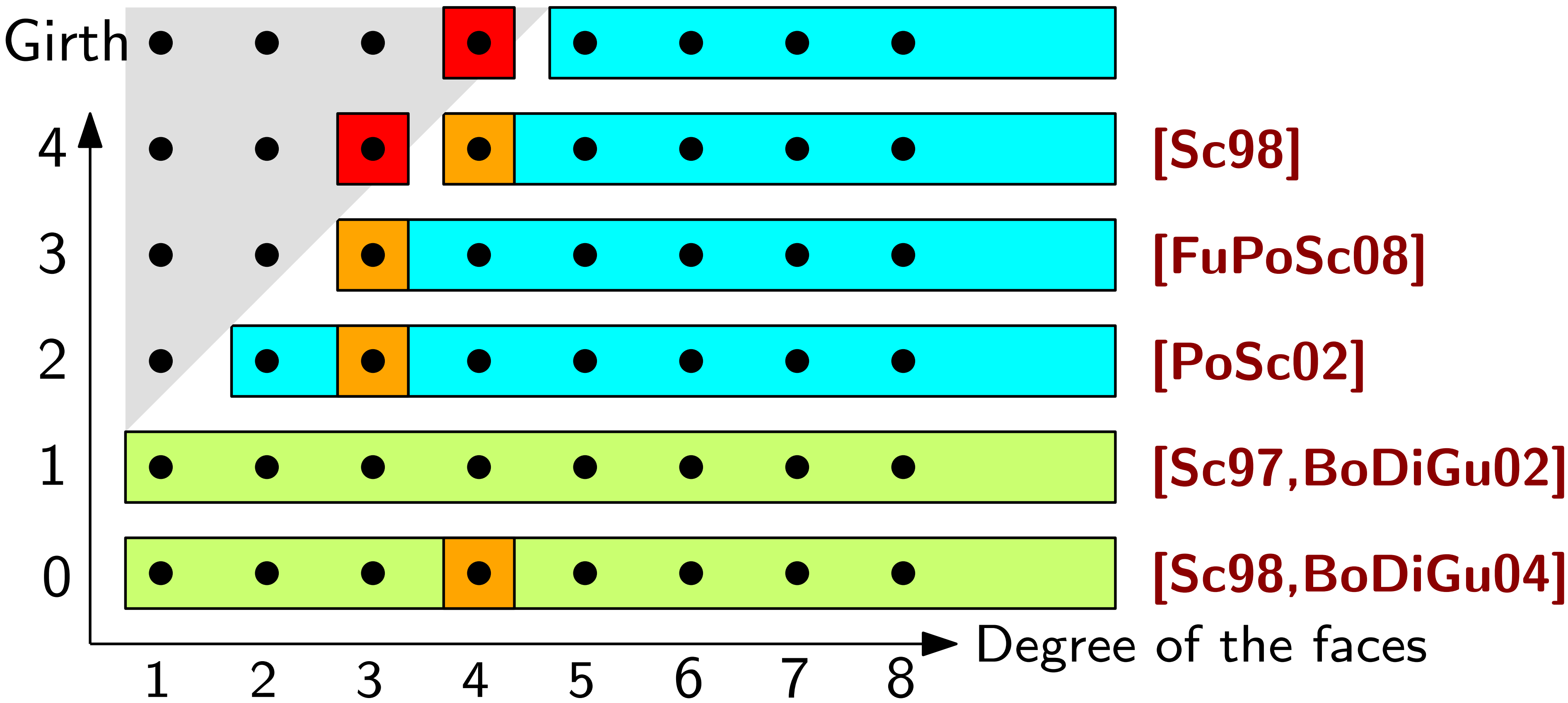

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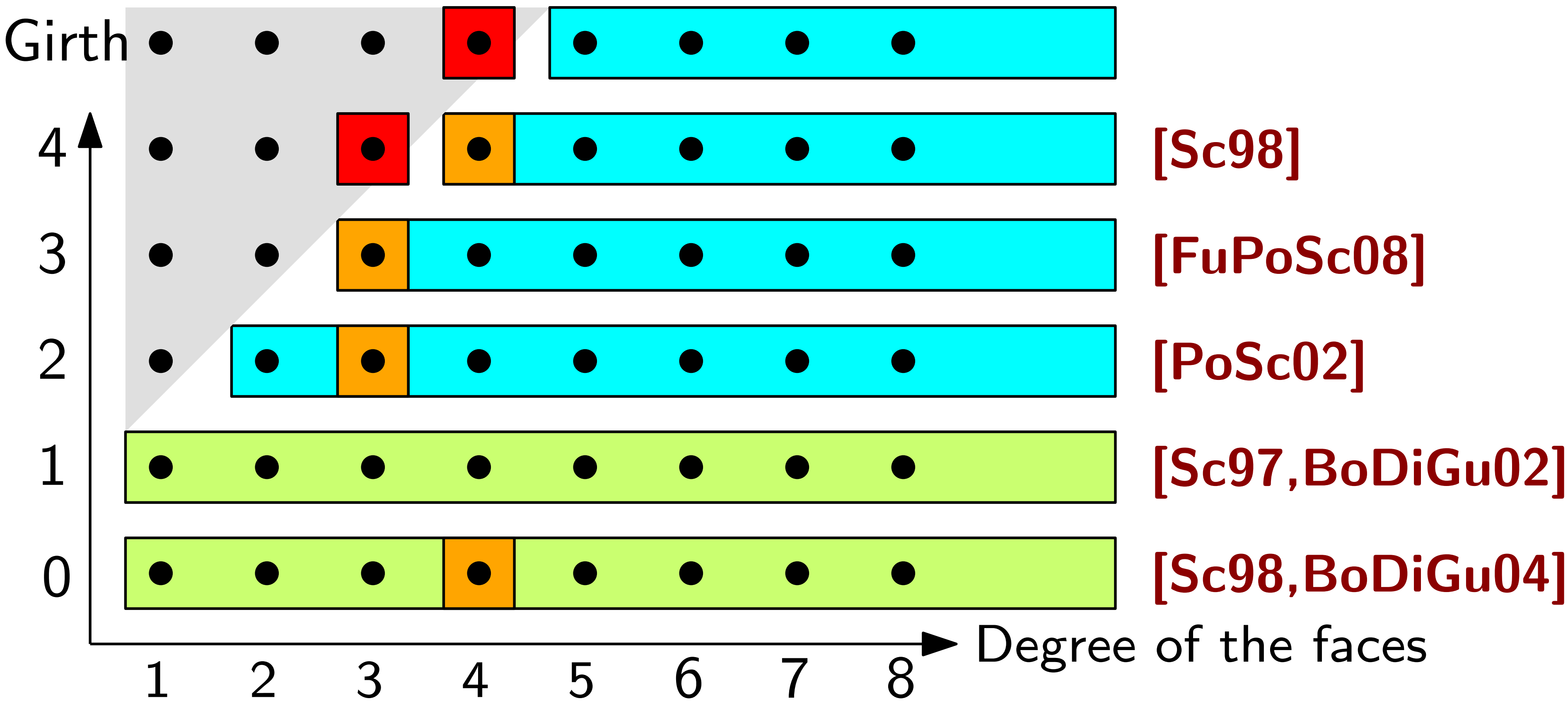
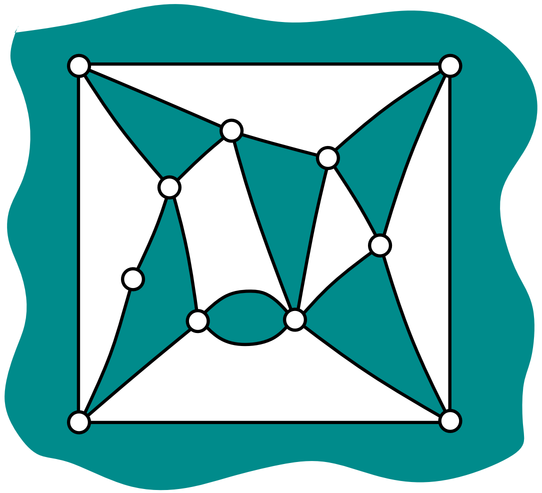
- Bipolar orientated maps[Fusy, Poulalhon, Schaeffer 09].



# Bijections for planar hypermaps

Master bijection approach extends to **hypermaps**.



# Bijections for planar hypermaps

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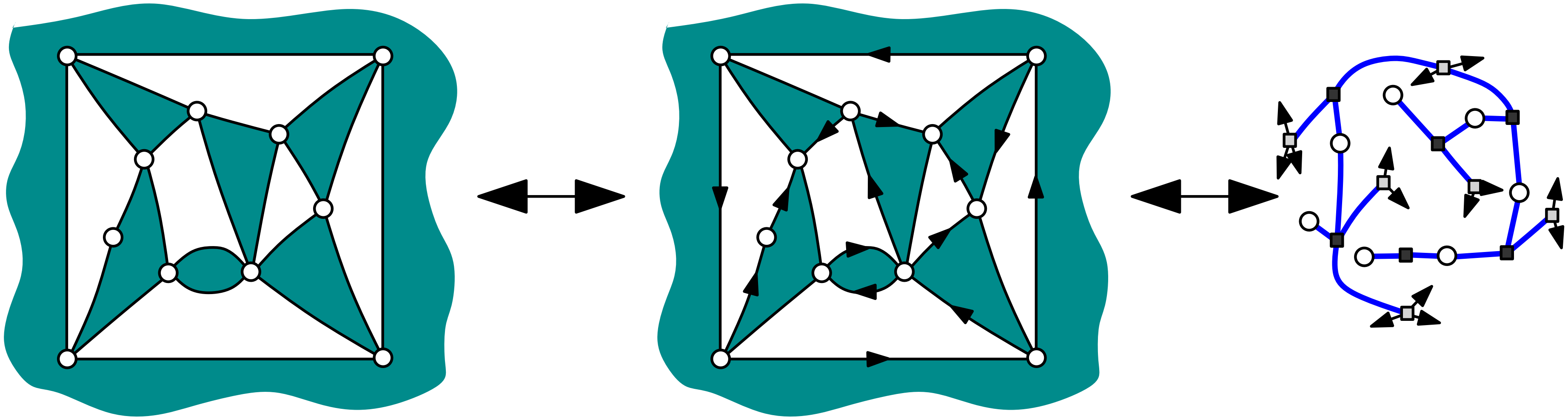
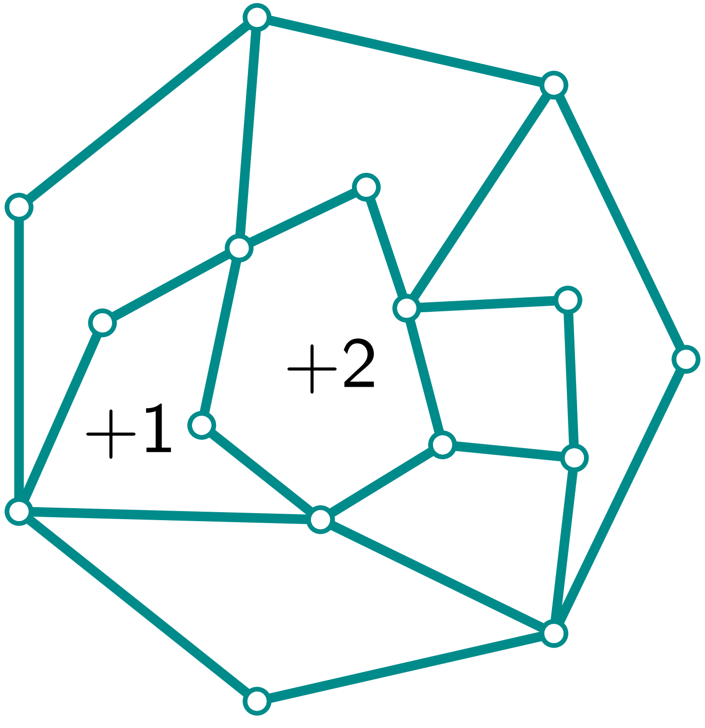


The master bijection generalizes bijections by
[Bousquet-Mélou, Schaeffer 00] (Constellations),
[Bousquet-Mélou, Schaeffer 02] (Ising model),
[Bouttier, Di Francesco, Guitter 04] (Distances).

## More fun with girth constraints?

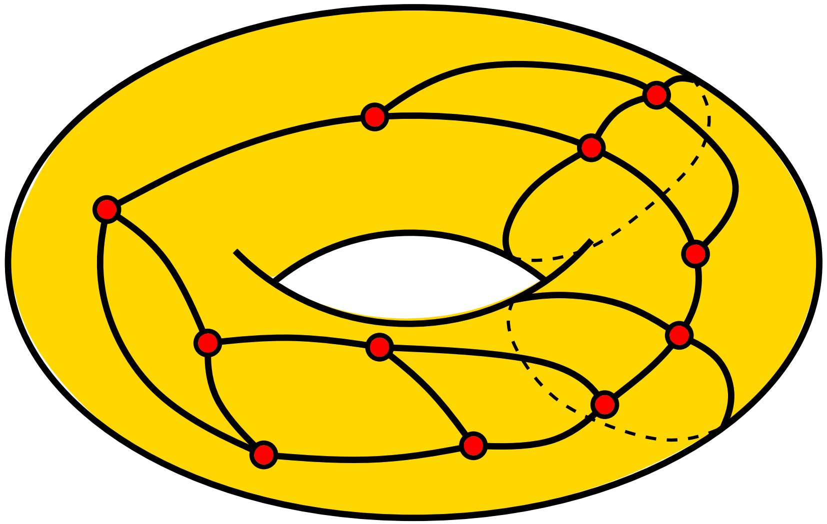
We know how to control more general girth constraints.
**Question:** Can we prove new probabilistic results on the cycle lengths in random maps?




$$|C| \geq d + \sum_{f \text{ inside } C} \sigma(f)$$

# Higher genus?

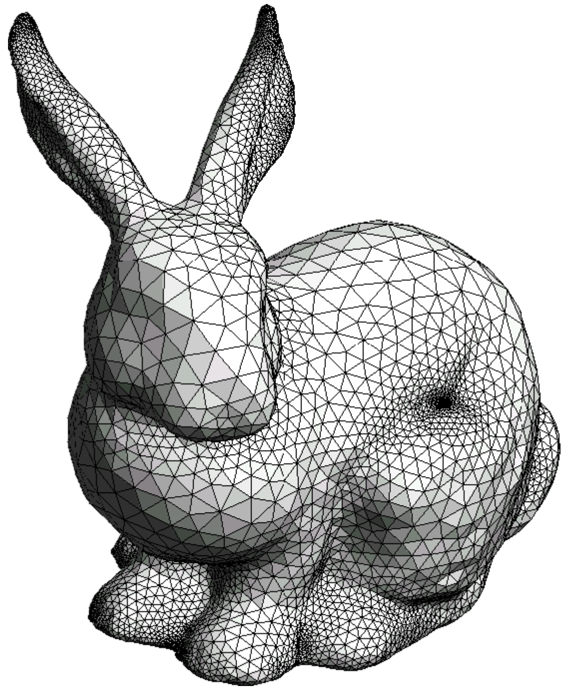
A version of the master bijection exists for maps on orientable surfaces [B., Chapuy 10].
**Question:** Can we find canonical orientations (hence bijections)?



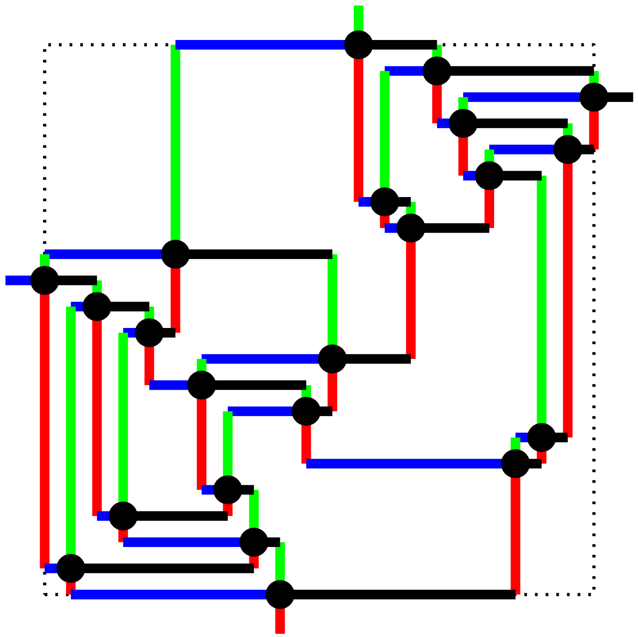
**Thanks.**

# 1. Algorithmic applications

1. Meshed surfaces.

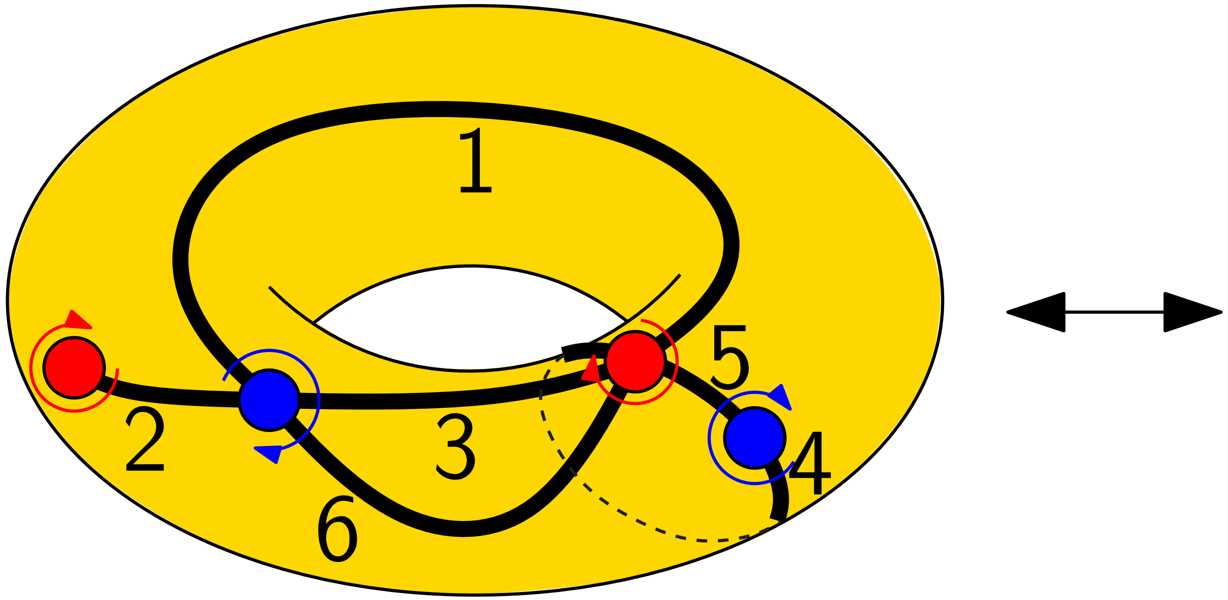


2. Graph drawing.



## 2. Relation with permutations.

Map with $n$ labelled edges $\longleftrightarrow$ pairs of permutation of $\{1, 2, \ldots, n\}$




$\pi = (1, 3, 6, 2)(4, 5)$

$\sigma = (1, 5, 6, 3, 4)(2)$

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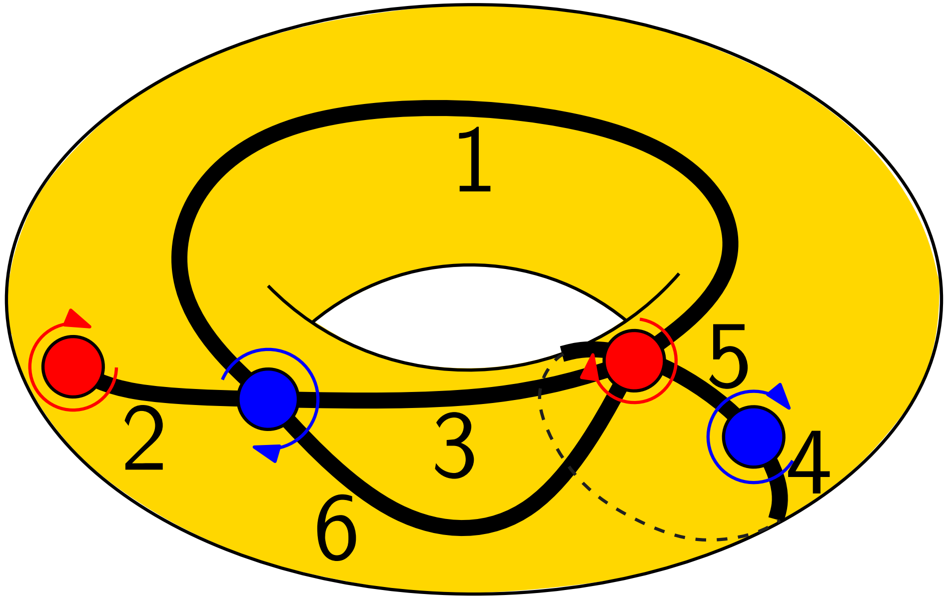


$\longleftrightarrow$

$\pi = (1, 3, 6, 2)(4, 5)$

$\sigma = (1, 5, 6, 3, 4)(2)$

Cycles of $\pi$ $\longleftrightarrow$ blue vertices

Cycles of $\sigma$ $\longleftrightarrow$ red vertices

Cycles of $\pi\sigma$ $\longleftrightarrow$ faces