Denominator Vectors and Compatibility Degrees in Cluster Algebras of Finite Type

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Theorem

Denominator vectors = vectors of compatibility degrees

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Denominator vectors = vectors of compatibility degrees (Cluster algebras)

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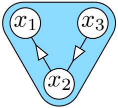
Denominator vectors = root function decompositions (Subword complexes)

Cluster algebras

Cluster algebras were introduced by Fomin and Zelevinsky in 2002. They are commutative algebras whose generators and relations are constructed in a recursive manner.

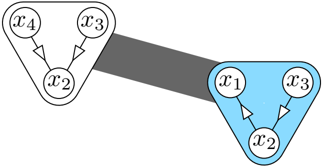
- Lie Theory: canonical bases / total positivity
- Representation theory
- Poisson geometry
- Combinatorics and discrete geometry
- Algebraic geometry

Cluster algebras



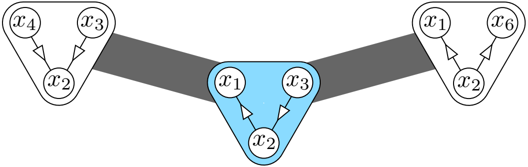
Cluster algebras

$$x_4 = \frac{1 + x_2}{x_1}$$



Cluster algebras

$$x_4 = \frac{1 + x_2}{x_1} \qquad x_6 = \frac{1 + x_2}{x_3}$$



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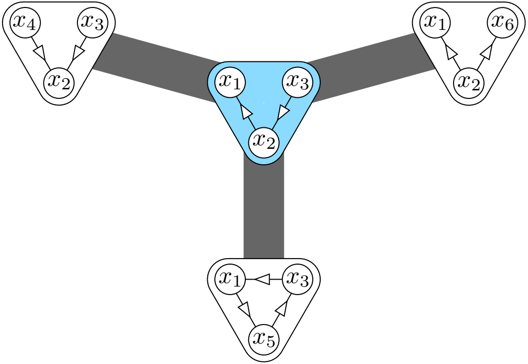
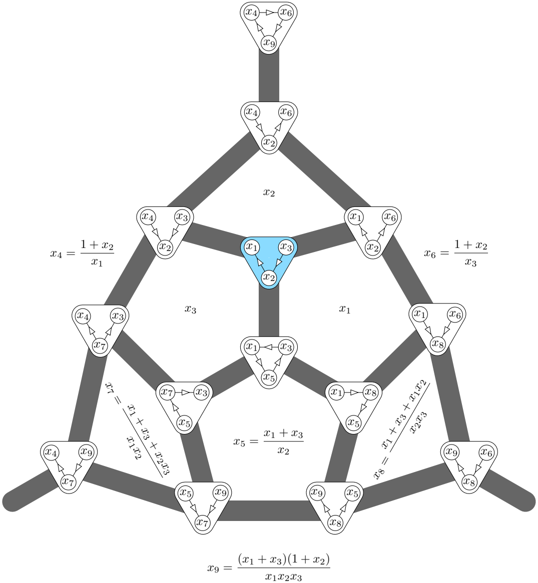


$$x_5 = \frac{x_1 + x_3}{x_2}$$

Cluster algebras



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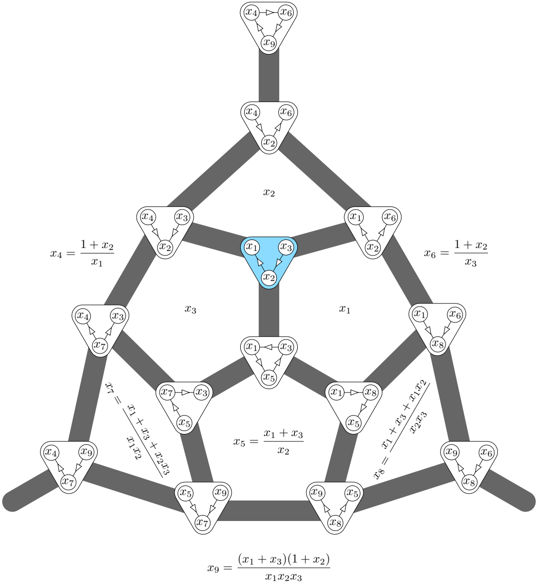
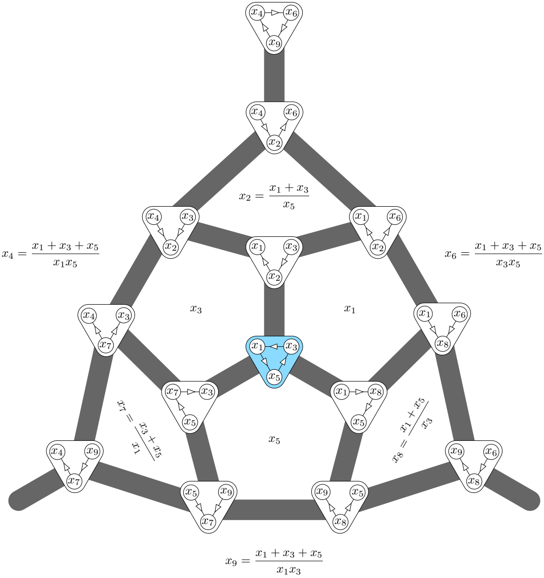


cluster: n-tuples obtained after mutations

cluster variables: elements of the clusters

cluster algebra $\mathcal{A}_Q$ **:** subalgebra of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by the cluster variables

Cluster algebras



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Cluster algebras

Theorem (Fomin & Zelevinsky '03)

- *Each cluster variable of* $\mathcal{A}_Q$ *is a Laurent polynomial with integer coefficients*

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Cluster algebras

Theorem (Fomin & Zelevinsky '03)

- *Each cluster variable of $\mathcal{A}_Q$ is a Laurent polynomial with integer coefficients*
- *Finite number of cluster variables $\Leftrightarrow$ Q is mutation equivalent to a Dynkin quiver*
- *If Q is a Dynkin quiver of type Δ, then the non-initial cluster variables are in bijection with the positive roots of the root system associated to Δ:*

$$\alpha = d_1\alpha_1 + \cdots + d_n\alpha_n \quad \Leftrightarrow \quad \textit{cluster variable } X_\alpha = \frac{F(x_1,\ldots,x_n)}{x_1^{d_1}\ldots x_n^{d_n}}$$

Denominator vectors

Theorem (Laurent phenomenon, Fomin & Zelevinsky '02)

Any cluster variable y viewed as a rational function in the variables $X = \{x_1, \ldots, x_n\}$ of any given cluster is a Laurent polynomial

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Denominator vector:

$$\mathbf{d}(X, y) = (d_1, \ldots, d_n)$$

In this talk

Three methods to compute denominator vectors in cluster algebras of finite type using:

- cluster variables
- almost positive roots
- positions in a word

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Our initial motivation: construct polytopal realizations of a subfamily of Knutson-Miller's subword complexes, which are simplicial complexes containing all cluster complexes of finite type.

Notation

$\mathcal{A}(W)$	cluster algebra of finite type W
W	Weyl group
Φ	root system
$\alpha_1, \ldots, \alpha_n$	simple roots
$\Phi_{\geq -1}$	almost positive roots

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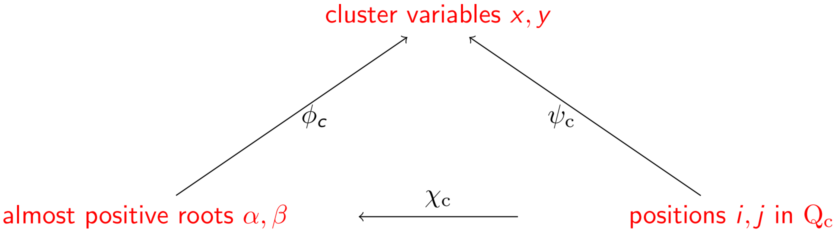
word $Q_c = cw_\circ(c)$

where $w_\circ(c) = (w_1, \ldots, w_N)$ first lexicographically subword of c^∞ which is a reduced expression of $w_\circ$

Three bijections



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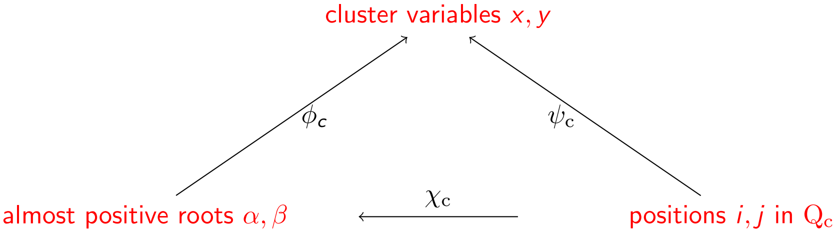

ϕ_c: Parametrizes cluster variables with respect to their denominator vectors with respect to a cluster seed associated to c.

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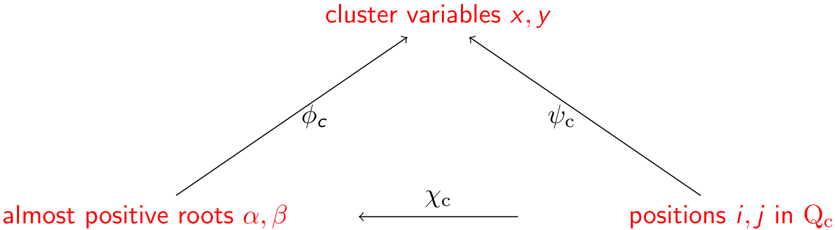

ϕ_c: Parametrizes cluster variables with respect to their denominator vectors with respect to a cluster seed associated to c.

χ_{c} : defined by $\psi_c(c_i) = -\alpha_i$ and $\psi(w_i) = w_1 \dots w_{i-1}(\alpha_{w_i})$

Three bijections: clusters

Theorem (C.–Labbé–Stump 2013)

A subset I of positions in Q_c forms a c-cluster in Q_c if and only if the subword of Q_c formed by the complement of I is a reduced expression for $w_\circ$.

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A subset I of positions in Q_c forms a c-cluster in Q_c if and only if the subword of Q_c formed by the complement of I is a reduced expression for $w_\circ$.

A similar result for

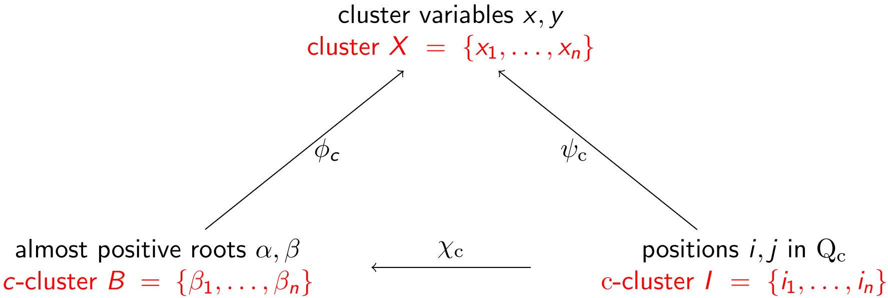
- bipartite Coxeter elements is due to Brady & Watt (2008)
- crystallographic types is due to Igusa & Schiffler (2010)

Three bijections: clusters

Clusters can be read in these three different contexts:

Cluster variables	clusters: n-tuples obtained by mutations
Almost positive roots	c-clusters: n-tuples of pairwise compatible roots
Positions in Q_c	c-clusters: complements of reduced expressions of $w_\circ$

Three bijections: clusters



Three notions of compatibility degrees

Compatibility degree on cluster variables:

$$\begin{array}{ccc} \{\text{var}\} \times \{\text{var}\} & \longrightarrow & \mathbb{Z} \\ (x, y) & \longrightarrow & d(x, y) \end{array}$$

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Lemma

$d(x, y)$ is independent of the choice of cluster X.

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c-**Compatibility degree on almost positive roots:**

$$\begin{array}{ccc} \Phi_{\geq -1} \times \Phi_{\geq -1} & \longrightarrow & \mathbb{Z} \\ (\alpha, \beta) & \longrightarrow & (\alpha \parallel_c \beta) \end{array}$$

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uniquely characterized by the following two properties:

$$(-\alpha_i \parallel_c \beta) = b_i, \qquad \text{for } \beta = \sum b_i \alpha_i \in \Phi_{\geq -1},$$

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Three notions of compatibility degrees

c-**Compatibility degree on positions in** $Q_c = (q_1, \ldots, q_m)$**:**

$$\begin{array}{ccc} [m] \times [m] & \longrightarrow & \mathbb{Z} \\ (i,j) & \longrightarrow & \{i \parallel_c j\} \end{array}$$

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given by

$$\{i \parallel_c j\} = \begin{cases} \rho_I(i,j) & \text{if } j > i, \\ -\rho_I(i,j) & \text{if } j \leq i. \end{cases}$$

for any c-cluster I in Q_c containing the position i.

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Lemma

$\{i \,\|_c\, j\}$ *is independent of the choice of c-cluster I.*

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$$\mathrm{r}(I,j) = \sum_{i \in I} \rho_I(i,j)\mathrm{r}(I,i).$$

Main results

Theorem (C.–Pilaud 2013)

The three notions of compatibility degrees coincide

cluster variables x, y
cluster $X = \{x_1, \ldots, x_n\}$
compatibility degree $d(x, y)$

ϕ_c

ψ_{c}

almost positive roots α, β
c-cluster $B = \{\beta_1, \ldots, \beta_n\}$
c-compatibility degree $(\alpha \parallel_c \beta)$

χ_{c}

positions i, j in $\mathrm{Q_c}$
c-cluster $I = \{i_1, \ldots, i_n\}$
c-compatibility degree $\{i \parallel_c j\}$

Main results

Corollary (C.–Pilaud 2013)

The denominator vector of an almost positive root β with respect to a c-cluster $B = \{\beta_1, \ldots, \beta_n\}$ is the compatibility degree vector

$$\mathbf{d}_c(B, \beta) = \big((\beta_1 \parallel_c \beta), \ldots, (\beta_n \parallel_c \beta)\big)$$

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Corollary (C.–Pilaud 2013)

The denominator vector of a position j in Q_c with respect to a c-cluster $I = \{i_1, \ldots, i_n\}$ is the coefficient vector

$$\mathbf{d}_c(I, j) = \big(\{i_1 \parallel_c j\}, \ldots, \{i_n \parallel_c j\}\big)$$

Main results

cluster variables x, y
cluster $X = \{x_1, \ldots, x_n\}$
compatibility degree $d(x, y)$
denominator vector $\mathbf{d}(X, y)$

ϕ_c (almost positive roots → cluster variables)

ψ_{c} (positions → cluster variables)

almost positive roots α, β
c-cluster $B = \{\beta_1, \ldots, \beta_n\}$
c-compatibility degree $(\alpha \,\|_c\, \beta)$
compatibility vector $\mathbf{d}_c(B, \beta)$

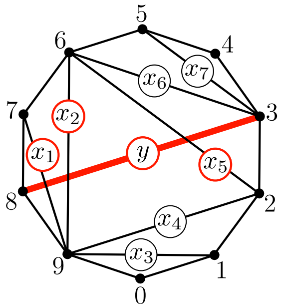
χ_{c} (positions → almost positive roots)

positions i, j in Q_{c}
c-cluster $I = \{i_1, \ldots, i_n\}$
c-compatibility degree $\{i \,\|_c\, j\}$
coefficient vector $\mathbf{d}_{\mathrm{c}}(I, j)$

Examples: classical types

TYPE A

$$y = \frac{\cdots}{x_1 x_2 x_5}$$



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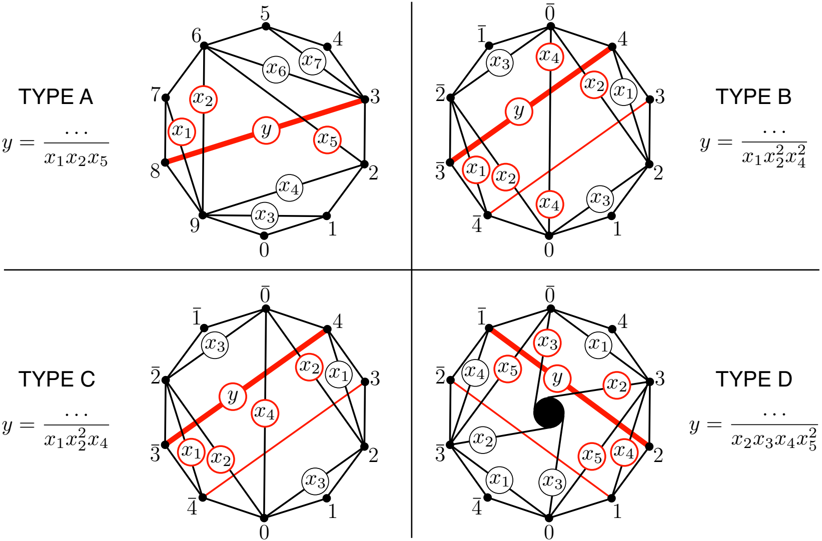

Thank you!