Structure and enumeration of $(3+1)$-free posets

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The story of a table of numbers

Number of vertices	1	2	3	4	5	6
All posets	1	2	5	16	63	318
. . . $(3+1)$-free	1	2	5	15	49	173
. . . and $(2+2)$-free	1	2	5	14	42	132

Number of vertices	7	8	9	10
All posets	2045	16999	183231	2567284
. . . $(3+1)$-free	639	2469	9997	43109
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The graded case: (Lewis-Zhang FPSAC 2012)

Colourings

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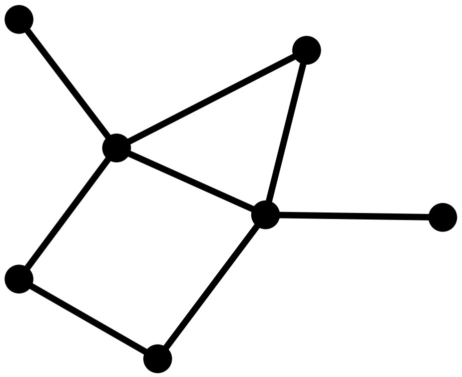

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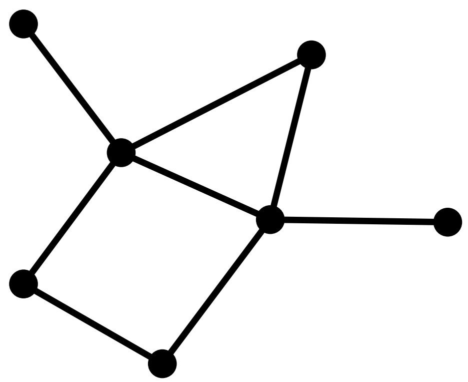

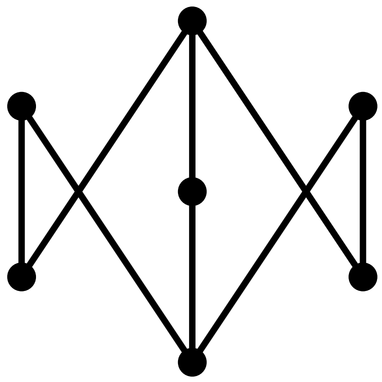

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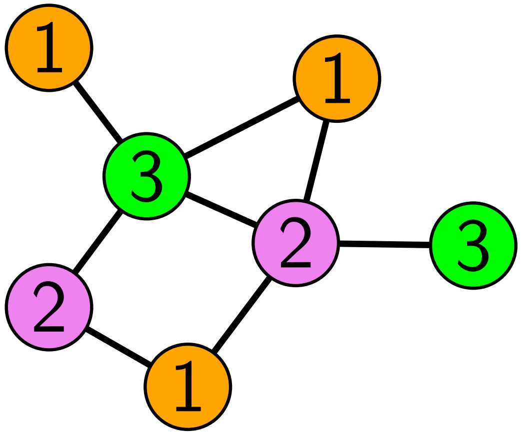

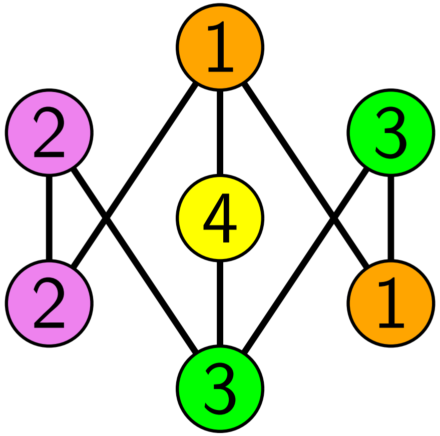

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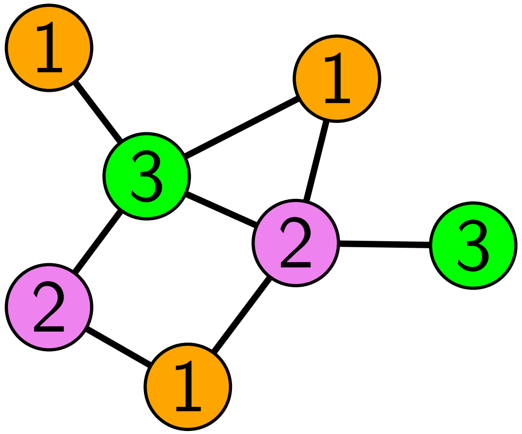

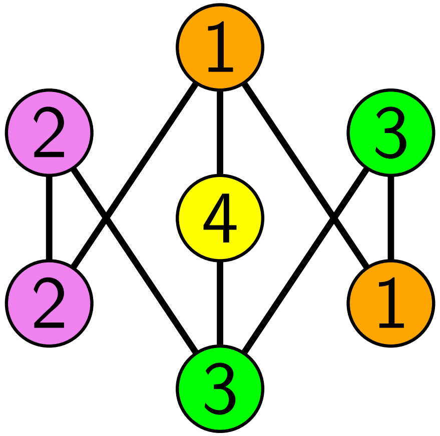

Graphs:
independent sets

Posets:
chains

Stanley's chromatic symmetric functions

- **Chromatic polynomial** $\chi_G(n)$:
  Polynomial function, counts the number of proper colourings with n colours.

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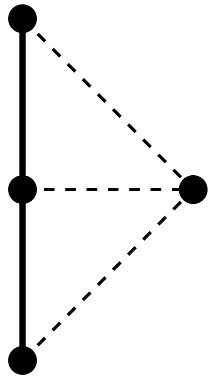
- **Chromatic symmetric function** $X_P(x_1, x_2, \ldots)$:
  Generating function for chain colourings, records the size of colour class i as the exponent of x_i.

Example



$(3+1)$

$$X_P(x_1, x_2, \ldots)$$

$$= x_1^3 x_2 + x_2^3 x_1 + x_1^3 x_3 + 6x_1^2 x_2 x_3 + \cdots$$

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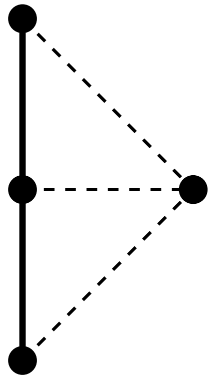

$(3+1)$

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X_P(&x_1, x_2, \ldots) \\
&= x_1^3 x_2 + x_2^3 x_1 + x_1^3 x_3 + 6x_1^2 x_2 x_3 + \cdots \\
&= m_{31} + 6m_{211} + 24m_{1111} \\
&= p_{1111} - 3p_{211} + 3p_{31} - p_4 \\
&= 8s_{1111} + 5s_{211} - s_{22} + s_{31} \\
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&= \ldots
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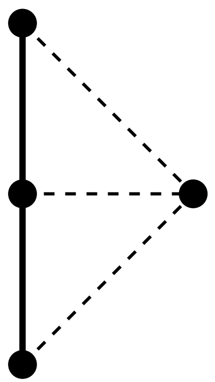

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Question:

Which posets have positive coefficients in which bases?

Stanley and Stembridge's conjecture (1993)

The data:

Contains $(3+1)$?	e-positive?	$n=4$	$n=5$	$n=6$	$n=7$
Yes	Yes	0	5	39	469
Yes	No	1	9	106	938
No	Yes	15	49	173	639
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The conjecture:

If a poset P is $(3+1)$-free, then its chromatic symmetric function $X_P(x_1, x_2, \ldots)$ is e-positive.

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Theorem (Gasharov 1996):

P is $(3+1)$-free implies $X_P(x_1, x_2, \ldots)$ is s-positive.

Numbers again

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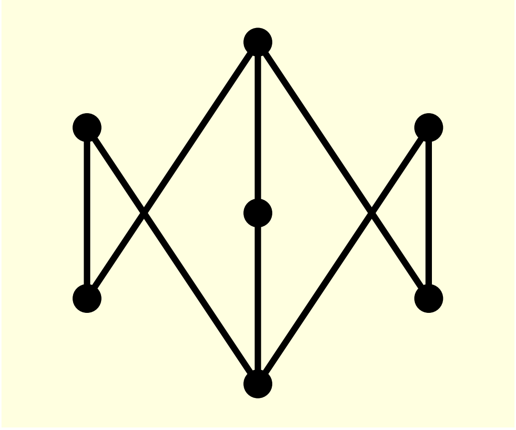
Generating posets: level by level

First idea:

Construct each poset one level at a time, starting from the minimal elements.



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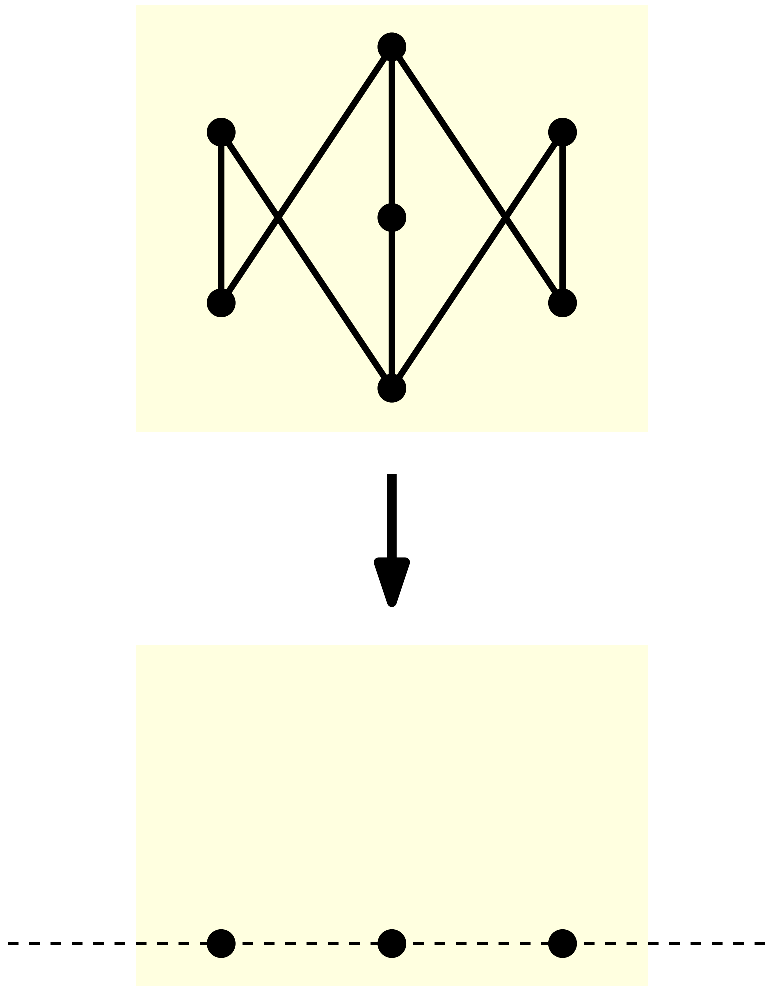

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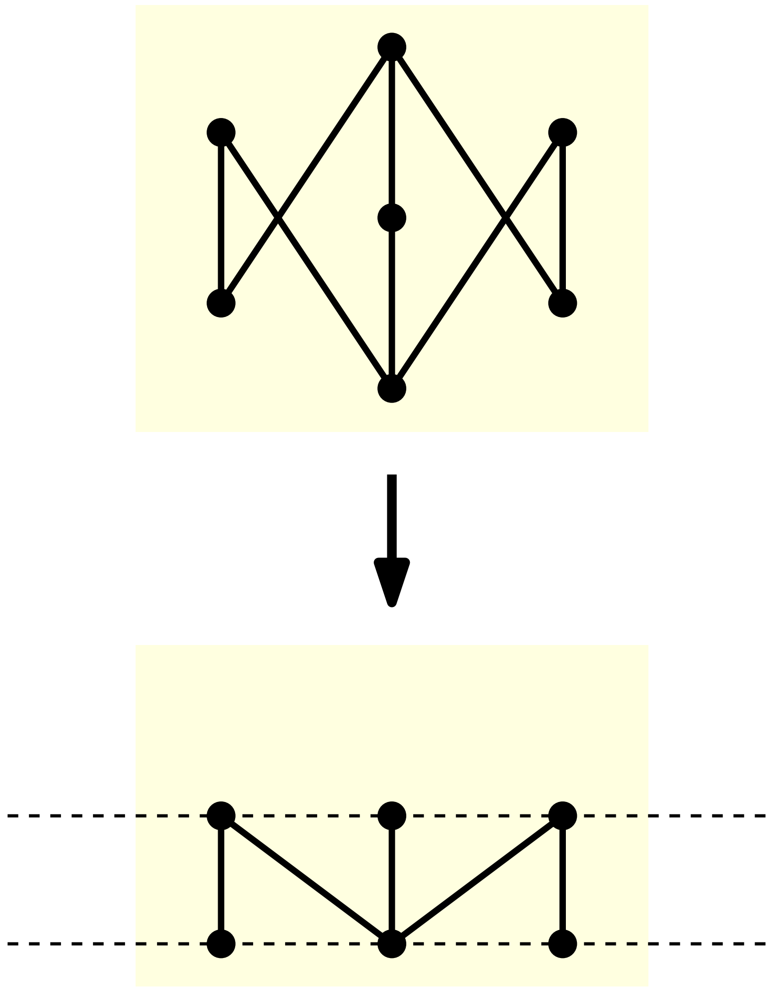

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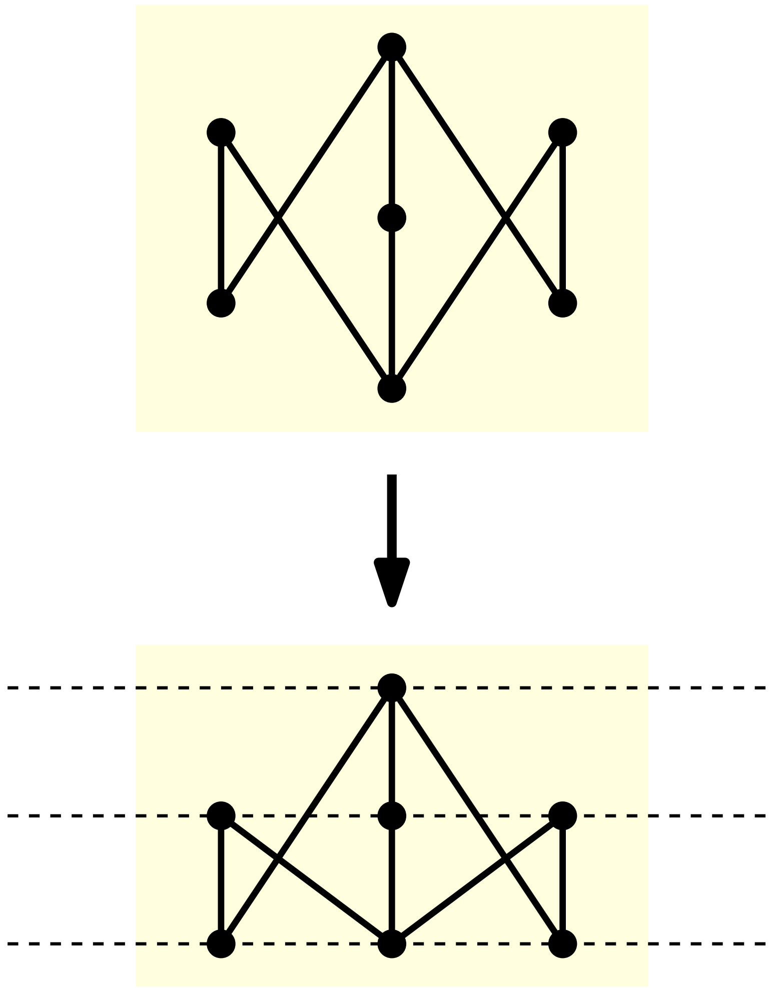

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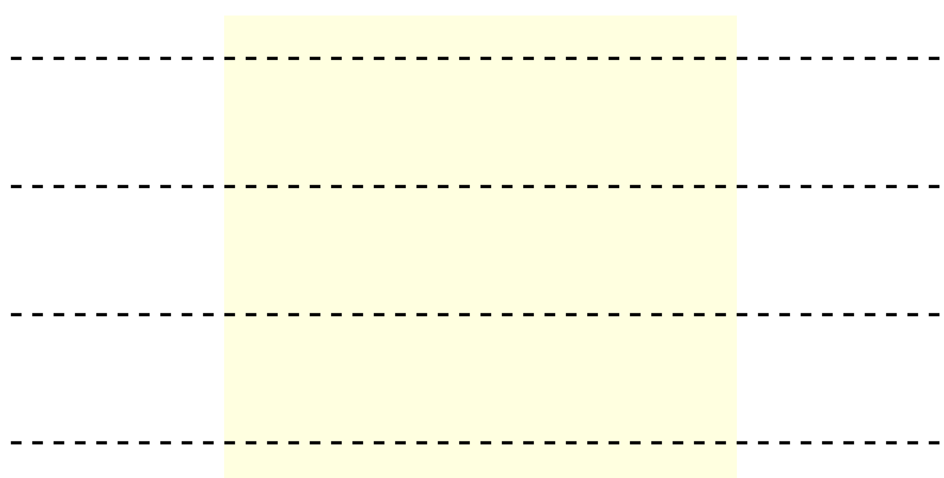

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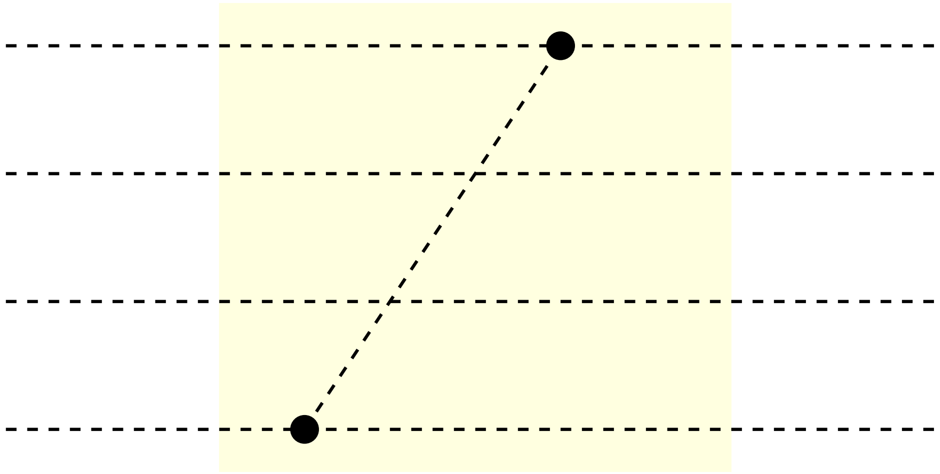

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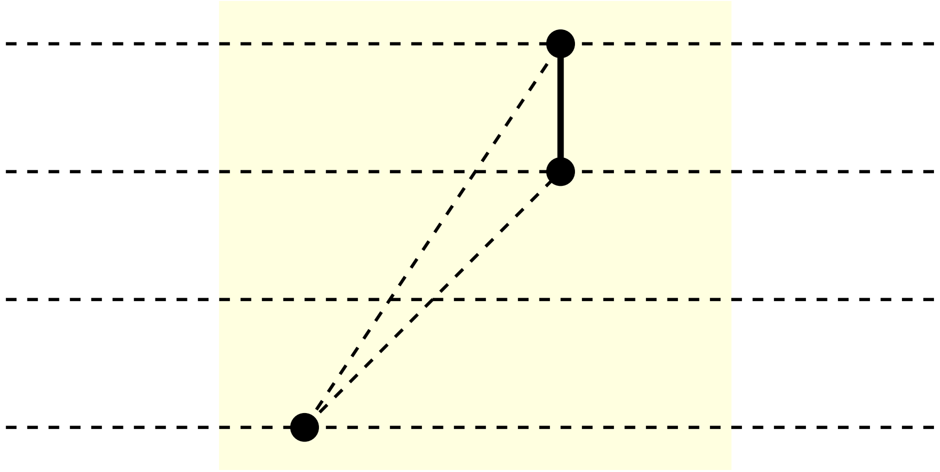

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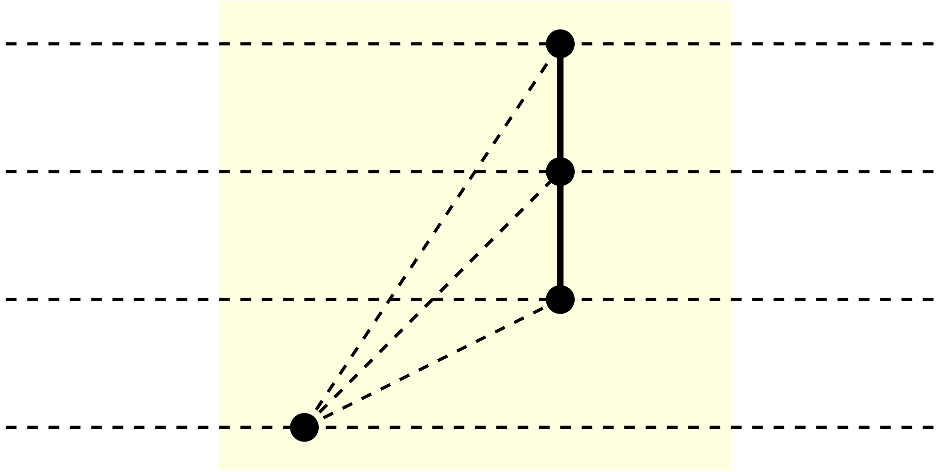

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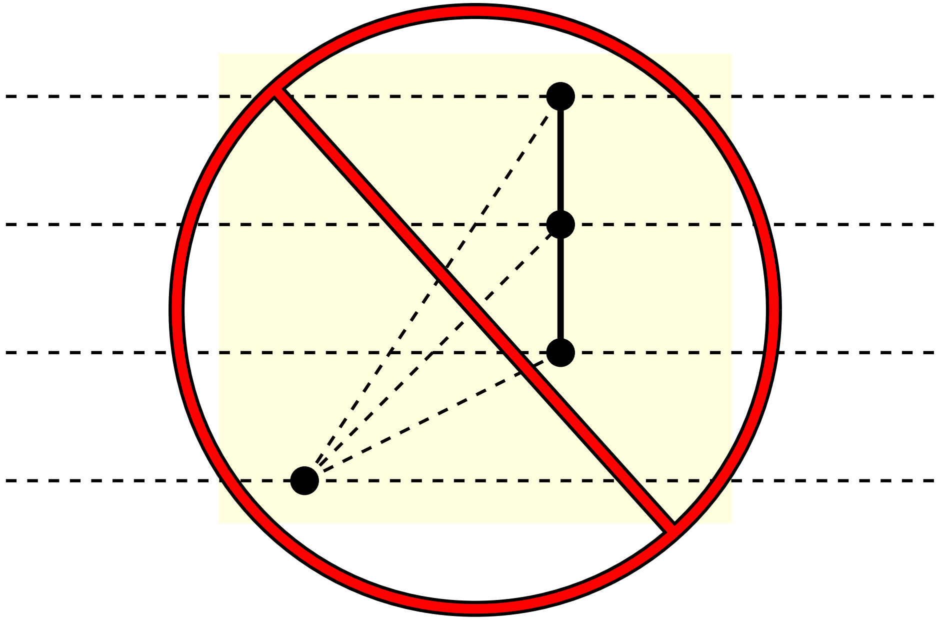
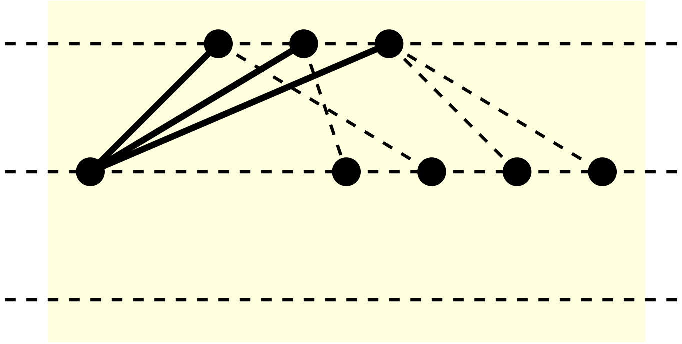


Generating posets: up-degree and down-degree

High up-degree:



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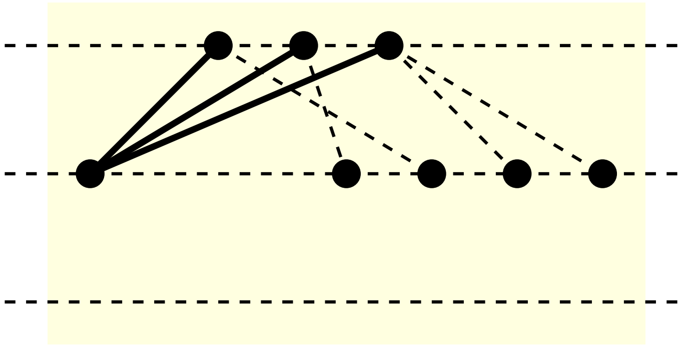
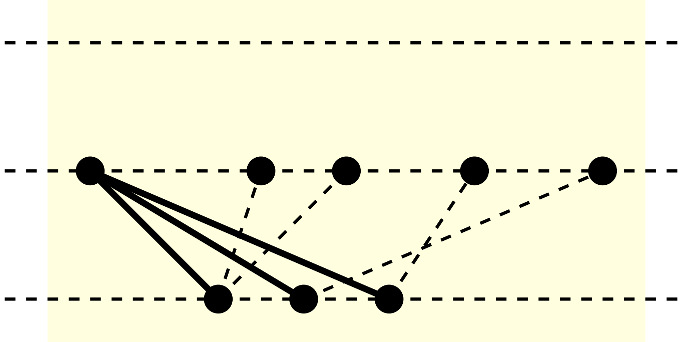


High down-degree:



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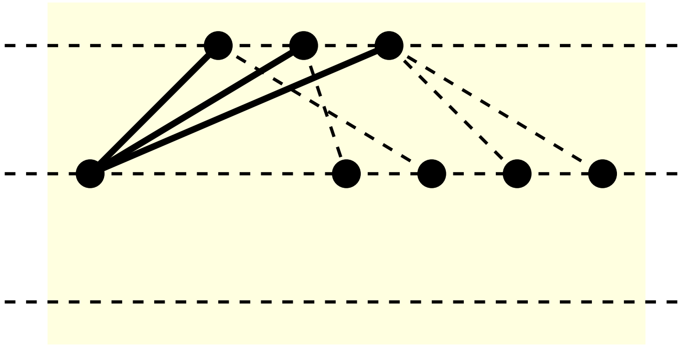

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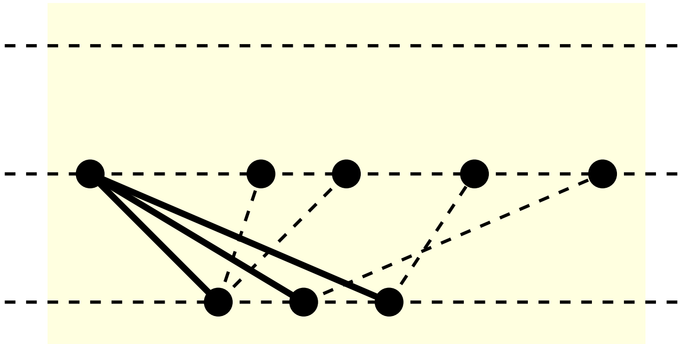
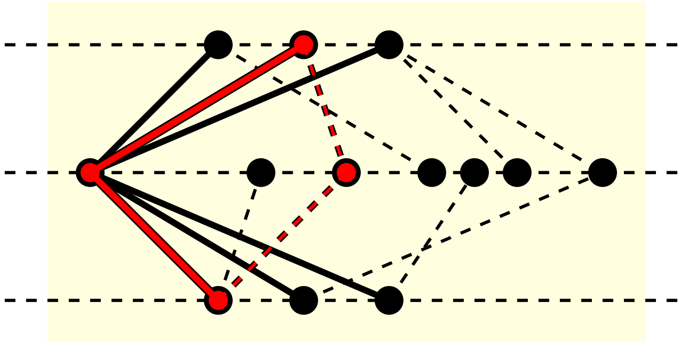
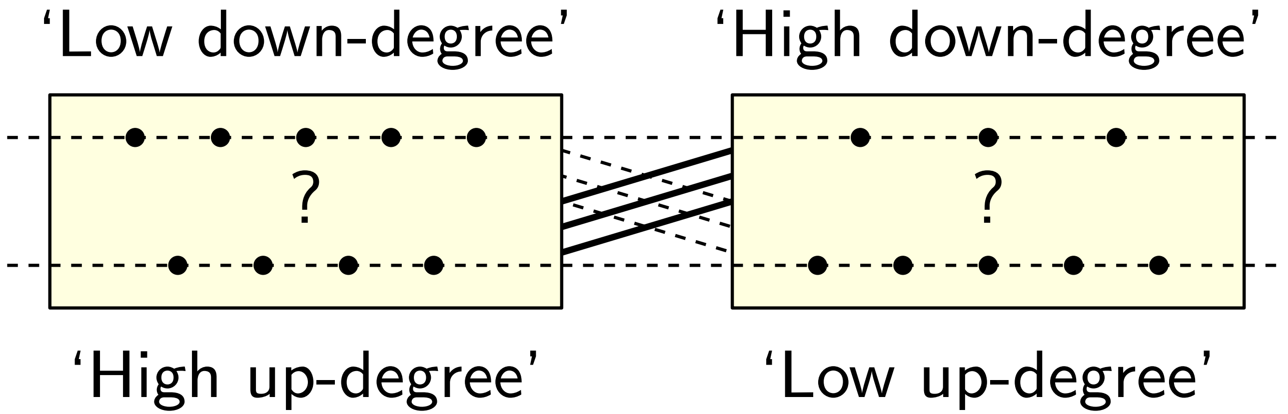


Both:



Generating posets: combing



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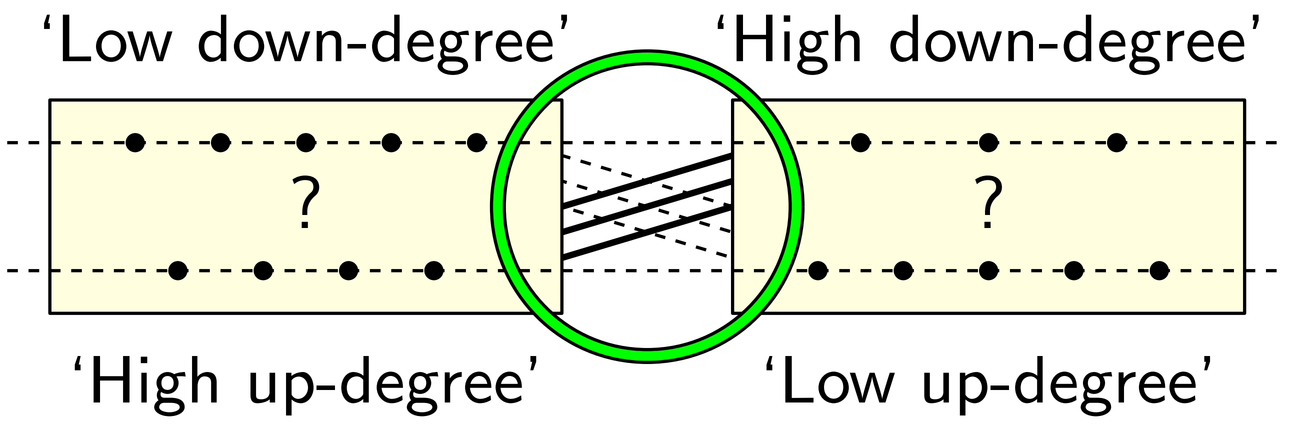

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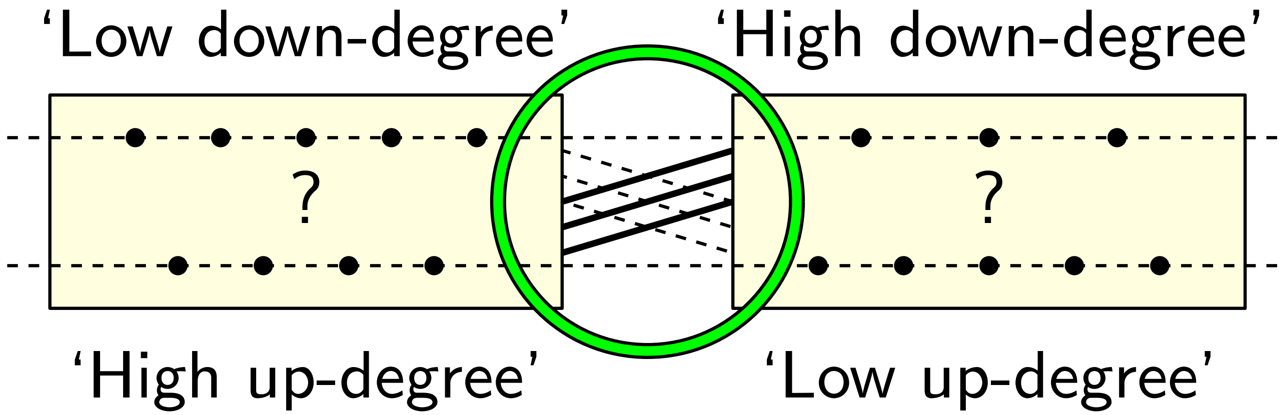
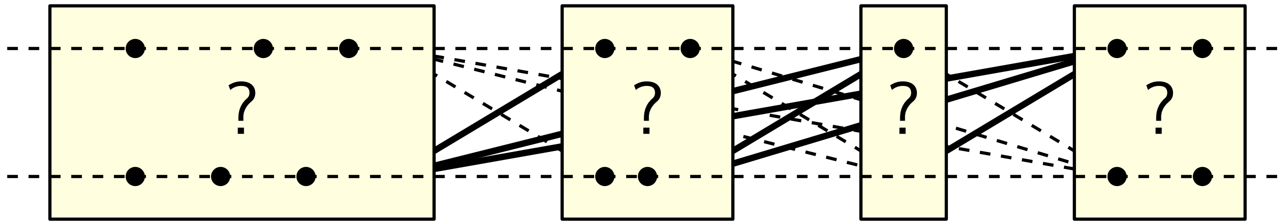


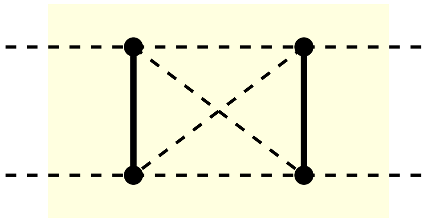
Sorted components for combing



Generating posets: tangles

Observation 1:

$(2+2)$ cannot be decomposed by combing.



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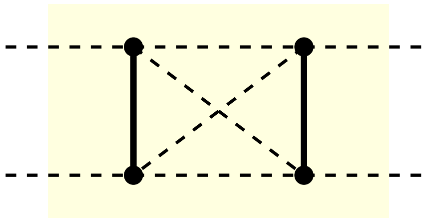
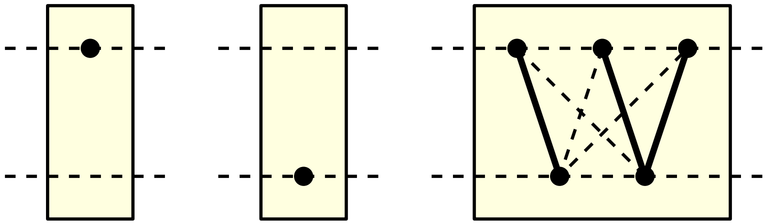


Observation 2:

Irreducible components are single vertices or connected by copies of $(2+2)$.



Generating function for tangles

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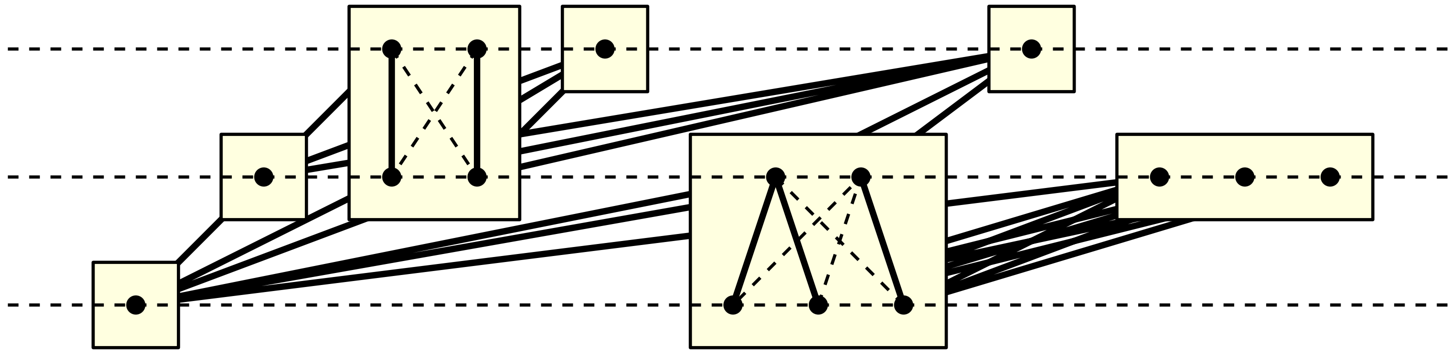
$$T(x,y) = 1 - x - y - \frac{1}{B(x,y)}$$

Generating poset: sorting all levels

Theorem:

Sorting the vertices of a level by combing with the level above or by combing with the level below gives compatible orderings.

In particular, tangles on different levels do not overlap.



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There is a bijection between skeleta of $(3+1)$-free posets and certain decorated Dyck paths.

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$$S(c,t) = \sum_{r,s\geq 0} \begin{pmatrix} \#\text{ of skeleta with} \\ r\text{ clone sets and} \\ s\text{ tangles} \end{pmatrix} c^r t^s$$

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$$S(c,t) = 1 + \frac{c}{1+c} S(c,t)^2 + tS(c,t)^3$$

Numbers once more

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Bonus

Theorem:

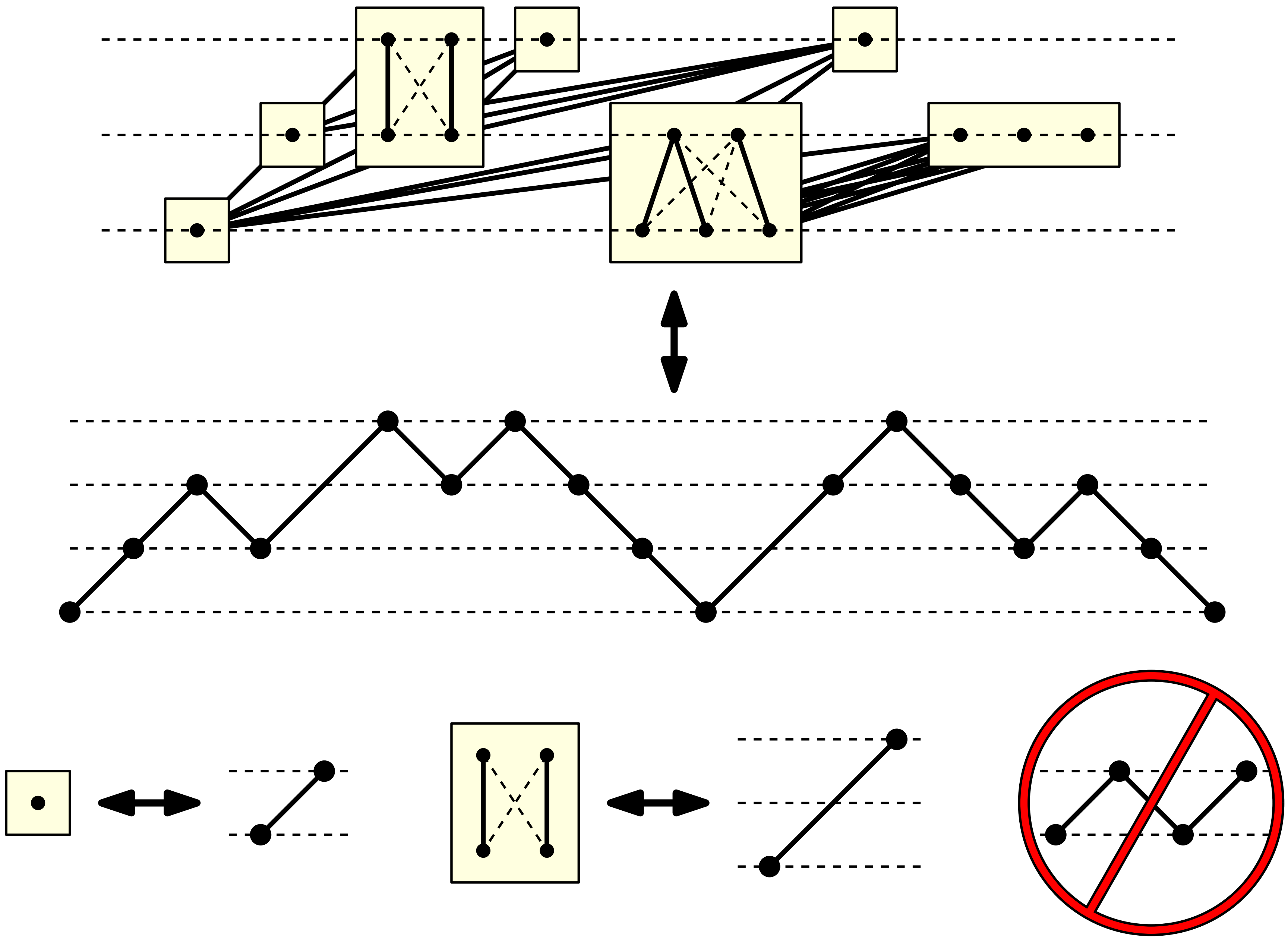
The e-positivity conjecture only needs to be checked for the smaller class of $(3+1)$-and-$(2+2)$-free posets.

Computation:

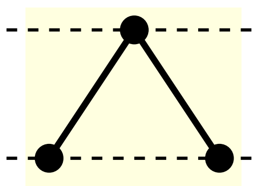
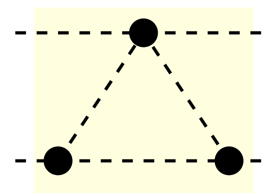
The e-positivity conjecture has been checked for all posets on up to 20 vertices.

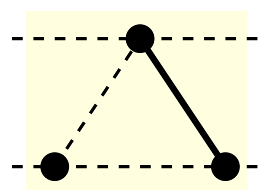
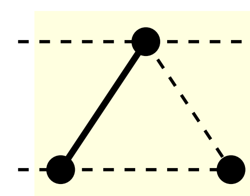
Thank you!

Bijection with (certain) Dyck paths



Modular relation

CSF () + CSF ()

= CSF () + CSF ()

Extra numbers

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... and basic	1	1	1	1	1	1

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