Euler flag enumeration of Whitney stratified spaces

Richard Ehrenborg UK & IAS
Mark Goresky IAS
Margaret Readdy UK & IAS

$P =$ n-dim polytope

$S \subseteq \{0, 1, \ldots, n-1\}$

$S = \{s_1 < s_2 < \cdots < s_k\}$

flag f-vector of P

$$f_S = \#(F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k) : \dim(F_i) = s_i$$

flag h-vector of P

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T$$

$P =$ n dim polytope

$S \subseteq \{0, 1, \ldots, n-1\}$

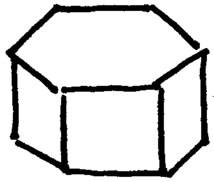
$S = \{ s_1 < s_2 < \cdots < s_k \}.$

flag f-vector of P

$$f_S = \#\,(F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k) : \quad \dim(F_i) = s_i$$

flag h-vector

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T$$

$P =$ 

S	f_S	h_S
$\emptyset$	1	1
0	12	11
1	18	17
2	8	7
01	36	7
02	36	17
12	36	11
012	72	1

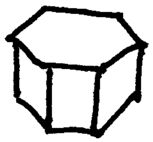
a + b noncommutative

$$u_S = u_0 u_1 \cdots u_{n-1}$$

where $$u_i = \begin{cases} a & i \notin S \\ b & i \in S \end{cases}$$

ab-index

$$\Psi(P) = \sum_S h_S \cdot u_S$$

$\Phi\Big($  $\Big) =$

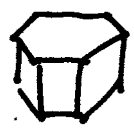
$$= 1\cdot aaa + 11\cdot baa + 17\cdot aba + 7\cdot aab + $$
$$+ 7\cdot bba + 17\cdot bab + 11\cdot abb + 1\cdot bbb$$

$$= (a+b)^3 + 10\,baa + 16\,aba + 6\,aab + $$
$$+ 6\,bba + 16\,bab + 10\,abb$$

$$= (a+b)^3 + 10\cdot(baa + aba + bab + abb)$$
$$+ 6\cdot(aba + aab + bba + bab)$$

$$= (a+b)^3 + 10\cdot(ab+ba)\cdot(a+b)$$
$$+ 6\cdot(a+b)\cdot(ab+ba)$$

Let $c = a + b$ and $d = ab + ba$

$\Phi($  $) = c^3 + 10\,dc + 6\,cd.$

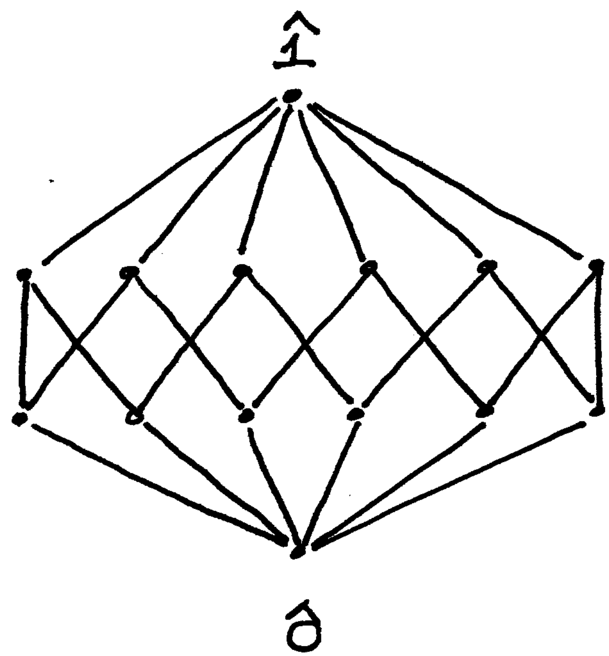
S	f_S	h_S	w_S	c^3	$10\,dc$	$6\,cd$
$\emptyset$	1	1	aaa	1	0	0
0	12	11	baa	1	10	0
1	18	17	aba	1	10	6
2	8	7	aab	1	0	6
01	36	7	bba	1	0	6
02	36	17	bab	1	10	6
12	36	11	abb	1	10	0
012	72	1	bbb	1	0	0

Theorem [Bayer-Klapper, Stanley]

For P polytope $\Psi(P) \in \mathbb{Z}\langle c,d \rangle$

For P Eulerian poset $\Psi(P) \in \mathbb{Z}\langle c,d \rangle$

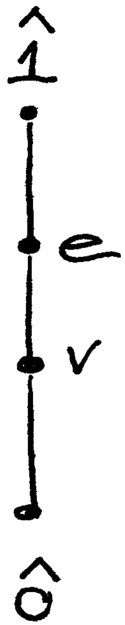
$P =$ n-gon 



S	f_S	h_S	w_S	c^2	$(n-2)\cdot d$
$\emptyset$	1	1	aa	1	0
0	n	$n-1$	ba	1	$n-2$
1	n	$n-1$	ab	1	$n-2$
01	$2n$	1	bb	1	0

$$\Phi(P) = c^2 + (n-2)\,d \qquad n \geq 2.$$

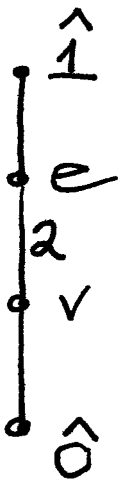
P = one-gon = 



s	f_s	h_s	w_s
$\emptyset$	1	1	aa
0	1	0	ba
1	1	0	ab
01	1	0	bb

$\Psi($  $) = aa \notin \mathbb{Z}\langle c, d\rangle.$

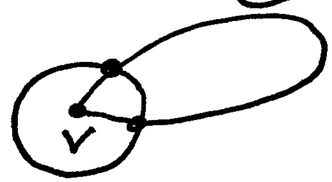
$P =$ one-gon $=$ 



S	$\overline{f}_S$	$\overline{h}_S$	w_S
$\emptyset$	1	1	aa
0	1	0	ba
1	1	0	ab
01	2	1	bb.

$\Phi($  $) = aa + bb$

$= c^2 - d.$



$\mathrm{link}_e(v) =$  $\qquad \chi($  $) = 2$

$$c = \{ \hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1} \}.$$

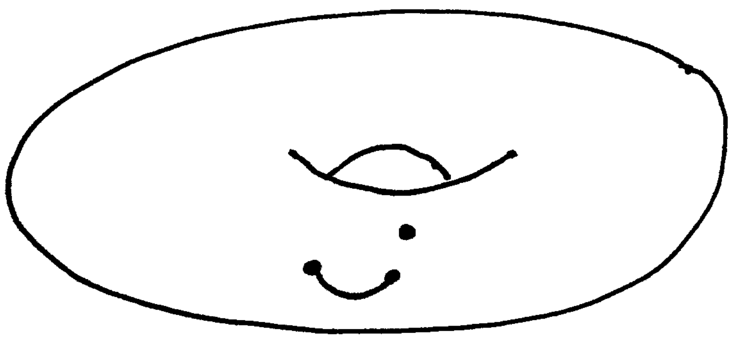
$$\bar{\xi}(c) = \chi(x_1) \cdot \chi(\mathrm{link}_{x_2}(x_1)) \cdot \cdots \cdot \chi(\mathrm{link}_{x_k}(x_{k-1}))$$

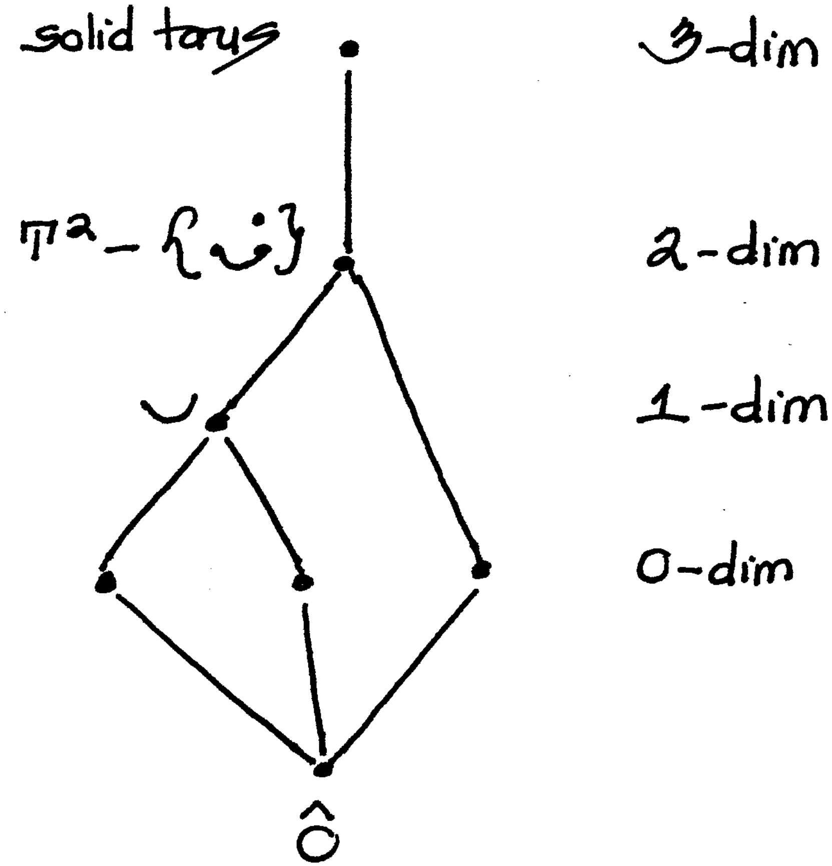
$$\bar{f}_S = \sum_{c} \bar{\xi}(c)$$

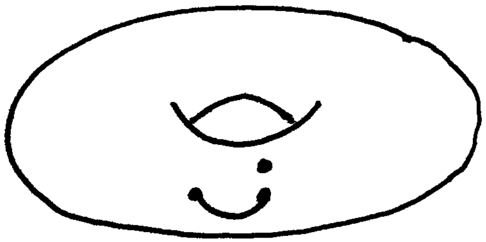
where $\dim(x_i) = s_i$.

$$\bar{h}_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot \bar{f}_T$$

XI.







S	$\bar{f}_S$	$\bar{h}_S$	w_S	$3dc$	$-2cd$
$\emptyset$	0	0	aaa	0	0
0	3	3	baa	3	0
1	1	1	aba	3	−2
2	−2	−2	aab	0	−2
01	2	−2	bba	0	−2
02	2	1	bab	3	−2
12	2	3	abb	3	0
012	4	0	bbb	0	0

Whitney stratifications:

Subdivide space into strata

$$W = \bigcup_{X \in P} X$$

Condition of the frontier

$$X \cap \overline{Y} \neq \emptyset \iff X \subseteq \overline{Y} \iff X \leq_P Y$$

Whitney conditions (A) and (B)

- No fractal behavior
- No infinite wiggling $\quad x \cdot \sin\left(\frac{1}{x}\right)$.

The links are well-defined.

THE FINE PRINT

Definition Let W be a closed subset of a smooth manifold M, and suppose W can be written as a locally finite disjoint union

$$W = \bigcup_{X \in \mathcal{P}} X$$

where $\mathcal{P}$ is a poset. Furthermore, suppose each $X \in \mathcal{P}$ is a locally closed subset of W satisfying the *condition of the frontier*:

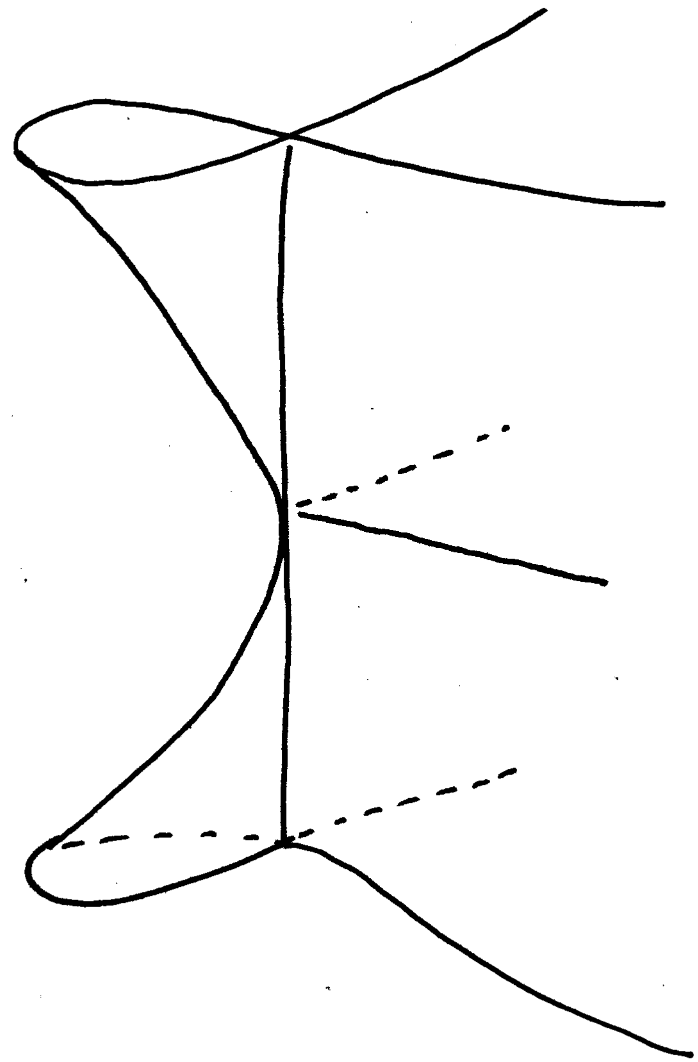
$$X \cap \overline{Y} \neq \emptyset \iff X \subseteq \overline{Y} \iff X \leq_{\mathcal{P}} Y.$$

This implies the closure of each stratum is a union of strata. We say W is a *Whitney stratification* if

1. Each $X \in \mathcal{P}$ is a locally closed smooth submanifold of M (not necessarily connected).
2. If $X <_{\mathcal{P}} Y$ then Whitney's conditions (A) and (B) hold: Suppose $y_i \in Y$ is a sequence of points converging to some $x \in X$ and that $x_i \in X$ converges to x. Also assume that (with respect to some local coordinate system on the manifold M) the secant lines $\ell_i = \overline{x_i y_i}$ converge to some limiting line ℓ and the tangent planes $T_{y_i}Y$ converge to some limiting plane τ. Then the inclusions

   $$\text{(A) } T_x X \subseteq \tau \qquad \text{and} \qquad \text{(B) } \ell \subseteq \tau$$

   hold.



Whitney cusp.

Quasi-graded poset

$(P, \rho, \bar{\zeta})$

P poset with $\hat{0}$ and $\hat{1}$

$\rho : P \to \mathbb{N}$ increasing

$\bar{\zeta} \in I(P)$ such that $\bar{\zeta}(x,x) = 1 \quad \forall x \in P.$

$(P, \rho, \bar{\zeta})$ Eulerian if

$$\sum_{x \leq y \leq z} (-1)^{\rho(y)-\rho(x)} \cdot \bar{\zeta}(x,y) \cdot \bar{\zeta}(y,z) = \delta_{x,z}$$

$$\Psi(P, \rho, \bar{\xi}) = \sum_{S} \bar{h}_S \cdot w_S.$$

Theorem: $(P, \rho, \bar{\xi})$ Eulerian $\Longrightarrow$

$$\Longrightarrow \Psi(P, \rho, \bar{\xi}) \in \mathbb{Z}\langle c, d\rangle.$$

Theorem: M manifold with a Whitney stratification of ~~its~~ boundary. Then the ~~quasi-graded~~ face ~~poset~~ is Eulerian with

$$\rho(x) = \dim(x) + 1.$$

$$\bar{\xi}(x, y) = \chi(\mathrm{link}_y(x)).$$

Open questions

1. Nonnegativity of cd-index

   S-shellable posets (polytopes) [Stanley]

   Gorenstein* posets [Karu]

   Conj : $\Psi(\mathcal{C}) \geq 0$ for regular cell complex

2. Inequalities – Kalai convolution (!)

   – lifting technique (?).

3. $(P, \rho, \overline{\xi}) \Rightarrow$ Find stratified space

4. Stratified analogue of Stanley-Reisner ring.

Merci Beaucoup!