Quiver mutation and combinatorial DT-invariants

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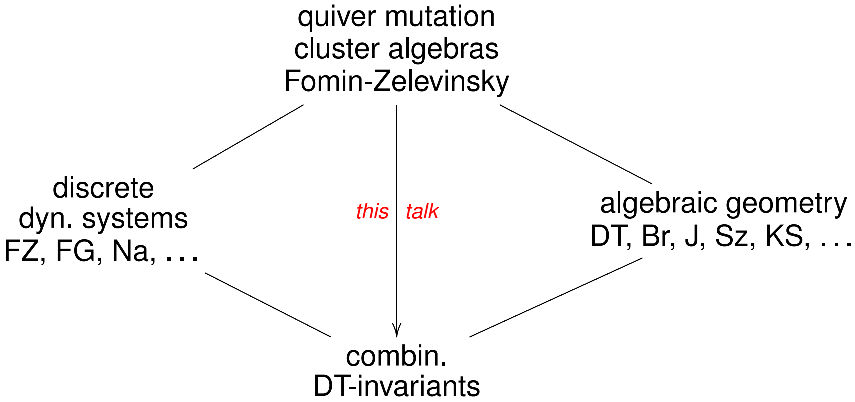
Context



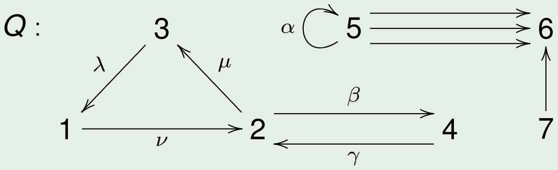
FG=Fock-Goncharov, Na=Nakanishi, DT=Donaldson-Thomas, Br=Bridgeland, J=Joyce, Sz=Szendröi, KS=Konts.-Soibelman

Plan

1. Quiver mutation
2. Combinatorial DT-invariants

Quivers: example and terminology

Example



We have $Q_0 = \{1, 2, 3, 4, 5, 6, 7\}$, $Q_1 = \{\alpha, \beta, \ldots\}$.
α is a *loop*, (β, γ) is a 2-*cycle*, (λ, μ, ν) is a 3-*cycle*.

Definition

A vertex i is a *source* of Q if no arrows stop at i.
A vertex i is a *sink* of Q if no arrows start at i.

Definition of quiver mutation

Let Q be a quiver without loops or 2-cycles.

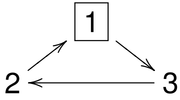
Definition (Fomin-Zelevinsky)

Let $j \in Q_0$. The *mutation* $\mu_j(Q)$ is the quiver obtained from Q as follows

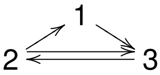
1) for each subquiver $i \xrightarrow{b} j \xrightarrow{a} k$, add a new arrow $i \xrightarrow{[ab]} k$;
2) reverse all arrows incident with j;
3) remove the arrows in a maximal set of pairwise disjoint 2-cycles (e.g. $\bullet \rightrightarrows\!\!\leftarrow \bullet$ yields $\bullet \longrightarrow \bullet$, '2-reduction').

Examples of quiver mutation

A simple example:



1)



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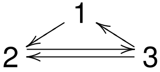

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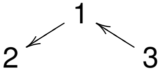
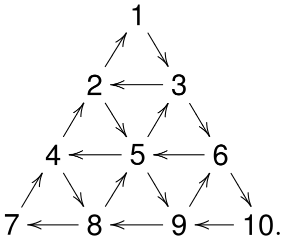
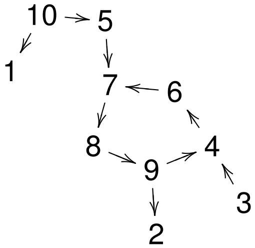
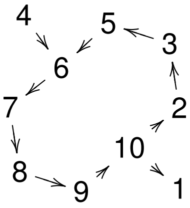


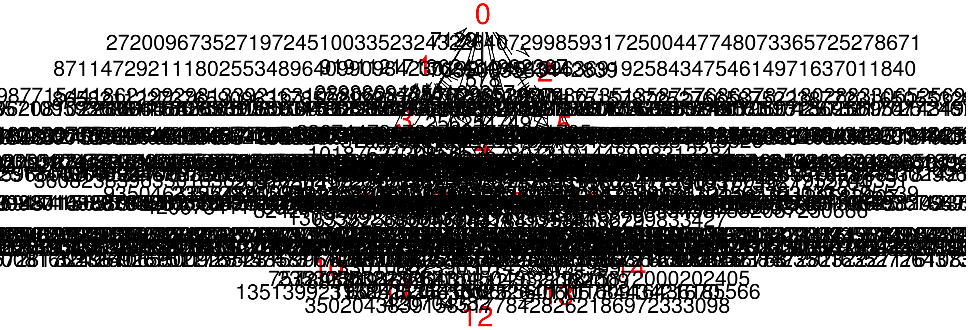
More complicated examples: Google ‘quiver mutation’!







Towards green quiver mutation



Next aim: Green quiver mutation!

Green quiver mutation: the framed quiver

Let Q be a quiver without loops or 2-cycles, $Q_0 = \{1, \ldots, n\}$.

Definition

The framed quiver $\widetilde{Q}$ is obtained from Q by adding, for each vertex i, a new vertex i' and a new arrow $i \to i'$.

Example

$$Q: \ 1 \longrightarrow 2 \qquad\qquad \widetilde{Q}: \ \begin{array}{ccc} 1 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ 1' & & 2' \end{array}$$

Definition

The vertices i' are frozen vertices, i.e. we never mutate at them.

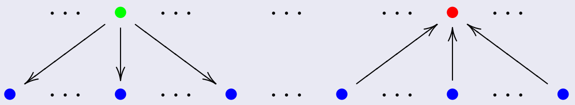
Green and red vertices

Suppose that we have transformed $\widetilde{Q}$ into $\widetilde{Q}'$ by a finite sequence of mutations (at non frozen vertices).

Definition

A vertex i of Q is green in $\widetilde{Q}'$ if there are no arrows $j' \to i$ in $\widetilde{Q}'$. It is red if there are no arrows $i \to j'$ in $\widetilde{Q}'$.

Green and red vertices in $\widetilde{Q}'$



Theorem (Derksen-Weyman-Zelevinsky 2010)

Each vertex of Q is either green or red in $\widetilde{Q}'$ and not both.

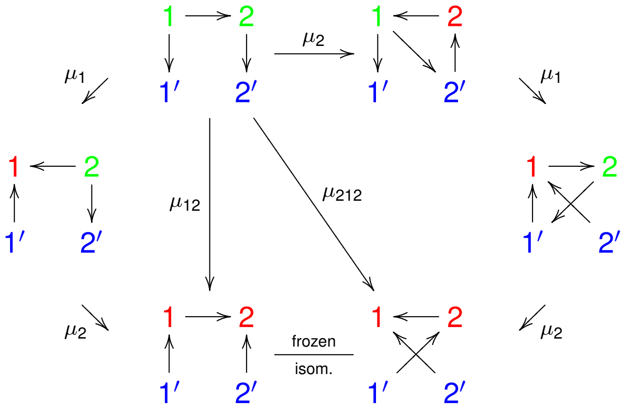
Green sequences

Definition

A sequence $\underline{i} = (i_1, \ldots, i_N)$ is green if, for each $1 \leq t \leq N$, the vertex i_t is green in the partially mutated quiver

$$\mu_{i_{t-1}} \ldots \mu_{i_2} \mu_{i_1} (\widetilde{Q}) =: \widetilde{Q}(\underline{i}, t).$$

Green quiver mutation: example



Reddening sequences

Theorem 1

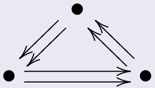
If $\underline{i}$ and $\underline{i}'$ are maximal green sequences, then there is a frozen isomorphism

$$\mu_{\underline{i}}(\widetilde{Q}) \xrightarrow{\sim} \mu_{\underline{i}'}(\widetilde{Q}).$$

Remarks

- More generally, this holds for reddening sequences, i.e. arbitrary sequences $\underline{i} = (i_1, \ldots, i_N)$ such that all vertices of Q are red in the final quiver

$$\mu_{\underline{i}}(\widetilde{Q}).$$

- Not all quivers admit reddening sequences: 

The exchange quiver

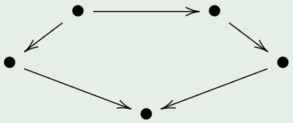
Definition

The exchange quiver $\mathcal{E}_Q$ has

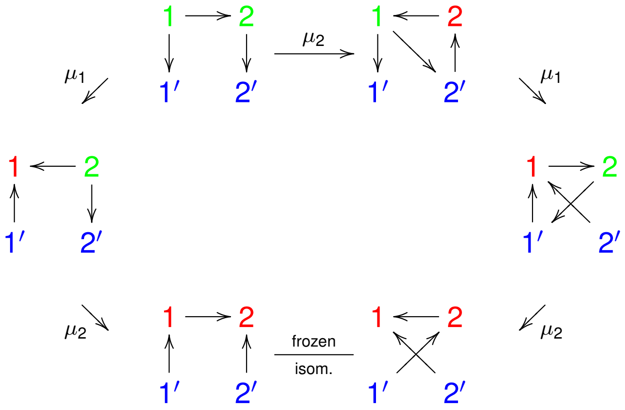
- vertices: frozen isomorphism classes $\mu_{\underline{i}}(\widetilde{Q})$, where $\underline{i}$ is any sequence of vertices,
- arrows: $\widetilde{Q}' \to \mu_j(\widetilde{Q}')$ if j is green in $\widetilde{Q}'$.

Example

$Q : 1 \to 2$ yields $\mathcal{E}_Q$:



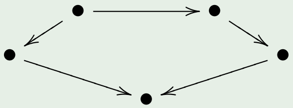
Reason



Key properties of the exchange quiver

Example (Reminder)

$Q: 1 \to 2$ yields $\mathcal{E}_Q$:



Remarks

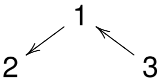
- green seq. = formal comp. of arrows in $\mathcal{E}_Q$ = path in $\mathcal{E}_Q$,
- any mut. seq. = formal comp. of arrows$^{\pm 1}$ = walk in $\mathcal{E}_Q$.
- $\mathcal{E}_Q$ has a sink iff Q admits a reddening sequence.
- This sink is then unique (final quiver of a reddening seq.).

Theorem (Nagao 2010)

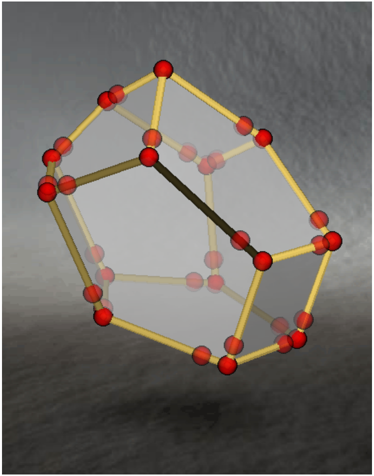
The exchange quiver $\mathcal{E}_Q$ is the Hasse diagram of a poset.

The 3-chain

Q :

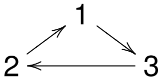


$\mathcal{E}_Q$:

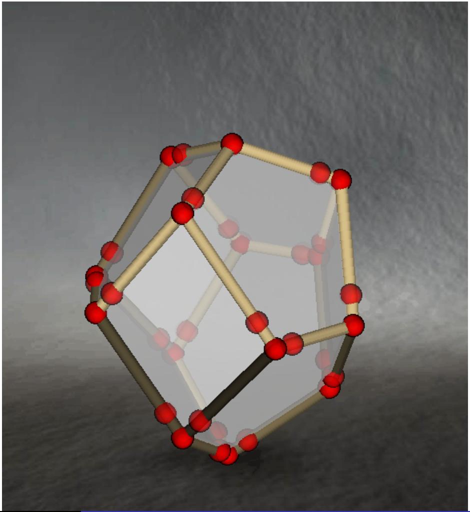


The 3-cycle

Q :

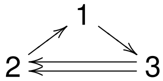


$\mathcal{E}_Q$:

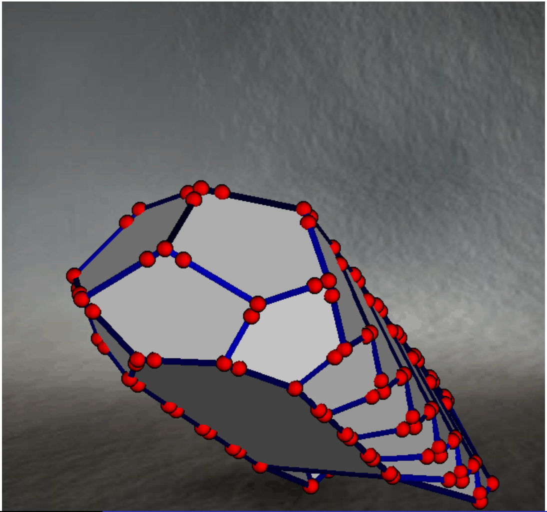


The 3-cycle with double arrow

Q :

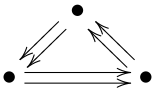


$\mathcal{E}_Q$:

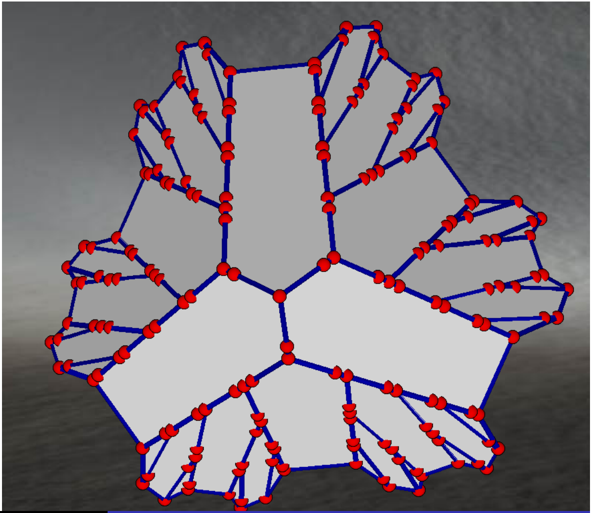


The doubled 3-cycle

Q :



$\mathcal{E}_Q$:



The quantum dilogarithm and the quiver $\vec{A}_1$

Aim

Define the combinatorial DT-invariant $\mathbb{E}_Q$ as a formal power series intrinsically associated with Q, whenever Q admits a reddening sequence.

Definition

The (exponential of the) quantum dilogarithm series is

$$\mathbb{E}(y) = 1 + \frac{q^{1/2}}{q-1} \cdot y + \cdots + \frac{q^{n^2/2} y^n}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})} + \cdots$$
$$\in \mathbb{Q}(q^{1/2})[[y]].$$

Remark: For the quiver $Q = \vec{A}_1 = \bullet$, we will have $\mathbb{E}_Q = \mathbb{E}(y)$.

The pentagon identity and the quiver $\vec{A}_2$

Pentagon identity (Faddeev–Kashaev–Volkov 1993)

$$y_1 y_2 = q y_2 y_1 \Longrightarrow \mathbb{E}(y_1)\mathbb{E}(y_2) = \mathbb{E}(y_2)\mathbb{E}(q^{-1/2} y_1 y_2)\mathbb{E}(y_1).$$

Combinatorial DT-invariant of $\vec{A}_2$

For

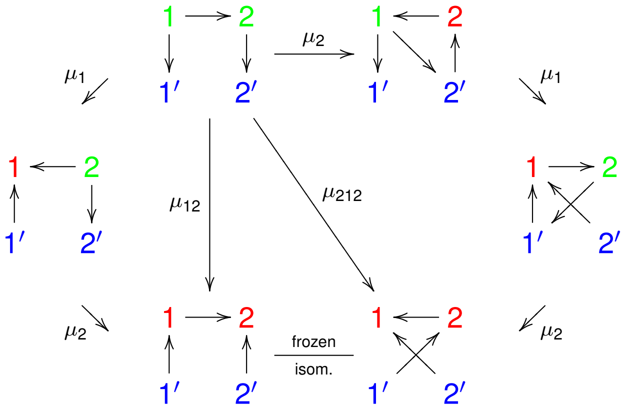
$$Q = \vec{A}_2 : \; 1 \longrightarrow 2 \,,$$

we will have

$$\mathbb{E}_Q = \mathbb{E}(y_1)\mathbb{E}(y_2) = \mathbb{E}(y_2)\mathbb{E}(q^{-1/2} y_1 y_2)\mathbb{E}(y_1).$$

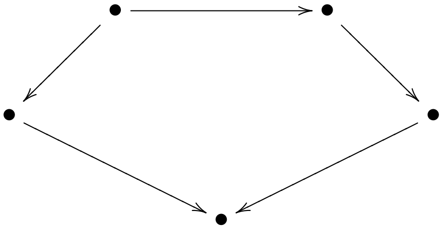
The pentagon and the quiver $\vec{A}_2$

$$\mathbb{E}(y_1)\mathbb{E}(y_2) = \mathbb{E}(y_2)\mathbb{E}(q^{-1/2}y_1y_2)\mathbb{E}(y_1)$$



In terms of the exchange quiver $\mathcal{E}_Q$

$$\mathbb{E}(y_1)\mathbb{E}(y_2) = \mathbb{E}_Q = \mathbb{E}(y_2)\mathbb{E}(q^{-1/2}y_1y_2)\mathbb{E}(y_1)$$



Each walk from the source to the sink of the exchange quiver yields a product expression for the combinatorial DT-invariant.

Key construction

Given any mutation sequence $\underline{i} = (i_1, \ldots, i_N)$, we need to construct a product $\mathbb{E}(\underline{i})$ of quantum dilogarithm series.
This product is taken in the algebra $\widehat{\mathbb{A}}_Q$ constructed as follows:
Let $\lambda_Q : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be bilinear antisymmetric such that

$$\lambda_Q(e_i, e_j) = \#(\text{arrows } i \to j \text{ in } Q) - \#(\text{arrows } j \to i \text{ in } Q).$$

Define

$$\widehat{\mathbb{A}}_Q := \mathbb{Q}(q^{1/2})\langle\langle y^\alpha, \alpha \in \mathbb{N}^n \mid y^\alpha y^\beta = q^{1/2\,\lambda(\alpha,\beta)} y^{\alpha+\beta} \rangle\rangle$$

$$\mathbb{E}(\underline{i}) := \mathbb{E}(y^{\alpha_1})^{\varepsilon_1} \ldots \mathbb{E}(y^{\alpha_N})^{\varepsilon_N} \,,$$

where

$$(\alpha_t)_j = \#(\text{arrows between } i_t \text{ and } j' \text{ in } \widetilde{Q}(\underline{i}, t))$$

and $\varepsilon_t = \pm 1$ depending on whether i_t is green or red in $\widetilde{Q}(\underline{i}, t)$.

The combinatorial DT-invariant

Main Theorem

Let $\underline{i}$ and $\underline{i}'$ be mutation sequences.

$$\exists \text{ frozen } \mu_{\underline{i}}(\widetilde{Q}) \cong \mu_{\underline{i}'}(\widetilde{Q}') \quad \Longrightarrow \quad \mathbb{E}(\underline{i}) = \mathbb{E}(\underline{i}').$$

Remark

In particular, if $\underline{i}$ and $\underline{i}'$ are reddening sequences, then by the first theorem, we have $\mathbb{E}(\underline{i}) = \mathbb{E}(\underline{i}')$.

Definition

If Q admits a reddening sequence $\underline{i}$, its combinatorial DT-invariant is

$$\mathbb{E}_Q = \mathbb{E}(\underline{i}) \in \widehat{\mathbb{A}}_Q.$$

The adjoint (combinatorial) DT-invariant

Definition (Reminder)

If Q admits a reddening sequence $\underline{i}$, its combinatorial DT-invariant is

$$\mathbb{E}_Q = \mathbb{E}(\underline{i}) \in \widehat{\mathbb{A}}_Q.$$

Definition

If Q admits a reddening sequence $\underline{i}$, its adjoint DT-invariant is

$$DT_Q : \mathrm{Frac}(\mathbb{A}_Q) \xrightarrow{\sim} \mathrm{Frac}(\mathbb{A}_Q), u \mapsto \mathbb{E}_Q\,(\Sigma u)\,\mathbb{E}_Q^{-1},$$

where $\Sigma(y^\alpha) = y^{-\alpha}$, $\alpha \in \mathbb{N}^n$.

Example

$Q = \vec{A}_1 \Longrightarrow DT_Q^2 = \mathrm{Id}, \qquad Q = \vec{A}_2 \Longrightarrow DT_Q^5 = \mathrm{Id}.$

Example 1: Dynkin quivers

Let Q be an alternating Dynkin quiver $\vec{\Delta}$, where Δ is an ADE Dynkin diagram, e. g.

$$Q = \vec{A}_5 : \bullet \longleftarrow \circ \longrightarrow \bullet \longleftarrow \circ \longrightarrow \bullet$$

Put

$$\begin{aligned} i_+ &= \text{ sequence of all sources } \circ \\ i_- &= \text{ sequence of all sinks } \bullet. \end{aligned}$$

Then $\underline{i} = i_+ i_-$ is maximal green and so is

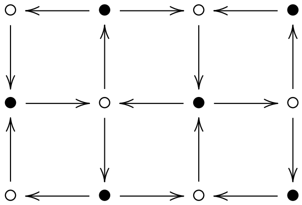
$$\underline{i}' = \underbrace{i_- i_+ i_- \ldots}_{h \text{ factors}} \quad ,$$

where h is the Coxeter number of the underlying graph of Q. Thus, we have $\mathbb{E}_Q = \mathbb{E}(\underline{i}) = \mathbb{E}(\underline{i}')$. These are Reineke's identities. They follow from the pentagon. We have $DT_Q^{h+2} = \mathrm{Id}$.

Example 2: Square products of Dynkin diagrams

$Q = \Delta\Box\Delta'$, where Δ and Δ' are ADE diagrams, e. g.

$Q = A_4 \Box A_3$:



$$i_+ = \text{ sequence of all } \circ$$

$$i_- = \text{ sequence of all } \bullet$$

$$\underline{i} \;= i_+ i_- i_+ \ldots \quad \text{with } h \text{ factors, is maximal green,}$$

$$\underline{i}' = i_- i_+ i_- \ldots \quad \text{with } h' \text{ factors, is maximal green.}$$

We get $\mathbb{E}_Q = \mathbb{E}(\underline{i}) = \mathbb{E}(\underline{i}')$. We have $DT_Q^{2(h+h')} = \mathrm{Id}$.

Two proofs of the main theorem

- ‘Proof’ based on Kontsevich-Soibelman’s theory (preprints from November 2008 and June 2010),
- Proof based on the ‘additive categorification’ of cluster algebras

Some contributors to ‘additive categorification’ (in reverse chronological order): Derksen–Weyman–Zelevinsky, Nagao, Plamondon, ..., Berenstein–Zelevinsky, ..., Buan–Marsh–Reineke–Reiten–Todorov, ..., Caldero–Chapoton, Fock–Goncharov, Fomin–Zelevinsky.

Main steps of the second proof

(1) $\mathbb{E}(\underline{i}) = \mathbb{E}(\underline{i}')$ in $\widehat{\mathbb{A}}_Q$ follows from
(2) $\mathbb{E}(\underline{i}) = \mathbb{E}(\underline{i}')$ in $\widehat{\mathbb{A}}_{\tilde{Q}}$ since $\widehat{\mathbb{A}}_Q \subset \widehat{\mathbb{A}}_{\tilde{Q}}$.
(3) The equality (2) is equivalent to

$$\mathrm{Ad}\,\mathbb{E}(\underline{i}) = \mathrm{Ad}\,\mathbb{E}(\underline{i}')$$

because the center of $\widehat{\mathbb{A}}_{\tilde{Q}}$ is $\mathbb{Q}(q^{1/2})$.
(4) The equality (3) is equivalent to its specialization at $q^{1/2} = 1$! (by a theorem on quantum cluster algebras due to Berenstein-Zelevinsky)
(5) The specialization of $\mathrm{Ad}\,\mathbb{E}(\underline{i})$ at $q^{1/2} = 1$ can be expressed in terms of Euler characteristics of quiver grassmannians of certain representations of Q (by the 'main theorem of additive categorification').
(6) One shows that these representations only depend on the class of $\mu_{\underline{i}}(\widetilde{Q})$ modulo frozen isomorphism.

Soon a third proof?

- Future proof based on mirror symmetry thanks to ongoing work by Gross–Keel–Kontsevich ?

Summary

- Quiver mutation yields combinatorial DT invariants.
- At the same time, it yields a host of quantum dilogarithm identities.
- Do these follow from the pentagon ?
- Google ‘quiver mutation in Java’ or ’quiver mutation in Sage’!