Spanning forests in regular planar maps

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Planar map = connected graph + embedding of this graph in the plane, considered up to continuous deformation.
**Planar Maps & Definition**

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Regular map = All vertices have the same degree. (In this talk: 4)
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We root the map at an outer corner.
Planar maps & Definition

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Regular map = All vertices have the same degree. (In this talk: 4)

We root the map at an outer corner.
Forest maps: Definition

Spanning forest of $M = \text{graph } F$ such that:
- $V(F) = V(M)$
- $E(F) \leq E(M)$ has no cycle.

Forested map $(M,F) = \text{Rooted map } M \text{ with a spanning forest } F$.

Some other structures: Spanning trees, colourings, percolation, Ising/Potts model, self-avoiding walks... [Tutte, Mullin, Kazakov, Borot, Bouchier, Guittet, Sportiello, Eynard, Duplantier, Bousquet-Mélou, Schaeffer, Bernardi, Angel...].
**FORESTED MAPS & DEFINITION**

Spanning forest of $M = \text{graph } F$ such that:
- $V(F) = V(M)$
- $E(F) \subseteq E(M)$ has no cycle.

**Forest map** $(M, F) = \text{Rooted map } M \text{ with a spanning forest } F$.

$$F(\beta, \mu) = \sum_{(M, F) \text{ 4-valent forested map}} \beta \mu$$
Forested Maps & Definition

Spanning forest of $M$ = graph $F$ such that:
- $V(F) = V(M)$
- $E(F) \leq E(M)$ has no cycle.

Forested map $(M, F) = $ Rooted map $M$ with a spanning forest $F$.

$$F_4(q, m) = \sum_{\text{forested map}} \binom{\text{# faces}}{\text{# components} - 1}$$
SPECIAL VALUES OF $\mu$

$$F(\mathfrak{g}, \mu) = \sum_{(M,F) \text{4-valent forested map}} \# \text{faces} \# \text{components} - 1$$

* $\mu = 1$: spanning forests

* $\mu = 0$: spanning trees  [Mullin, 1967]

* $\mu = -1$: root-connected acyclic orientations on (dual) quadrangulations.  [Las Vergnas, 1984]
**GENERIC VALUES OF** $\mu$

1) Connected subgraphs on quadrangulations (counted by cycles)

2) Tutte polynomial $T_\mu(\mu+1, 1)$

3) Sandpile model $\begin{cases} \text{Merino Lopez,} \\ \text{Cori, Le Borgne} \end{cases}$

4) Limit $q \to 0$ of the Potts model

3)

$$F(g, \mu) = \sum_{\text{quadrangulation}} g \# \text{vertices} (\mu + 1)^{\text{level}(C)}$$

natural domain $\mu \in [-1, +\infty)$
**Generic Values of \( \mu \)**

1) Connected subgraphs on quadrangulations (counted by cycles)

2) Tutte polynomial \( T_{\mu}(\mu + 1, 1) \)

3) Sandpile model \([\text{Merino Lopez, Cori, Le Borgne}]\) \( \mu \in [-1, +\infty) \)

4) Limit \( q \to 0 \) of the Potts model

3)

\[
F(g, \mu) = \sum_{\text{quadrangulation with recurrent configuration } C} g^{|\text{vertices}|} (\mu + 1)^{\text{level}(C)}
\]
**Generic Values of** \( \mu \)

1) Connected subgraphs on quadrangulations (counted by cycles)
2) Tutte polynomial \( T_\mu (\mu + 1, 1) \)
3) Sandpile model \[ \text{Merino Lopez, Cori, Le Borgne} \]
   \( \mu \in [-1, +\infty) \)
4) Limit \( q \to 0 \) of the Potts model

\[
F(\gamma, \nu) = \sum_{\text{quadrangulation with recurrent configuration } C} \gamma \# \text{vertices} (\mu + 1)^{\text{level}(C)}
\]
OBJECTIVES

EXACT ENUMERATION

→ Generating function $F$ of forested maps.
→ Nature of $F$:
  D-finite?  D-algebraic?

ASYMPTOTIC ENUMERATION

→ For $u \geq -1$, asymptotic behaviour
  of $f_n(u) = [z^n] F(z,u)$
OBJECTIVES

EXACT ENUMERATION

→ Generating function $F$ of forested maps.

→ Nature of $F$:
  D-finite?
  i.e. satisfies a linear differential equation.
  (non-trivial equations with coefficients in $\mathbb{Q}(z,y)$)
  D-algebraic?
  i.e. satisfies a polynomial differential equation.

ASYMPTOTIC ENUMERATION

→ For $\mu \geq -1$, asymptotic behaviour
  of $f_n(\mu) = [z^n] F(z,\mu)$
EXACT ENUMERATION
From Forested To General Maps
From Forested To General Maps
FROM FORESTED TO GENERAL MAPS

tree contraction
FROM FORESTED TO GENERAL MAPS

tree contraction
From Forested to General Maps

tree contraction
FROM FORESTED TO GENERAL MAPS

tree contraction

tree extraction
FROM FORESTED TO GENERAL MAPS

tree contraction

tree extraction
FROM FORESTED TO GENERAL MAPS

- Tree contraction
- Tree extraction
FROM FORESTED TO GENERAL MAPS

tree contraction

tree extraction
FROM FORESTED TO GENERAL MAPS

Tree contraction

BIJECTION!

Tree extraction
Translation into Generating Functions

\[ M(\gamma, \mu; g_1, g_2, g_3, \ldots; h_1, h_2, h_3, \ldots) = \]

Generating function of rooted maps with a weight:

- \( \gamma \) per face,
- \( \mu g_k \) per non-root vertex of degree \( k \),
- \( h_k \) if the root vertex has degree \( k \).

\[ \mathcal{F}(\gamma, \mu) = M(\gamma, \mu; t_4, t_2, t_3, \ldots; t_4^c, t_2^c, t_3^c, \ldots) \]

\( t_k = \# 4\text{-valent leaf-rooted trees with } k \text{ leaves} \)

\( t_k^c = \# 4\text{-valent corner-rooted trees with } k \text{ leaves} \)
Generating Function For General Maps

\[
M (r, \mu; g_1, g_2, g_3, \ldots; h_1, h_2, h_3, \ldots) =
\]
Generating function of rooted maps with a weight:
\begin{itemize}
  \item \( r \) per face,
  \item \( \mu \) per non-root vertex of degree \( k \),
  \item \( h_k \) if the root vertex has degree \( k \).
\end{itemize}

This generating function is known of [Bouttier - Guitter, 2012]

\((M' \text{ is even nicer})\)

Notation: \( X' = \frac{3X}{\partial X} \)
The Generating Function of Forested Maps

**Theorem**

There exists a unique series $R$ in $x$ with constant term 0 and coefficients in $\mathbb{Q}[x]$ such that

$$R = x + \sum_{i \geq 2} \frac{(3i-3)!}{(i-1)!^2 i!} R^i$$

Then:

$$F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)! i!^2} R^i$$
The Generating Function of Forested Maps

**Theorem**

There exists a unique series \( R \) in \( z \) with constant term 0 and coefficients in \( \mathbb{Q}[w] \) such that

\[
R = z_8 + w \sum_{i \geq 2} \frac{(3i-3)!}{(i-1)!^2 i!} R^i
\]

Then:

\[
F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)! \ i!^2} R^i
\]

For \( w = 0, \) [Mullin]

\[
R = z_8 \quad \text{and} \quad F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)! \ i!^2} z_8^i \quad \text{D-finite.}
\]
The Generating Function of Forested Maps

**Theorem**

There exists a unique series \( R \) in \( zy \) with constant term 0 and coefficients in \( \mathbb{Q}[u] \) such that

\[
R = zy + u \phi(R)
\]

Then:

\[
F' = \Theta(R)
\]

where

\[
\phi(x) = \sum_{i \geq 2} \frac{(3i-3)!}{(i-1)!i!} x^i, \quad \Theta(x) = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)!i!^2} x^i.
\]
A Differential Equation For $F$

\[ R = z + \omega \phi(R) \quad F' = \Theta(R) \]

\text{Prop} \quad F \text{ is D-algebraic.}

(Fundamental reason: $\phi$ and $\Theta$ are D-finite.)
A Differential Equation For $F$

$$R = y + u \phi(R) \quad F' = \Theta(R)$$

Prop $F$ is $D$-algebraic.

(Fundamental reason: $\phi$ and $\Theta$ are $D$-finite.)

Can a differential equation for $F$ be explicitly computed?
A DIFFERENTIAL EQUATION FOR F

\[ R = \gamma + \omega \phi(R) \quad F' = \Theta(R) \]

Prop F is D-algebraic.

(Fundamental reason: \( \phi \) and \( \Theta \) are D-finite.)

Can a differential equation for F be explicitly computed?

YES!
A Differential Equation For $F$

\[ 9F'^2F'F''F'''F''''F'''''F'''F'' - 12(21\eta - 1)F'F''F''''F'''F''''F'''F'' - 48(24\eta - 4)F'F''F'''F''''F'''F'''''F''''F''' + 864F'^2F''F'''F''''F'''F'''''F'''F'' - 36(27\eta - 2)F'F''F'''F''''F'''F'''''F''''F''' + 4(27\eta - 4)(15\eta - 1)F''F'''F''''F'''F'''''F''''F''' + 1728F'^2F''F''F'''F''''F'''F'''''F''''F''' - 288(21\eta - 2)F'^2F''F'''F''''F'''F'''''F''''F''' + 10368F'F''F'''F''''F'''F'''''F''''F''' + 16(27\eta - 4)(21\eta - 1)F''F'''F''''F'''F'''''F''''F''' + 2304F'^2F''F'''F''''F'''F'''''F''''F''' - 288(31\eta - 4)F'^2F'''F''''F'''F'''''F''''F''' + 64(6\eta + 162\eta^2 + 33\eta - 1)F''F'''F''''F'''F'''''F'''' + 2804F'F''F'''F''''F'''F'''''F''''F''' - 2804(6\eta - 4)F'^2F'''F''''F'''F'''''F''''F''' - 152(8\eta - 54\eta^2 + 25\eta - 1)F''F'''F''''F'''F'''''F''''F''' + 768(2\eta + 18\eta^2 - 7)F''F'''F''''F'''F'''''F''''F''' + 2304F'^2F''F'''F''''F'''F'''''F''''F''' - 3072(3\eta - 1)F'^2F''F'''F''''F'''F'''''F''''F''' - 152(24\eta - 27\eta^2 + 55\eta - 2)F''F'''F''''F'''F'''''F'''' - 1536(21\eta - 4)F''F'''F''''F'''F'''''F'''' - 768(12\eta + 81\eta^2 + 24\eta - 1)F''F'''F''''F'''F'''''F'''' + 1536(9\eta + 2)F''F'''F''''F'''F'''''F'''' - 512(39\eta + 81\eta^2 + 54\eta - 2)F''F'''F''''F'''F'''''F'''' + 36864F'F''F'''F''''F'''F'''''F'''' + 1024(12\eta - 162\eta^2 + 33\eta - 1)F''F'''F''''F'''F'''''F'''' - 1024(36\eta + 27\eta - 4)F''F'''F''''F'''F'''''F'''' - 24576\eta = 0. \]

Differential equation of order 2 in $F'$ and degree 7. (but not in $F$)
A Differential Equation For \( F \)

\[
R = z + \alpha \phi(R) \quad F' = \Theta(R)
\]

\( \overset{\text{Prop}}{\text{Prop}} \quad F \text{ is } D\text{-algebraic.} \)

(Fundamental reason: \( \phi \) and \( \Theta \) are \( D\text{-finite.} \))

- A differential equation for \( F \) can be explicitly computed.

- cf Bernardi - Bousquet-Mélou's result:

  The Potts generating function of planar maps is \( D\text{-algebraic.} \)
  (established in a more painful way.)
Asymptotic Results
Fix $\mu$,

$\rho \mu = \text{radius of convergence of } F(z, \mu) = \sum_{n} f_n(\mu) z^n$.

\[ \rho_0 = \frac{1}{2\pi} \]

\[
\begin{cases}
\rho \mu = \mathbf{z} \mu - \mu \phi(\mathbf{z} \mu) \\
\phi'(\mathbf{z} \mu) = \frac{1}{\mu} 
\end{cases} \quad (\mu > 0)
RADIUS OF CONVERGENCE

Fix $\mu$ in $[-1, +\infty)$,

$\rho_\mu = \text{radius of convergence of } F(z, \mu) = \sum \limits_n f_n(\mu) z^n$.

$\rho_\mu$ is affine on $[-1, 0]$!

$\rho_0 = \frac{1}{27}$

$\rho_1$

\[
\begin{cases}
\rho_\mu = z_\mu - \mu \phi(z_\mu) \\
\phi'(z_\mu) = \frac{1}{\mu}
\end{cases} \quad (\mu > 0)
\]
Fix $\mu$ in $[-1, +\infty)$, 
$\rho_\mu = \text{radius of convergence of } F(z, \mu) = \sum_{n} f_n(\mu) z^n$.  

$\rho_{-1} = \frac{\sqrt{3}}{12\pi}$

$\rho_\mu$ is affine on $[-1, 0]$!

$\rho_\mu = \frac{1}{2\pi} (1+\mu) - \frac{\sqrt{3}}{12\pi} \mu$.  

$\rho_0 = \frac{1}{2\pi}$

$\rho_1$

\[
\begin{cases}
\rho_\mu = z\mu - \mu \Phi(z\mu) \\
\Phi'(z\mu) = \frac{1}{\mu} 
\end{cases}
\quad (\mu > 0)
\]
Fix \( \mu \) in \([-1, +\infty)\),
\[ p_\mu = \text{radius of convergence of } F(z, \mu) = \sum_{n} f_n(\mu) z^n. \]

\[ p_{-1} = \frac{\sqrt{3}}{12\pi} \]

\[ p_0 = \frac{1}{27} \]

\[ p_1 \]

\( p_\mu \) is affine on \([-1, 0]\)!
\[ p_\mu = \frac{1}{27}(1+\mu) - \frac{\sqrt{3}}{12\pi} \mu. \]

\[ \begin{cases} 
q_\mu = z_\mu - \mu \phi(z_\mu) \\
\phi'(z_\mu) = \frac{1}{\mu} 
\end{cases} \quad (\mu > 0) \]

\( p_{-1} \) is transcendental:
\[ F(z, -1) \] is not D-finite.
Phase Transition At $0$

\[ f_n(\mu) = [z^\mu] F(z, \mu) \]

$-1 \leq \mu < 0$

\[ f_n(\mu) \sim \frac{c_\mu \mu^{-n}}{n^3 \ln^2 n} \]

New "Universality class" for maps

$\mu = 0$

\[ f_n(\mu) \sim \frac{c_\mu \mu^{-n}}{n^3} \]

maps with a spanning tree

$0 < \mu$

\[ f_n(\mu) \sim \frac{c_\mu \mu^{-n}}{n^{5/2}} \]

standard
Phase Transition At 0

\[ f_n(m) = \left[ z^n \right] F(z, m) \]

\(-1 \leq m < 0\)

\[ f_n(m) \sim \frac{c m n^{-m}}{n^3 \ln^2 n} \]

"Universality class" for maps

\[ m = 0 \]

\[ f_n(m) \sim \frac{c m n^{-m}}{n^3} \]

maps with a spanning tree

\[ 0 < m \]

\[ f_n(m) \sim \frac{c m n^{-m}}{n^{5/2}} \]

standard

Cor

For \( m \in [-1, 0) \), \( F(z, m) \) is not D-finite.
**IDEA OF THE PROOF**

Singularity analysis [Flajolet - Odlyzko]

A link between the singular behaviour of $F(g, \mu)$ near $\mu$ and the asymptotic behaviour of $f_m(\mu)$.
IDEA OF THE PROOF

Singularity analysis [Flajolet - Odlyzko]

A link between the singular behaviour of $F(g, \mu)$ near $\mu$ and the asymptotic behaviour of $f_m(\mu)$.

$$\begin{cases} R = g + \mu \phi(R) \\ R(0) = 0 \end{cases}$$

radius of convergence of $\phi = \frac{1}{2\pi}$
**IDEA OF THE PROOF**

Singularity analysis [Flajolet - Odlyzko]

A link between the singular behaviour of $F(z, \mu)$ near $\rho_\mu$ and the asymptotic behaviour of $f_m(\mu)$.

\[
\begin{cases}
R = z_\mu + \mu \phi(R) \\
R(0) = 0
\end{cases}
\]

radius of convergence of $\phi = \frac{1}{2\pi}$

\[
\mu > 0
\]

singularity in $(\rho_\mu - z_\mu)^{1/2}$
IDEA OF THE PROOF

Singularity analysis [Flajolet - Odlyzko]

A link between the singular behaviour of $F(x, \mu)$ near $\mu$ and the asymptotic behaviour of $f_m(\mu)$.

\[
\begin{align*}
R &= x + \mu \phi(R) \\
R(0) &= 0
\end{align*}
\]

radius of convergence of $\phi = \frac{1}{2\pi}

\mu < 0 \quad \mu > 0

singularity in \( \frac{\mu - x}{\ln(\mu - x)} \)

singularity in \( (\mu - x)^{\frac{1}{2}} \)
**IDEA OF THE PROOF**

Singularity analysis \[ \text{[Flajolet - Odlyzko]} \]
A link between the singular behaviour of \( F(\gamma, \mu) \) near \( \rho_\mu \) and the asymptotic behaviour of \( h_m(\mu) \).

\[
\begin{align*}
R &= \frac{\gamma + \mu \phi(R)}{\mu} \\
R(0) &= 0
\end{align*}
\]

radius of convergence of \( \phi = \frac{1}{2\pi} \)

\( \mu < 0 \) \hspace{1cm} \( \mu > 0 \)

singularity in \( \frac{\rho_\mu - \gamma}{\ln(\rho_\mu - \gamma)} \)

singularity in \( (\rho_\mu - \gamma)^{\frac{1}{2}} \)
MORE RESULTS

→ Exact enumeration of $p$-valent forested maps, $p > 3$.

→ Asymptotic behaviour of $p$-valent forested maps, $p = 3, p$ even.

→ Random forested maps:
  - size of the root component
  - number of components
Future Work

- Enumeration of all (non-regular) forested maps.

- Random forested maps:
  - size of the largest component.
    (Conjecture: mean $\sim C \ln n$)
  - Random sampling.
THANK YOU!