The Kronecker product and the partition algebra

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$$S(\lambda) \otimes S(\mu) = \sum_{\nu} g^{\nu}_{\lambda,\mu} S(\nu),$$

where $g^{\nu}_{\lambda,\mu}$ are the Kronecker coefficients.

Combinatorial description of $g^{\nu}_{\lambda,\mu}$?

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Partition algebra

Idea: Use the partition algebra to study the Kronecker problem.

Structure of the talk

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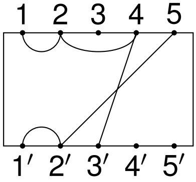

and **multiplication** given by concatenation and scalar multiplication by n^t where t is the number of connected components consisting of middle vertices only.

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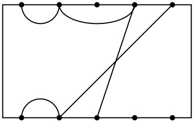
$X =$ 

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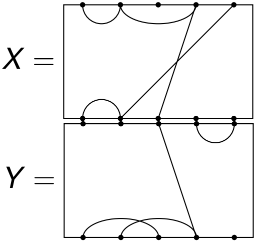

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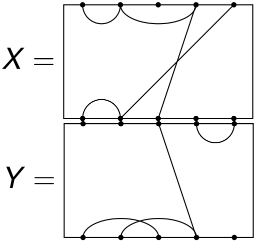

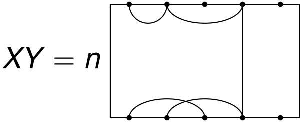

Remark: The group algebra $\mathbb{C}\mathfrak{S}_r$ appears naturally as a subalgebra of $P_r(n)$ (as the span of all diagrams having precisely r 'propagating blocks').

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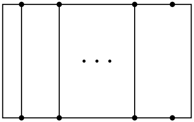
$e = \frac{1}{n}$ 

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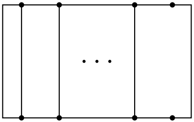
$e = \frac{1}{n}$  $e^2 = e.$

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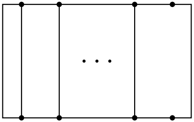
$e = \frac{1}{n}$  $e^2 = e.$

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Let L be a simple P_r-module. Then either $eL = 0$ and so L is a simple $\mathbb{C}\mathfrak{S}_r$-module, or $eL \neq 0$ and so eL is a simple P_{r-1}-module.

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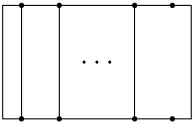

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Thus we have that the simple P_r-modules are indexed by partitions of degree $\leq r$.

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A complete set of non-isomorphic simple P_r-modules is given by

$$\{L_r(\lambda) := \mathrm{hd}\, \Delta_r(\lambda), \quad \lambda \in \Lambda_{\leq r}\}.$$

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Definition: Let λ, μ be partitions. We say that (μ, λ) form an *n-pair* and we write $\mu \hookrightarrow_n \lambda$ if $\mu \subset \lambda$ and λ/μ consists of a single row of boxes of which the last (rightmost) one has content $n - |\mu|$.

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Example: $((2, 1), (4, 1))$ form a 6-pair (with $6 - |\mu| = 3$).

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline -1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \subset \quad \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

Proposition: The set $\Lambda_{\leq r}$ splits into maximal chains of n-pairs:

$$\lambda^{(0)} \hookrightarrow_n \lambda^{(1)} \hookrightarrow_n \lambda^{(2)} \hookrightarrow_n \ldots \hookrightarrow_n \lambda^{(t)}$$

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Each cell module $\Delta_r(\lambda^{(i)})$ $(0 \leq i \leq t-1)$ has Loewy structure

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In the Grothendieck group we have

$$[L_r(\lambda^{(i)})] = \sum_{j=i}^{t} (-1)^{j-i} [\Delta_r(\lambda^{(j)})].$$

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Back to Schur-Weyl duality: As a $(\mathfrak{S}_n, P_r(n))$-bimodule we have

$$V_n^{\otimes r} = \sum S(\lambda) \otimes L_r(\lambda_{>1})$$

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Theorem: Let $\lambda, \mu, \nu \vdash n$ with $\lambda_{>1} \vdash r$ and $\mu_{>1} \vdash s$ then we have

$$g_{\lambda,\mu}^{\nu} = \begin{cases} [L_{r+s}(\nu_{>1})\downarrow_{P_r \otimes P_s} : L_r(\lambda_{>1}) \otimes L_s(\mu_{>1})] & \text{if } \nu_{>1} \in \Lambda_{\leq r+s} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \sum_{i=0}^{t} (-1)^i [\Delta_{r+s}(\eta^{(i)})\downarrow_{P_r \otimes P_s} : L_r(\lambda_{>1}) \otimes L_s(\mu_{>1})] & \text{if } \nu_{>1} \in \Lambda_{\leq r+s} \\ 0 & \text{otherwise} \end{cases}$$

where $\nu_{>1} = \eta^{(0)} \hookrightarrow_n \eta^{(1)} \hookrightarrow_n \eta^{(2)} \hookrightarrow_n \ldots \hookrightarrow_n \eta^{(t)}$ is the chain containing $\nu_{>1}$.

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Then for all $n \geq 4$ we have

$$S(n-1,1) \otimes S(n-1,1) = S(n) \oplus S(n-1,1) \oplus S(n-2,1^2) \oplus S(n-2,2).$$

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THANK YOU

Let $\lambda_{[n]}, \mu_{[n]}, \nu_{[n]}$ be partitions of n with $|\lambda| = r$, $|\mu| = s$ and $|\nu| = r + s - l$.
(i) Suppose $\nu_{[n]} = (n-k, k)$ is a two-part partition. Then we have

$$g_{\lambda_{[n]},\mu_{[n]}}^{(n-k,k)} = \overline{g}_{\lambda,\mu}^{(k)} = \sum_{\substack{l_1,l_2 \\ l=l_1+2l_2}} \sum_{\substack{\sigma\vdash l_1 \\ \gamma\vdash l_2}} c_{(r-l_1-l_2),\sigma,\gamma}^{\lambda} c_{\gamma,\sigma,(s-l_1-l_2)}^{\mu}$$

for all $n \geq \min\{|\lambda| + \mu_1 + k, |\mu| + \lambda_1 + k\}$.
(ii) Suppose $\nu_{[n]} = (n-k, 1^k)$ is a hook partition. Then we have

$$g_{\lambda_{[n]},\mu_{[n]}}^{(n-k,1^k)} = \overline{g}_{\lambda,\mu}^{(1^k)} = \sum_{\substack{l_1,l_2 \\ l=l_1+2l_2}} \sum_{\substack{\sigma\vdash l_1 \\ \gamma\vdash l_2}} c_{(1^{r-l_1-l_2}),\sigma,\gamma}^{\lambda} c_{\gamma,\sigma',(1^{s-l_1-l_2})}^{\mu}$$

for all $n \geq \min\{|\lambda| + |\mu| + 1, |\mu| + \lambda_1 + k, |\lambda| + \mu_1 + k\}$ and where σ' denotes the transpose of σ.